SIMULTANEOUSLY SELECTING NORMAL POPULATIONS CLOSE TO A CONTROL

by

TaChen Liang
Wayne State University

Technical Report # 95-26C

Department of Statistics
Purdue University

June 1995
Revised June 1996
SIMULTANEOUSLY SELECTING NORMAL POPULATIONS CLOSE TO A CONTROL*

TaChen Liang

Department of Mathematics
Wayne State University
Detroit, MI 48202

Abstract

We study the problem of selecting populations close to a control from among $k$ normal populations using the parametric empirical Bayes approach. A Bayes selection rule is derived, which depends on certain parameters. When those parameters are unknown, using the empirical Bayes idea, we first present estimators, based on information collected from the $k$ populations for the unknown parameters. Then, mimicking the behavior of the Bayes selection rule, an empirical Bayes selection rule is constructed. The relative regret Bayes risk is used as a measure of performance of the empirical Bayes selection rule. It is shown that the relative regret Bayes risk of the proposed empirical Bayes selection rule converges to zero at a rate of order $O(k^{-1})$. A simulation study is also carried out to investigate the performance of the proposed empirical Bayes selection rule for small to moderate values of $k$.

AMS 1991 Subject Classification: Primary 62F07; Secondary 62C12.

Keywords and phrases: Asymptotically optimal; empirical Bayes, rate of convergence; relative regret Bayes risk; simultaneous selection.

*This research was supported in part by US Army Research Office, Grant DAAH04-95-1-0165 under the direction of Professor S. S. Gupta
1. Introduction

Problems of selecting populations close to a control arise frequently in many applications. Assume that there are \( k \) populations and a target value used as a control. Our goal is to select those populations which are sufficiently close to the control. To motivate such a study, for example, consider the matching parts problem in industrial production described in Burr (1976). A diesel engine plant has to make plunger rods for forcing fuel through small holes. The diameter of the plunger rods should meet certain specification limits. Suppose there are several plunger rods. Then, we may wish to select one or all those rods which meet the specification limits. Also, as described by Wellek and Michaelis (1991), such selection problem arises in clinical trials and bioavailability trials, for instance, to identify the equivalence of a newly developed formulation of a drug with different administration methods to a standard formulation.

In the literature, Gupta and Singh (1979) and Gupta and Hsiao (1981) have derived Bayes, \( \Gamma \)-minimax and minimax procedures for selecting populations close to a control. Mee, Shah and Lefante (1987) have developed multiple testing procedures to compare the means of \( k \) normal populations with respect to a control. Giani and Straßburger (1994) have studied testing and selection procedures for equivalence of \( k \) populations with respect to a control. It should be noted that comparing populations with a control under different types of formulation has been investigated in the literature. To mention a few, for example, Bechhofer and Turnbull (1978), Dunnett (1984), Wilcox (1984) and Gupta, Liang and Rau (1994) have discussed problems of selecting the best population provided that the best is better than a control. Paulson (1952) and Gupta and Sobel (1958) have studied problems of selecting a subset containing all populations better than a control. Randles and Hollander (1971) and Miescke (1981) have derived optimal selection rules via the \( \Gamma \)-minimax and minimax approaches for selecting good populations. Huang (1975) has derived Bayes selection rules to partition normal populations.

In this paper, we study the problem of selecting populations close to a control from among \( k \) normal populations according to the Kullback-Leibler discrimination information. We will derive empirical Bayes simultaneous selection rules for this selection problem and investigate the corresponding optimality of the empirical Bayes selection rule. The paper
is organized as follows. The framework of the selection problem is given in Section 2. An empirical Bayes selection rule is proposed is Section 3. The asymptotic optimality of the empirical Bayes selection rule is investigated in Section 4. The relative regret Bayes risk is used as a measure of performance of the empirical Bayes selection rule. It is shown that the relative regret Bayes risk of the concerned empirical Bayes selection rule converges to zero with a rate of order $O(k^{-1})$. A simulation study is carried out to investigate the performance of the empirical Bayes selection rule for small to moderate values of $k$.

2. Formulation of the Selection Problem and A Bayes Selection Rule

Consider $k$ independent normal populations $\pi_1, \ldots, \pi_k$ with unknown means $\theta_1, \ldots, \theta_k$, respectively, and a common variance $\sigma^2$. Let $\theta_0$ be a known control value. Also, let $X_i$ denote a random variable arising from population $\pi_i$ and $f(x|\theta_i, \sigma^2)$ denote the density of a $N(\theta_i, \sigma^2)$ distribution. Then, the distance between population $\pi_i$ and the control $\theta_0$ is defined as:

$$
\delta_i = E_{\theta_i} \left[ \frac{\ln f(X_i|\theta_i, \sigma^2)}{f(X_i|\theta_0, \sigma^2)} \right] = \frac{(\theta_i - \theta_0)^2}{2\sigma^2} \quad (2.1)
$$

the Kullback-Leibler discrimination information between two normal distributions $N(\theta_i, \sigma^2)$ and $N(\theta_0, \sigma^2)$. Note that $\delta_i$ is increasing in $|\theta_i - \theta_0|$ and $\delta_i = 0$ as $\theta_i = \theta_0$.

For a given constant $c > 0$, population $\pi_i$ is said to be good if $\delta_i \leq c$, and bad otherwise. Our selection goal is to select all good populations (or all populations with at most distance $c$ from the control $\theta_0$) and to exclude all bad populations.

Let $\Omega = \{ \theta = (\theta_1, \ldots, \theta_k, \sigma^2) | \theta_i \in \mathbb{R}, i = 1, \ldots, k; \sigma^2 > 0 \}$ be the parameter space. Let $a = (a_1, \ldots, a_k)$ denote an action, where $a_i = 0, 1, i = 1, \ldots, k$. Whenever action $a$ is taken, it means that population $\pi_i$ is selected as good if $a_i = 1$ and excluded as bad if $a_i = 0$. The following loss function is adopted:

$$
L(\theta, a) = \sum_{i=1}^{k} L_i(\theta, a_i) \quad (2.2)
$$

where for each $i = 1, \ldots, k$,

$$
L_i(\theta, a_i) = a_i(\delta_i - c)I_{(c,\infty)}(\delta_i) + (1 - a_i)(c - \delta_i)I_{[0,c]}(\delta_i), \quad (2.3)
$$

where $I_s$ denotes the indicator function of the set $S$. 

3
In (2.3), the first term is the loss of selecting population \( \pi_i \) as good while \( \pi_i \) is at least \( c \) distance away from the control \( \theta_0 \), and the second term is the loss of wrongly excluding \( \pi_i \) as bad one while \( \pi_i \) is within \( c \) distance from the control \( \theta_0 \).

For each \( i = 1, \ldots, k \), let \( Y_{i1}, \ldots, Y_{im} \) be a sample of size \( m \) taken from population \( \pi_i \). It is assumed that \( \theta_i \) is a realization of a random variable \( \Theta_i \) which has a \( N(\theta_0, \tau^2) \) prior distribution with unknown common variance \( \tau^2 \). The random variables \( \Theta_1, \ldots, \Theta_k \) are assumed to be mutually independent.

Let \( Y_i = (Y_{i1}, \ldots, Y_{im}) \), \( i = 1, \ldots, k \), and \( Y = (Y_1, \ldots, Y_k) \) and let \( \mathcal{Y} \) denote the sample space of \( Y \). A selection rule \( d = (d_1, \ldots, d_k) \) is a mapping defined on the sample space \( \mathcal{Y} \) into \( [0,1]^k \), such that for each \( y \in \mathcal{Y} \), \( d(y) = (d_1(y), \ldots, d_k(y)) \) where \( d_i(y) \) is the probability of selecting population \( \pi_i \) as good.

Under the preceding statistical model, the Bayes risk of a selection rule \( d \) is:

\[
R_k(d) = \sum_{i=1}^{k} R_{ki}(d_i)
\]

where

\[
R_{ki}(d_i) = \int_{\mathcal{Y}} d_i(y)[\varphi_i(y_i) - c]\prod_{j=1}^{k} f_j(y_j)dy + C_i
\]

and

\[
\left\{
\begin{array}{l}
C_i = E[(c - \frac{(\Theta_i - \theta_0)^2}{2\sigma^2})I_{[0,c]}(\frac{(\Theta_i - \theta_0)^2}{2\sigma^2})], \\
f_j(y_j) \text{ is the marginal probability density of } Y_j, \\
\varphi_i(y_i) = E[(\frac{(\Theta_i - \theta_0)^2}{2\sigma^2})|Y_i = y_i].
\end{array}
\right.
\]

Since given \( Y_i = y_i \), \( \Theta_i \) has a posterior normal distribution with mean \( B\theta_0 + (1-B)\bar{y}_i \) and variance \( B\tau^2 \), where \( \bar{y}_i = \frac{1}{m}\sum_{j=1}^{m} y_{ij} \) and \( B = \frac{\sigma^2}{m}(\frac{\sigma^2}{m} + \tau^2) \), it follows that

\[
\varphi_i(y_i) = \frac{\text{Var}(\Theta_i|Y_i = y_i)}{2\sigma^2} + \frac{E[\Theta_i|Y_i = y_i] - \theta_0}{2\sigma^2}
\]

\[
= \frac{\frac{\sigma^2}{m}\tau^2}{2\sigma^2(\frac{\sigma^2}{m} + \tau^2)} + \frac{(\bar{y}_i - \theta_0)^2}{2\sigma^2} \times \frac{\tau^4}{(\frac{\sigma^2}{m} + \tau^2)^2}
\]

\[
= \frac{1}{2m}[1-B] + \frac{(\bar{y}_i - \theta_0)^2}{2\sigma^2}(1-B)^2 \equiv \psi_i(\bar{y}_i).
\]
Hence, a Bayes selection rule \( d_B = (d_{B1}, \ldots, d_{Bk}) \), which minimizes the Bayes risks \( R_k(d) \) among all selection rules, is given as follows:

For each \( y \in \mathcal{Y} \), and \( i = 1, \ldots, k \),

\[
d_{Bi}(y) = \begin{cases} 
1 & \text{if } \psi_i(\overline{y}_i) \leq c, \\
0 & \text{otherwise}.
\end{cases} \quad (2.7)
\]

From (2.6) and (2.7), we see that for each component \( i \), the Bayes selection rule \( d_{Bi} \) is independent of \( y_j \), for all \( j \neq i \), and depends on \( y_i \) only through the sample mean value \( \overline{y}_i \). Therefore, it can be written as \( d_{Bi}(\overline{y}) \). That is, \( d_{Bi}(\overline{y}_i) = d_{Bi}(y_i) \). The minimum Bayes risk is:

\[
R_k(d_B) = \sum_{i=1}^{k} R_{ki}(d_{Bi}) \quad (2.8)
\]

and

\[
R_{ki}(d_{Bi}) = \int_{-\infty}^{\infty} d_{Bi}(\overline{y}_i) \left| \psi_i(\overline{y}_i) - c \right| g_i(\overline{y}_i) d\overline{y}_i + C_i \quad (2.9)
\]

where \( g_i(\overline{y}_i) \) is the marginal pdf of the sample mean \( \overline{Y}_i = \frac{1}{m} \sum_{j=1}^{m} Y_{ij} \). According to the statistical model described previously, it is known that \( \overline{Y}_1, \ldots, \overline{Y}_k \) are iid, with a normal distribution \( N(\theta_0, \frac{\sigma^2}{m} + \tau^2) \).

Let \( c^* = 2\sigma^2 [c - \frac{1-B}{2m}] / (1-B)^2 \). Then \( \psi_i(\overline{y}_i) - c \leq 0 \) if and only if \( (\overline{y}_i - \theta_0)^2 \leq c^* \). If \( c^* < 0 \), then \( \psi_i(\overline{y}_i) - c > 0 \) for all \( \overline{y}_i, i = 1, \ldots, k \). Hence \( d_{Bi}(\overline{y}_i) = 0 \) for all \( \overline{y}_i, i = 1, \ldots, k \). When \( c^* > 0 \), the Bayes selection rule \( d_B \) can be rewritten as follows: For each \( i = 1, \ldots, k \),

\[
d_{Bi}(\overline{y}_i) = \begin{cases} 
1 & \text{if } \overline{y}_i \in I, \\
0 & \text{otherwise},
\end{cases} \quad (2.10)
\]

where \( I = [\theta_0 - \sqrt{c^*}, \theta_0 + \sqrt{c^*}] \).

Finally, we note that for each \( i \), \( \psi_i(\overline{y}_i) \) is increasing in \( |\overline{y}_i - \theta_0| \) and \( d_{Bi}(\overline{y}_i) \) is nonincreasing in \( |\overline{y}_i - \theta_0| \). Also, since when \( c^* < 0 \), \( d_{Bi}(\overline{y}_i) = 0 \) for all \( \overline{y}_i, i = 1, \ldots, k \), and this may be an extreme case. Hence, in the following analysis, it is assumed that \( c > \frac{1}{2m} \) and therefore \( c^* > 0 \).
3. An Empirical Bayes Selection Rule

It should be noted that the Bayes selection rule \( d_B \) depends on \( \psi_i(\overline{y}_i), i = 1, \ldots, k \), which are dependent on \( \sigma^2 \) and \( \tau^2 \). When the parameters are unknown, the Bayes selection rule \( d_B \) cannot be implemented for the selection problem at hand. The unknown parameters should be estimated. In the following, the parametric empirical Bayes approach is employed for estimating the unknown parameters and deriving a selection rule.

For each \( i = 1, \ldots, k \), let \( S_i = \sum_{j=1}^{m}(Y_{ij} - \overline{Y}_i)^2 \) and \( S = \sum_{i=1}^{k} S_i \). It is known that \( S_i/\sigma^2 \sim \chi^2_{m-1}, i = 1, \ldots, k; S_1, \ldots, S_k \) are mutually independent and hence \( S/\sigma^2 \sim \chi^2_{k(m-1)} \). Let \( W = \sum_{i=1}^{k}(\overline{Y}_i - \theta_0)^2 \). Since \( \overline{Y}_1, \ldots, \overline{Y}_k \) are iid, having a \( N(\theta_0, \sigma^2/m + \tau^2) \) distribution, \( W/(\sigma^2/m + \tau^2) \sim \chi^2_k \). Note that \( E[\frac{S}{k(m-1)}] = \sigma^2, E[\frac{W}{k}] = \frac{\sigma^2}{m} + \tau^2 \). Hence we may use \( \hat{B} = \left( \frac{S/(k(m-1)m)}{W/k} \wedge 1 \right) \) to estimate \( B \) by noting that \( B = \frac{\sigma^2}{m}/(\frac{\sigma^2}{m} + \tau^2) < 1 \) where \( a \wedge b = \min(a, b) \). Also, we use \( \hat{\sigma}^2 = \frac{S}{k(m-1)} \) to estimate \( \sigma^2 \).

Define

\[
\psi_i^{*}(\overline{y}_i) = \frac{1}{2m}[1 - \hat{B}] + \frac{(\overline{y}_i - \theta_0)^2}{2\hat{\sigma}^2}[1 - \hat{B}]^2, i = 1, \ldots, k. \tag{3.1}
\]

\( \psi_i^{*}(\overline{y}_i) \) is a mimicry of \( \psi_i(\overline{y}_i) \) with the unknown parameters \( B \) and \( \sigma^2 \) being replaced by the corresponding estimators \( \hat{B} \) and \( \hat{\sigma}^2 \), respectively. Now, an empirical Bayes simultaneous selection rule \( d_k^{*} = (d_{k1}^{*}, \ldots, d_{kk}^{*}) \) is proposed as follows.

For each \( i = 1, \ldots, k \); and each \( y \in \mathcal{Y} \), define

\[
d^{*}_{ki}(\overline{y}_i) = d^{*}_{ki}(\overline{y}_i|\hat{B}, \hat{\sigma}^2) = \begin{cases} 
1 & \text{if } \psi_i^{*}(\overline{y}_i) \leq c, \\
0 & \text{otherwise.} \end{cases} \tag{3.2}
\]

Note that for each \( i = 1, \ldots, k \), \( d^{*}_{ki} \) depends on \( y_i \) as well as \( y_j, j \neq i \), through \( \overline{y}_i, \hat{B} \), and \( \hat{\sigma}^2 \). Also, it can be seen that \( \psi_i^{*}(\overline{y}_i) \) is increasing in \( |\overline{y}_i - \theta_0| \) and therefore, \( d^{*}_{ki}(\overline{y}_i|\hat{B}, \hat{\sigma}^2) \) is nonincreasing in \( |\overline{y}_i - \theta_0| \).

The Bayes risk of the empirical Bayes selection rule \( d^{*}_{k} \) is

\[
R_k(d^{*}_{k}) = \sum_{i=1}^{k} R_{ki}(d^{*}_{ki}). \tag{3.3}
\]
where
\[
R_{ki}(d_{ki}^*) = \int E_i[d_{ki}^*(\bar{Y}_i|\hat{B}, \hat{\sigma}^2)|\bar{Y}_i = \bar{y}_i][\psi_i(\bar{y}_i) - c]g_i(\bar{y}_i)d\bar{y}_i + C_i
\]
\[= \int P_i[d_{ki}^*(\bar{y}_i|\hat{B}, \hat{\sigma}^2) = 1|\bar{Y}_i = \bar{y}_i][\psi_i(\bar{y}_i) - c]g_i(\bar{y}_i)d\bar{y}_i + C_i. \tag{3.4}
\]
In (3.4), \(P_i\) is the conditional probability measure generated by \(\hat{B}\) and \(\hat{\sigma}^2\) conditioning on \(\bar{Y}_i = \bar{y}_i\) and \(E_i\) is the expectation taken with respect to the conditional probability measure \(P_i\) given \(\bar{Y}_i = \bar{y}_i\).

**Example.** The example of Romano (1977, page 248) is used to illustrate the application of the empirical Bayes selection rule \(d_{ki}^*\). Four product lines in an industrial corporation are set to manufacture a specific type of ball bearing with a diameter of 1mm. An experimenter is interested in finding out all those product lines for which the associated Kullback Leibler discrimination information from the control value \(\theta_0 = 1\)mm is at most 0.1. For this purpose, at the end of a day’s production, ten ball bearings are randomly and independently selected from each of the four lots manufactured by the product lines. The data is given below.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4 (mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\bar{Y}_i)</td>
<td>1.194</td>
<td>1.406</td>
<td>1.129</td>
<td>1.226</td>
</tr>
<tr>
<td>(S_i)</td>
<td>0.7552</td>
<td>1.6510</td>
<td>1.5319</td>
<td>0.5328</td>
</tr>
</tbody>
</table>

Note that \(\theta_0 = 1\)mm, \(c = 0.1\), \(k = 4\) and \(m = 10\). Then, \(S = 4.4709\), \(W = 0.270189\), \(\hat{\sigma}^2 = 0.124192\) and \(\hat{B} = 0.183859\). Hence,

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\psi_i(\bar{y}_i))</td>
<td>0.1417</td>
<td>0.4828</td>
<td>0.0854</td>
<td>0.1778</td>
</tr>
<tr>
<td>(d_{ki}^*(\bar{y}_i))</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

That is, the empirical Bayes selection rule \(d_{ki}^*\) selects product line 3 only and excludes the other three product lines.

4. **Asymptotic Optimality**

For a selection rule \(d = (d_1, \ldots, d_k)\) let \(R_k(d)\) denote the Bayes risk of \(d\). Since \(d_B\) is the Bayes selection rule, \(D_{ki}(d_i) = R_{ki}(d_i) - R_{ki}(d_{Bi}) \geq 0\) for each \(i = 1, \ldots, k\). Hence,
\[ D_k(\tilde{d}) = R_k(\tilde{d}) - R_k(d_B) = \sum_{i=1}^{k} D_{ki}(d_i) \geq 0. \] 

\( D_k(\tilde{d}) \) and \( \rho_k(\tilde{d}) = D_k(\tilde{d})/R_k(d_B) \) are called regret Bayes risk and relative regret Bayes risk, respectively, of the selection rule \( \tilde{d} \). In the following, the relative regret Bayes risk \( \rho_k(\tilde{d}) \) is used as a measure of performance of the selection rule \( \tilde{d} \).

A selection rule \( \tilde{d} \) is said to be asymptotically optimal of order \( \{\beta_k\} \) if \( \rho_k(\tilde{d}) = O(\beta_k) \) where \( \{\beta_k\} \) is a sequence of positive numbers such that \( \lim_{k \to \infty} \beta_k = 0 \).

In the following, we will investigate the asymptotic optimality of the empirical Bayes selection rule \( d_{k}^* \). For doing so, note that under the previously described statistical model, for the Bayes and empirical Bayes selection rules \( d_B \) and \( d_k^* \), we have: \( R_{k1}(d_{B1}) = \cdots = R_{kk}(d_{Bk}) \) and \( R_{k1}(d_{k1}^*) = \cdots = R_{kk}(d_{k1}^*) \). Hence, \( \rho_k(d_k^*) = \left[ R_{k1}(d_{k1}^*) - R_{k1}(d_{B1}) \right]/R_{k1}(d_{B1}) \). Since \( R_{k1}(d_{B1}) \) is fixed for all \( k \), therefore, it suffices to investigate the asymptotic behavior of the regret Bayes risk \( R_{k1}(d_{k1}^*) - R_{k1}(d_{B1}) \).

Let \( c_1(\overline{y}_1) = \psi_1(\overline{y}_1) - c \). Also, let \( J = R - I \), the complement of the interval \( I = [\theta_0 - c^*, \theta_0 + c^*] \). Note that \( d_{B1}(\overline{y}_1) = 1, c_1(\overline{y}_1) \leq 0 \) if \( \overline{y}_1 \in I \); and \( d_{B1}(\overline{y}_1) = 0, c_1(\overline{y}_1) > 0 \) for \( \overline{y}_1 \in J \). From (2.9), (3.4) and the fact that \( d_B \) is a Bayes selection rule, we have that

\[ 0 \leq R_{k1}(d_{k1}^*) - R_{k1}(d_{B1}) \]

\[ = \int E_1[d_k^*(\overline{y}_1|\hat{B}, \hat{\sigma}^2) - d_{B1}(\overline{y}_1)|\overline{Y}_1 = \overline{y}_1]c_1(\overline{y}_1)g_1(\overline{y}_1)d\overline{y}_1 \]

\[ = \int_J P_1[d_k^*(\overline{y}_1|\hat{B}, \hat{\sigma}^2) = 0|\overline{Y}_1 = \overline{y}_1]|c_1(\overline{y}_1)]g_1(\overline{y}_1)d\overline{y}_1 \]

\[ + \int_J P_1[d_k^*(\overline{y}_1|\hat{B}, \hat{\sigma}^2) = 1|\overline{Y}_1 = \overline{y}_1]c_1(\overline{y}_1)g_1(\overline{y}_1)d\overline{y}_1. \]
By the definitions of $d_{k1}^*, \psi_1^*$ and by an application of Bonferroni inequality, for each $\bar{y}_1 \in I$,

$$P_1\{d_{k1}^*(\bar{y}_1 | \hat{B}, \hat{\sigma}^2) = 0 | \bar{Y}_1 = \bar{y}_1\}$$

$$= P_1\{\psi_1^*(\bar{y}_1) > c | \bar{Y}_1 = \bar{y}_1\}$$

$$= P_1\left\{\left[\frac{1}{2m} (1 - \hat{B}) + \frac{(\bar{y}_1 - \theta_0)^2}{2\hat{\sigma}^2} (1 - \hat{B})^2\right] - \frac{1}{2m} (1 - B) + \frac{(\bar{y}_1 - \theta_0)^2}{2\sigma^2} (1 - B)^2\right\} > -c_1(\bar{y}_1) | \bar{Y}_1 = \bar{y}_1\}$$

$$\leq P_1\left\{\frac{B - \hat{B}}{2m} > -\frac{c_1(\bar{y}_1)}{3} | \bar{Y}_1 = \bar{y}_1\right\}$$

$$+ P_1\left\{\frac{(\bar{y}_1 - \theta_0)^2}{2\sigma^2} [2 - B - B][B - \hat{B}] > -\frac{c_1(\bar{y}_1)}{3} | \bar{Y}_1 = \bar{y}_1\right\}$$

$$+ P_1\left\{\frac{c_1(\bar{y}_1)}{3} | \bar{Y}_1 = \bar{y}_1\right\}$$

$$\leq P_1\{\hat{B} - B < \frac{2mc_1(\bar{y}_1)}{3} | \bar{Y}_1 = \bar{y}_1\}$$

$$+ P_1\{\hat{B} - B < \frac{2\sigma^2 c_1(\bar{y}_1)}{3(\bar{y}_1 - \theta_0)^2} | \bar{Y}_1 = \bar{y}_1\}$$

$$+ P_1\{\hat{\sigma}^2 - \sigma^2 < \frac{\sigma^4 c_1(\bar{y}_1)}{2\sigma^2[-c_1(\bar{y}_1)] + 3(\bar{y}_1 - \theta_0)^2} | \bar{Y}_1 = \bar{y}_1\} \right\}$$

(4.2)

Similarly, for each $\bar{y}_1 \in J$,

$$P_1\{d_{k1}^*(\bar{y}_1 | \hat{B}, \hat{\sigma}^2) = 1 | \bar{Y}_1 = \bar{y}_1\}$$

$$= P_1\{\psi_1^*(\bar{y}_1) \leq c | \bar{Y}_1 = \bar{y}_1\}$$

$$= P_1\left\{\frac{B - \hat{B}}{2m} + \frac{(\bar{y}_1 - \theta_0)^2}{2\sigma^2} (1 - \hat{B})^2 - \frac{(\bar{y}_1 - \theta_0)^2}{2\sigma^2} (1 - B)^2 < -c_1(\bar{y}_1) | \bar{Y}_1 = \bar{y}_1\right\}$$

$$\leq P_1\{\hat{B} - B > \frac{2mc_1(\bar{y}_1)}{3} | \bar{Y}_1 = \bar{y}_1\}$$

$$+ P_1\{\hat{B} - B > \frac{2\sigma^2 c_1(\bar{y}_1)}{3(\bar{y}_1 - \theta_0)^2} | \bar{Y}_1 = \bar{y}_1\}$$

$$+ P_1\{\hat{\sigma}^2 - \sigma^2 > \frac{2\sigma^4 c_1(\bar{y}_1)}{2\sigma^2[-c_1(\bar{y}_1)] + 3(\bar{y}_1 - \theta_0)^2} | \bar{Y}_1 = \bar{y}_1\} \right\}$$

(4.3)

Combining (4.1)-(4.3) together yields that

$$0 \leq R_{k1}(d_{k1}^*) - R_{k1}(d_{B1}) \leq A_1 + A_2 + A_3 + B_1 + B_2 + B_3$$

(4.4)
where
\[
A_1 = \int_J P_1 \{ \hat{Y} - B < \frac{2mc_1(\bar{y}_1)}{3} | \bar{Y}_1 = \bar{y}_1 \} [-c_1(\bar{y}_1)]g_1(\bar{y}_1) d\bar{y}_1,
\]
\[
A_2 = \int_J P_1 \{ \hat{Y} - B < \frac{2\sigma^2 c_1(\bar{y}_1)}{3(\bar{y}_1 - \theta_0)^2} | \bar{Y}_1 = \bar{y}_1 \} [-c_1(\bar{y}_1)]g_1(\bar{y}_1) d\bar{y}_1,
\]
\[
A_3 = \int_J P_1 \{ \hat{Y} - B < \frac{\sigma^4 c_1(\bar{y}_1)}{2\sigma^2 [-c_1(\bar{y}_1)] + 3(\bar{y}_1 - \theta_0)^2} | \bar{Y}_1 = \bar{y}_1 \} [-c_1(\bar{y}_1)]g_1(\bar{y}_1) d\bar{y}_1,
\]
\[
B_1 = \int_J P_1 \{ \hat{Y} - B > \frac{2mc_1(\bar{y}_1)}{3} | \bar{Y}_1 = \bar{y}_1 \} c_1(\bar{y}_1)g_1(\bar{y}_1) d\bar{y}_1,
\]
\[
B_2 = \int_J P_1 \{ \hat{Y} - B > \frac{2\sigma^2 c_1(\bar{y}_1)}{3(\bar{y}_1 - \theta_0)^2} | \bar{Y}_1 = \bar{y}_1 \} c_1(\bar{y}_1)g_1(\bar{y}_1) d\bar{y}_1,
\]
and
\[
B_3 = \int_J P_1 \{ \hat{Y} - B > \frac{2\sigma^4 c_1(\bar{y}_1)}{2\sigma^2 [-c_1(\bar{y}_1)] + 3(\bar{y}_1 - \theta_0)^2} | \bar{Y}_1 = \bar{y}_1 \} c_1(\bar{y}_1)g_1(\bar{y}_1) d\bar{y}_1.
\]

Therefore, it suffices to investigate the asymptotic behavior for each of the six terms in (4.4). For this purpose, certain useful lemmas are introduced as follows.

**Lemma 4.1** For a random variable $S \sim \chi^2_n$.

(a) $P \{ \frac{S}{n} - 1 \leq C \} \leq \exp \{ -\frac{n}{2}[C - \ln(1 + C)] \}$ for $-1 < C < 0$;

(b) $P \{ \frac{S}{n} - 1 \geq C \} \leq \exp \{ -\frac{n}{2}[C - \ln(1 + C)] \}$ for $C > 0$.

Note: Lemma 4.1 is from Corollary 4.1 of Gupta, Liang and Rau (1994).

For each real value $b$ and $y$, define $\alpha_1(b) = \frac{(k-1)b}{4kB}$, $\alpha_2(b) = \frac{-b}{4(B+b)}$, $c_2(y,b) = \frac{(k-1)b}{kB}$, $c_3(y,b) = \frac{(k-1)b}{kB} + \frac{m(B+b)(y-\theta_0)^2}{k\sigma^2} - \frac{1}{k}$, and $c_3(y,b) = \frac{(k-1)b}{kB} + \frac{m(B+b)(y-\theta_0)^2}{k\sigma^2}$. Also, let $W_1 = \sum_{j=2}^{k} (\bar{Y}_j - \theta_0)^2$. Note that $W_1/(\sigma^2 + \tau^2) \sim \chi^2_{k-1}$. Finally, set $h(c) = c - \ln(1 + c)$.

**Lemma 4.2** For $\bar{y}_1 \in J$ and $b > 0$ such that $(k-1)b > 2B$, we have
\[
P_1 \{ \hat{Y} - B > b | \bar{Y}_1 = \bar{y}_1 \} \leq \exp \{ -\frac{k(m-1)}{2} h(\alpha_1(b)) \} + \exp \{ -\frac{k-1}{2} h(\alpha_2(b)) \}.
\]

Proof: First note that $c_2(y,b) \geq \frac{(k-1)b}{kB} - \frac{1}{k} \geq \frac{(k-1)b}{2kB} = 2\alpha_1(b) > 0$, since $(k-1)b > 2B$. 

10
Then, by the definition of \( \hat{B} \) and the preceding inequality, we can obtain

\[
\begin{align*}
P_1\{ \hat{B} - B > b | \overline{Y}_1 = \overline{y}_1 \} & \leq P_1\left\{ \frac{S}{km(m-1)} - \frac{W(B+b)}{k} > 0 | \overline{Y}_1 = \overline{y}_1 \right\} \\
& = P_1\left\{ \frac{S}{km(m-1)} - \frac{W_1(B+b)}{k} > \frac{(\overline{y}_1 - \theta_0)^2 (B+b)}{2} | \overline{Y}_1 = \overline{y}_1 \right\} \\
& = P_1\left\{ \frac{S}{km(m-1)} - \frac{\sigma^2}{m} - [W_1 - (k-1)\frac{\sigma^2}{m} + \tau^2] \frac{B+b}{k} > \frac{\sigma^2}{m} c_2(\overline{y}_1, b) | \overline{Y}_1 = \overline{y}_1 \right\} \\
& \leq P_1\left\{ \frac{S}{\sigma^2 k(m-1)} - 1 > \frac{1}{2} c_2(\overline{y}_1, b) | \overline{Y}_1 = \overline{y}_1 \right\} \\
& \quad + P_1\left\{ \frac{W_1}{(\frac{\sigma^2}{m} + \tau^2)(k-1)} - 1 < \frac{1}{2} c_2(\overline{y}_1, b) \frac{kB}{(k-1)(B+b)} | \overline{Y}_1 = \overline{y}_1 \right\} \\
& \leq P_1\left\{ \frac{S}{\sigma^2 k(m-1)} - 1 > \alpha_1(b) | \overline{Y}_1 = \overline{y}_1 \right\} \\
& \quad + P_1\left\{ \frac{W_1}{(\frac{\sigma^2}{m} + \tau^2)(k-1)} - 1 < \alpha_2(b) | \overline{Y}_1 = \overline{y}_1 \right\} \\
& \leq \exp\left\{ -\frac{k(m-1)}{2} [\alpha_1(b) - \ln(1 + \alpha_1(b))] \right\} \\
& \quad + \exp\left\{ -\frac{k-1}{2} [\alpha_2(b) - \ln(1 + \alpha_2(b))] \right\} \\
& = \exp\left\{ -\frac{k(m-1)}{2} h(\alpha_1(b)) \right\} + \exp\left\{ -\frac{k-1}{2} h(\alpha_2(b)) \right\}.
\end{align*}
\] (4.5)

In (4.5), the last inequality is obtained from an application of Lemma 4.1 by noting that \( \alpha_1(b) > 0 \) and \( \alpha_2(b) < 0 \) and \( S \) and \( W_1 \) are independent of \( \overline{Y}_1 \).

\[ \square \]

**Lemma 4.3** For each \( \overline{y}_1 \in I \) and \( b < 0 \) such that \( B + b > 0 \) and \( -b > 2mB^2(\overline{y}_1 - \theta_0)^2 / [(k-1)\sigma^2 + 2mB(\overline{y}_1 - \theta_0)^2] \), we have that

\[
P_1\{ \hat{B} - B < b | \overline{Y}_1 = \overline{y}_1 \} \leq \exp\left\{ -\frac{k(m-1)}{2} h(\alpha_1(b)) \right\} + \exp\left\{ -\frac{k-1}{2} h(\alpha_2(b)) \right\}.
\]
Proof: Note that \( c_2(\bar{y}_1, b) = c_3(\bar{y}_1, b) - \frac{1}{k} \leq c_3(\bar{y}_1, b) \). Also, under the assumption of the Lemma, \( c_3(\bar{y}_1, b) < 2\alpha_1(b) < 0 \). Following an argument similar to the proof of Lemma 4.2, and by noting the preceding inequality, we obtain that

\[
P_1\{\hat{B} - B < b | Y_1 = \bar{y}_1\} \\
\leq P_1\{ \frac{S}{\sigma^2 k(m-1)} - 1 < \frac{1}{2} c_2(\bar{y}_1, b) | Y_1 = \bar{y}_1\} \\
\quad + P_1\{ \frac{W_1}{(\sigma^2_m + \sigma^2_r)(k-1)} > -\frac{1}{2} c_2(\bar{y}_1, b) \frac{kB}{(k-1)(B+b)} | Y_1 = \bar{y}_1\} \\
\leq P_1\{ \frac{S}{\sigma^2 k(m-1)} - 1 < \alpha_1(b) | Y_1 = \bar{y}_1\} \\
\quad + P_1\{ \frac{W_1}{(\sigma^2_m + \sigma^2_r)(k-1)} - 1 > \alpha_2(b) | Y_1 = \bar{y}_1\} \\
\leq \exp\left\{ -\frac{k(m-1)}{2} [\alpha_1(b) - \ln(1 + \alpha_1(b))] \right\} \\
\quad + \exp\left\{ -\frac{k-1}{2} [\alpha_2(b) - \ln(1 + \alpha_2(b))] \right\} \\
\leq \exp\left\{ -\frac{k(m-1)}{2} h(\alpha_1(b)) \right\} + \exp\left\{ -\frac{k-1}{2} h(\alpha_2(b)) \right\}. \quad \Box
\]

**Lemma 4.4** For fixed \( t_1 > 0 \), and \( n > 0 \),

\[
\int_0^{t_1} x \exp\left\{ -\frac{n}{2} [x - \ln(1 + x)] \right\} dx = O(n^{-1}).
\]

(b) For \( 0 < t_0 < 1 \) and \( n > 0 \),

\[
\int_0^{t_0} x \exp\left\{ \frac{n}{2} [x + \ln(1 - x)] \right\} dx = O(n^{-1}).
\]

Proof: These results can be obtained through straightforward computation. The details are omitted here. \quad \Box

Now we are going to investigate the asymptotic behavior of \( A_i \) and \( B_i, i = 1, 2, 3 \) for \( k \) being sufficiently large.

**Lemma 4.5**

\[
\int P_1\{\hat{B} - B > \frac{2mc_1(\bar{y}_1)}{3} | Y_1 = \bar{y}_1\} c_1(\bar{y}_1) g_1(\bar{y}_1) d\bar{y}_1 = O(k^{-1}).
\]

12
Proof: Assume $k$ being sufficiently large so that $2\sigma^2(\frac{1-B}{m}+c)/(1-B)^2 > \frac{2B}{k-1}$. Note that by the definition of $\hat{B}, P_1\{\hat{B} - B > \frac{2mc_1(\bar{y}_1)}{3} \mid \bar{Y}_1 = \bar{y}_1\} = 0$ if $\frac{2mc_1(\bar{y}_1)}{3} > 1 - B$, which is equivalent to that $(\bar{y}_1 - \theta_0)^2 \geq 2\sigma^2(\frac{1-B}{m}+c)/(1-B)^2$.

Let
\[
J_1 = \{\bar{y}_1 \in J \mid 0 < \frac{2mc_1(\bar{y}_1)}{3} \leq \frac{2B}{k-1}\},
\]
\[
J_2 = \{\bar{y}_1 \in J \mid \frac{2B}{k-1} < \frac{2mc_1(\bar{y}_1)}{3} < 1 - B\}.
\]

Then,
\[
B_1 = \int_{J_1} P_1\{\hat{B} - B > \frac{2mc_1(\bar{y}_1)}{3} \mid \bar{Y}_1 = \bar{y}_1\}c_1(\bar{y}_1)g_1(\bar{y}_1)d\bar{y}_1
\]
\[
+ \int_{J_2} P_1\{\hat{B} - B > \frac{2mc_1(\bar{y}_1)}{3} \mid \bar{Y}_1 = \bar{y}_1\}c_1(\bar{y}_1)g_1(\bar{y}_1)d\bar{y}_1
\]
\[
\equiv B_{11} + B_{12},
\]

where
\[
B_{11} \leq \int_{J_1} \frac{3B}{m(k-1)}g_1(\bar{y}_1)d\bar{y}_1 \leq \frac{3B}{m(k-1)} = O(k^{-1});
\]

and by Lemma 4.2,
\[
B_{12} \leq \int_{J_2} \exp\{-\frac{k(m-1)}{2}h(\alpha_1(\frac{2mc_1(\bar{y}_1)}{3}))\}c_1(\bar{y}_1)g_1(\bar{y}_1)d\bar{y}_1
\]
\[
+ \int_{J_2} \exp\{-\frac{k-1}{2}h(\alpha_2(\frac{2mc_1(\bar{y}_1)}{3}))\}c_1(\bar{y}_1)g_1(\bar{y}_1)d\bar{y}_1
\]
\[
\equiv B_{121} + B_{122}.
\]

For $\bar{y}_1 \in J_2, \frac{g_1(\bar{y}_1)}{|\bar{y}_1 - \theta_0|} \leq [8c^*\pi(\frac{3}{m} + \tau^2)]^{-\frac{1}{2}} \exp \{-\frac{c^*}{2(\frac{3}{m} + \tau^2)}\} \equiv M^*.$

Let $M^*_1 = M^* \times \frac{2\sigma^2}{(1-B)^2}$. Hence,
\[
B_{121} = \int_{J_2} \exp\{-\frac{k(m-1)}{2}h(\alpha_1(\frac{2mc_1(\bar{y}_1)}{3}))\}c_1(\bar{y}_1)g_1(\bar{y}_1)\times \frac{2\sigma^2}{(1-B)^2}d\bar{c}\bar{1}(\bar{y}_1)
\]
\[
\leq \int_{J_2} M^*_1 c_1(\bar{y}_1)\exp\{-\frac{k(m-1)}{2}h(\alpha_1(\frac{2mc_1(\bar{y}_1)})\})dc_1(\bar{y}_1)
\]
\[
= \int_{\frac{1}{m(k-1)}}^{1-B} \left(\frac{3}{2m}\right)^2 M^*_1 z \exp\{-\frac{k(m-1)}{2}h(\alpha_1(z))\}dz
\]
\[
= M^* \left[\frac{6kB}{m(k-1)}\right]^2 \int_{\frac{1}{4kB}}^{\frac{(k-1)(1-B)}{4kB}} \alpha \exp\{-\frac{k(m-1)}{2}[\alpha - \ln(1 + \alpha)]\}d\alpha
\]
\[
= O(k^{-1}) \text{ by Lemma 4.4(a).}
\]
Also,
\[
B_{122} \leq M^* \left( \frac{3}{2m} \right)^2 \int_{1-\frac{b}{m(k-1)}}^{1-B} \exp\left\{ -\frac{k-1}{2} h(\alpha_2(z)) \right\} z \, dz \\
= M^* \left( \frac{6B}{m} \right)^2 \int_{\frac{1-\frac{b}{m(k-1)+1}}{4}}^{1-B} \frac{\alpha}{(1-4\alpha)^2} \exp\left\{ -\frac{k-1}{2} [\alpha + \ln(1-\alpha)] \right\} \, d\alpha \\
\leq M^* \left( \frac{6B}{m} \right)^2 \int_{\frac{1-\frac{b}{m(k-1)+1}}{4}}^{1-B} \frac{\alpha}{B^2} \exp\left\{ -\frac{k-1}{2} [\alpha + \ln(1-\alpha)] \right\} \, d\alpha \\
= O(k^{-1}) \text{ by Lemma 4.4(b).} (4.10)
\]

Now, combining (4.6)-(4.10) together concludes the result of the lemma.

\[\square\]

**Lemma 4.6**

\[
\int_J P_1 \{ \hat{B} - B > \frac{2\sigma^2 c_1(\bar{y}_1)}{3(\bar{y}_1 - \theta_0)^2} \} |\bar{Y}_1 = \bar{y}_1 \} c_1(\bar{y}_1) g_1(\bar{y}_1) \, d\bar{y}_1 = O(k^{-1}).
\]

Proof: For $\bar{y}_1 \in J$, $0 < \frac{2\sigma^2 c_1(\bar{y}_1)}{3(\bar{y}_1 - \theta_0)^2} = \frac{(1-B)^2}{9} + \frac{2\sigma^2}{9^2} \times \frac{1}{(\bar{y}_1 - \theta_0)^2}$, which is increasing in $|\bar{y}_1 - \theta_0|$ and bounded above by $\frac{(1-B)^2}{9}$ since $\frac{1}{2m} (1 - B) - c < 0$ by the assumption that $c > \frac{1}{2m}$ (see the end of Section 2). Assume $k$ being sufficiently large so that $\frac{2\sigma^2 c_1(\theta_0 + 2\sqrt{c^*})}{12c^*} > \frac{2B}{k-1}$.

Let

\[
J_1^* = \{ \bar{y}_1 \in J \mid 0 < \frac{2\sigma^2 c_1(\bar{y}_1)}{3(\bar{y}_1 - \theta_0)^2} \leq \frac{2B}{k-1} \},
\]

\[
J_2^* = \{ \bar{y}_1 \in J \mid |\bar{y}_1 - \theta_0| < 2\sqrt{c^*} \text{ and } \frac{2\sigma^2 c_1(\bar{y}_1)}{3(\bar{y}_1 - \theta_0)^2} > \frac{2B}{k-1} \},
\]

\[
J_3^* = \{ \bar{y}_1 \in J \mid |\bar{y}_1 - \theta_0| \geq 2\sqrt{c^*} \}.
\]

Note that $\bar{y}_1 \in J_2^*$ iff $\frac{2\sigma^2 c_1(\bar{y}_1)}{3(\bar{y}_1 - \theta_0)^2} \geq \frac{2\sigma^2 c_1(\theta_0 + 2\sqrt{c^*})}{12c^*} \equiv \beta^*$. Therefore,

\[
B_2 = \int_{J_1^*} P_1 \{ \hat{B} - B > \frac{2\sigma^2 c_1(\bar{y}_1)}{3(\bar{y}_1 - \theta_0)^2} \} |\bar{Y}_1 = \bar{y}_1 \} c_1(\bar{y}_1) g_1(\bar{y}_1) \, d\bar{y}_1 \\
+ \int_{J_2^*} P_1 \{ \hat{B} - B > \frac{2\sigma^2 c_1(\bar{y}_1)}{3(\bar{y}_1 - \theta_0)^2} \} |\bar{Y}_1 = \bar{y}_1 \} c_1(\bar{y}_1) g_1(\bar{y}_1) \, d\bar{y}_1 \\
+ \int_{J_3^*} P_1 \{ \hat{B} - B > \frac{2\sigma^2 c_1(\bar{y}_1)}{3(\bar{y}_1 - \theta_0)^2} \} |\bar{Y}_1 = \bar{y}_1 \} c_1(\bar{y}_1) g_1(\bar{y}_1) \, d\bar{y}_1 \\
\equiv B_{21} + B_{22} + B_{23}, (4.11)
\]

14
where,
\[
B_{21} \leq \int_{J^*_1} \frac{3B}{(k-1)\sigma^2} (\bar{y}_1 - \theta_0)^2 g_1(\bar{y}_1) d\bar{y}_1 \\
\leq \frac{3B}{(k-1)\sigma^2} \left( \frac{\sigma^2}{m} + \tau^2 \right) = O(k^{-1});
\]

\[
B_{23} \leq \int_{J^*_2} P_1 \{ \hat{B} - B > \beta^* | \bar{Y}_1 = \bar{y}_1 \} c_1(\bar{y}_1) g_1(\bar{y}_1) d\bar{y}_1 \\
\leq \int_{J^*_2} \left[ \exp\left\{ -\frac{k(m-1)}{2} h(\alpha_1(\beta^*)) \right\} + \exp\left\{ -\frac{k-1}{2} h(\alpha_2(\beta^*)) \right\} \right] \frac{(1-B)^2(\bar{y}_1 - \theta_0)^2}{6\sigma^2} g_1(\bar{y}_1) d\bar{y}_1 \\
\leq \left[ \exp\left\{ -\frac{k(m-1)}{2} h(\alpha_1(\beta^*)) \right\} + \exp\left\{ -\frac{k-1}{2} h(\alpha_2(\beta^*)) \right\} \right] \frac{(1-B)^2(\frac{\sigma^2}{m} + \tau^2)}{6\sigma^2} = O(k^{-1})
\]

and by Lemma 4.2,
\[
B_{22} \leq \int_{J^*_2} \exp\left\{ -\frac{k(m-1)}{2} h(\alpha_1(\frac{2\sigma^2 c_1(\bar{y}_1)}{3(\bar{y}_1 - \theta_0)^2})) \right\} c_1(\bar{y}_1) g_1(\bar{y}_1) d\bar{y}_1 \\
+ \int_{J^*_2} \exp\left\{ -\frac{k-1}{2} h(\alpha_2(\frac{2\sigma^2 c_1(\bar{y}_1)}{3(\bar{y}_1 - \theta_0)^2})) \right\} c_1(\bar{y}_1) g_1(\bar{y}_1) d\bar{y}_1 \\
\leq \int_{J^*_2} \exp\left\{ -\frac{k(m-1)}{2} h(\alpha_1(\frac{2\sigma^2 c_1(\bar{y}_1)}{12e^*})) \right\} c_1(\bar{y}_1) g_1(\bar{y}_1) d\bar{y}_1 \\
+ \int_{J^*_2} \exp\left\{ -\frac{k-1}{2} h(\alpha_2(\frac{2\sigma^2 c_1(\bar{y}_1)}{3e^*})) \right\} c_1(\bar{y}_1) g_1(\bar{y}_1) d\bar{y}_1 = O(k^{-1}).
\]

In (4.14), the second inequality is obtained by the fact that for \( \bar{y}_1 \in J^*_2 \), \( h(\alpha_1(\frac{2\sigma^2 c_1(\bar{y}_1)}{3(\bar{y}_1 - \theta_0)^2})) \geq h(\alpha_1(\frac{2\sigma^2 c_1(\bar{y}_1)}{3(\bar{y}_1 - \theta_0)^2})) \) and \( h(\alpha_2(\frac{2\sigma^2 c_1(\bar{y}_1)}{3(\bar{y}_1 - \theta_0)^2})) \geq h(\alpha_2(\frac{2\sigma^2 c_1(\bar{y}_1)}{3(\bar{y}_1 - \theta_0)^2})). \) Also, the last equality is obtained by an argument similar to that for \( B_{12} \) by noting that \( J^*_2 \) is a bounded set. Combining (4.11)-(4.14) together leads to the result of the lemma. \( \square \)

**Lemma 4.7**

\[
\int_J P_1 \{ \sigma^2 - \sigma^2 > \frac{2\sigma^4 c_1(\bar{y}_1)}{2\sigma^2[-c_1(\bar{y}_1)] + 3(\bar{y}_1 - \theta_0)^2} \bar{Y}_1 = \bar{y}_1 \} c_1(\bar{y}_1) g_1(\bar{y}_1) d\bar{y}_1 = O(k^{-1}).
\]

**Proof:** For \( \bar{y}_1 \in J \), \( c_1(\bar{y}_1) > 0 \) and \( 0 < 2\sigma^2[-c_1(\bar{y}_1)] + 3(\bar{y}_1 - \theta_0)^2 < 3(\bar{y}_1 - \theta_0)^2 \).
Hence, \( \frac{2\sigma^4 c_1(\bar{y}_1)}{2\sigma_0^2(-c_1(\bar{y}_1)) + 3(\bar{y}_1 - \theta_0)^2} > \frac{2\sigma^4 c_1(\bar{y}_1)}{3(\bar{y}_1 - \theta_0)^2} \). Therefore,

\[
B_3 \leq \int J P_1 (\hat{\sigma}^2 - \tilde{\sigma}^2 > \frac{2\sigma^4 c_1(\bar{y}_1)}{3(\bar{y}_1 - \theta_0)^2} | \bar{Y}_1 = \bar{y}_1 \}) c_1(\bar{y}_1) g_1(\bar{y}_1) d\bar{y}_1 \\
= \int J P_1 (\hat{\sigma}^2 - \tilde{\sigma}^2 < \frac{2\sigma^2 c_1(\bar{y}_1)}{3(\bar{y}_1 - \theta_0)^2} | \bar{Y}_1 = \bar{y}_1 \}) c_1(\bar{y}_1) g_1(\bar{y}_1) d\bar{y}_1 \\
\equiv B_3^*,
\]

which is a form similar to that of \( B_2 \). Therefore, the technique used to treat \( B_2 \) can be applied here and one can conclude that \( B_3^* = O(k^{-1}) \).

\[\square\]

**Lemma 4.8**

\[
\int I P_1 (\hat{B} - B < \frac{2mc_1(\bar{y}_1)}{3} | \bar{Y}_1 = \bar{y}_1 \}) [-c_1(\bar{y}_1)] g_1(\bar{y}_1) d\bar{y}_1 = O(k^{-1}).
\]

Proof: Note that \( \ell(\bar{y}_1 - \theta_0) = \frac{2mB^2(\bar{y}_1 - \theta_0)^2}{(k-1)\sigma^2 + 2m(\bar{y}_1 - \theta_0)^2} \) is increasing in \( |\bar{y}_1 - \theta_0| \), and \( 0 \leq \ell(\bar{y}_1 - \theta_0) \leq \frac{2mB^2c^*}{(k-1)\sigma^2 + 2mc^*} \) for all \( \bar{y}_1 \in I \). Thus, for \( k \) being sufficiently large, \(-b^* \equiv -\frac{2m}{3} c_1(\theta_0 + \frac{\sqrt{c^*}}{2}) > \ell(\bar{y}_1 - \theta_0) \) for all \( \bar{y}_1 \in I \). Let

\[
I_1 = [\theta_0 - \frac{\sqrt{c^*}}{2}, \theta_0 + \frac{\sqrt{c^*}}{2}], \\
I_2 = \{ \bar{y}_1 \in I - I_1 | 0 \leq \frac{-2mc_1(\bar{y}_1)}{3} \leq \ell(\bar{y}_1 - \theta_0) \}, \\
I_3 = \{ \bar{y}_1 \in I - I_1 | \frac{-2mc_1(\bar{y}_1)}{3} > \ell(\bar{y}_1 - \theta_0) \}.
\]

Then,

\[
A_1 = \int_{I_1} P_1 (\hat{B} - B < \frac{2mc_1(\bar{y}_1)}{3} | \bar{Y}_1 = \bar{y}_1 \}) [-c_1(\bar{y}_1)] g_1(\bar{y}_1) d\bar{y}_1 \\
+ \int_{I_2} P_1 (\hat{B} - B < \frac{2mc_1(\bar{y}_1)}{3} | \bar{Y}_1 = \bar{y}_1 \}) [-c_1(\bar{y}_1)] g_1(\bar{y}_1) d\bar{y}_1 \\
+ \int_{I_3} P_1 (\hat{B} - B < \frac{2mc_1(\bar{y}_1)}{3} | \bar{Y}_1 = \bar{y}_1 \}) [-c_1(\bar{y}_1)] g_1(\bar{y}_1) d\bar{y}_1 \\
\equiv A_{11} + A_{12} + A_{13}.
\]

For \( \bar{y}_1 \in I_1 \), \( \frac{-2mc_1(\bar{y}_1)}{3} \leq \frac{2mc_1(\theta_0 + \frac{\sqrt{c^*}}{2})}{3} = b^* < 0 \). Also, \( |c_1(\bar{y}_1)| \leq c \) for all \( \bar{y}_1 \in I \).
Therefore, by Lemma 4.3,

\[
A_{11} \leq \int_{I_1} P_1\{\hat{B} - B < b^*|Y_1 = \bar{y}_1\}cg_1(\bar{y}_1)d\bar{y}_1
\leq \int_{I_1} c|\exp\{-\frac{k(m-1)}{2}h(\alpha_1(b^*)))\} + \exp\{-\frac{k-1}{2}h(\alpha_2(b^*)))\]|g_1(\bar{y}_1)d\bar{y}_1
\leq c\left|\exp\{-\frac{k(m-1)}{2}h(\alpha_1(b^*)))\} + \exp\{-\frac{k-1}{2}h(\alpha_1(b^*)))\}\right|
= O(k^{-1}).
\]

Also,

\[
A_{12} \leq \int_{I_2} 3\ell(\bar{y}_1 - \theta_0)g_1(\bar{y}_1)d\bar{y}_1
\leq \int_{I_2} \frac{2mB^2c^*}{(k-1)\sigma^2 + 2mc^*}g_1(\bar{y}_1)d\bar{y}_1
= O(k^{-1}).
\]

By Lemma 4.3 and by a proof similar to that of (4.9) and (4.10), we have

\[
A_{13} \leq \int_{I_3} \exp\{-\frac{k(m-1)}{2}h(\alpha_1(\frac{2mc_1(\bar{y}_1)})\})[-c_1(\bar{y}_1)]g_1(\bar{y}_1)d\bar{y}_1
+ \int_{I_3} \exp\{-\frac{k-1}{2}h(\alpha_2(\frac{2mc_1(\bar{y}_1)})\})[-c_1(\bar{y}_1)]g_1(\bar{y}_1)d\bar{y}_1
= O(k^{-1}).
\]

Combining (4.15)-(4.18) together leads to the conclusion of the lemma. \qed

Lemma 4.9

\[
\int_{I} P_1\{\hat{B} - B < \frac{2\sigma^2c_1(\bar{y}_1)}{3(\bar{y}_1 - \theta_0)^2}|Y_1 = \bar{y}_1\}[-c_1(\bar{y}_1)]g_1(\bar{y}_1)d\bar{y}_1 = O(k^{-1}).
\]

Proof: For \(\bar{y}_1 \in I, \frac{2\sigma^2c_1(\bar{y}_1)}{3(\bar{y}_1 - \theta_0)^2} \leq \frac{2\sigma^2c_1(\bar{y}_1)}{3c^*} < 0\). Therefore,

\[
A_2 \leq \int_{I} P_1\{\hat{B} - B < \frac{2\sigma^2c_1(\bar{y}_1)}{3c^*}|Y_1 = \bar{y}_1\}[-c_1(\bar{y}_1)]g_1(\bar{y}_1)d\bar{y}_1
= \equiv A_2^*
\]

which has a form similar to that of \(A_1\). Hence we conclude that \(A_2 = O(k^{-1})\) by Lemma 4.8. \qed

Lemma 4.10

\[
\int_{I} P_1\{\hat{\sigma}^2 - \sigma^2 < \frac{\sigma^4c_1(\bar{y}_1)}{2\sigma^2[-c_1(\bar{y}_1)] + 3(\bar{y}_1 - \theta_0)^2}|Y_1 = \bar{y}_1\}[-c_1(\bar{y}_1)]g_1(\bar{y}_1)d\bar{y}_1 = O(k^{-1}).
\]

17
Proof: For $\bar{y}_1 \in I$, 
\[
\frac{\sigma^4 c_1(\bar{y}_1)}{2\sigma^2[-c_1(\bar{y}_1)] + 3(\bar{y}_1 - \theta_0)^2} \leq \frac{\sigma^4 c_1(\bar{y}_1)}{2\sigma^2 c + 3c^*} < 0.
\]
Hence, similar to that of Lemma 4.8, we can conclude that
\[
A_3 \leq \int I \{ \hat{\sigma}^2 - \sigma^2 < \frac{\sigma^4 c_1(\bar{y}_1)}{2c\sigma^2 + 3c^*} | \bar{Y}_1 = \bar{y}_1 \}[ -c_1(\bar{y}_1)] g_1(\bar{y}_1) d\bar{y}_1
\]
\[
= O(k^{-1}). \quad \square
\]

We summarize the preceding results as a theorem as follows.

**Theorem 4.1** Under the statistical model described in Section 2, the empirical Bayes selection rule $d^*_k$ is asymptotically optimal with $\rho_k(d^*_k) = O(k^{-1}).$

5. Small Sample Performance: Simulation Study

We carried out a simulation study to investigate the small sample performance of the empirical Bayes selection rule $d^*_k$. Note that
\[
\rho_k(d^*_k) = \frac{R_k(d^*_k) - R_k(d_B)}{kR_{k1}(d_{B1})},
\]
where $R_{k1}(d_{B1})$ is a fixed value, independent of the value $k$. Let
\[
B_k(Y) = \sum_{i=1}^{k} [d^*_k(\bar{Y}_i, \hat{B}, \hat{\sigma}^2) - d_{B_i}(\bar{Y}_i)][\psi_i(\bar{Y}_i) - c].
\]

Then,
\[
EB_k(Y) = \sum_{i=1}^{k} E\{[d^*_k(\bar{Y}_i, \hat{B}, \hat{\sigma}^2) - d_{B_i}(\bar{Y}_i)][\psi_i(\bar{Y}_i) - c]\}
\]
\[
= \sum_{i=1}^{k} E_i \{ E_{(i)} \{ E_i \{ d^*_k(\bar{Y}_i, \hat{B}, \hat{\sigma}^2) - d_{B_i}(\bar{Y}_i) \} \psi_i(\bar{Y}_i) - c \} \} \}
\]
\[
= R_k(d^*_k) - R_k(d_B),
\]
where the expectation $E_{(i)}$ is taken with respect to the probability measure generated by $\bar{Y}_i$, and $E_i$ is the expectation taken with respect to the conditional probability measure generated by $\hat{B}$ and $\hat{\sigma}^2$ given $\bar{Y}_i$. Hence,
\[
\rho_k(d^*_k) = E\left[ \frac{B_k(Y)}{kR_{k1}(d_{B1})} \right] = \frac{1}{R_{k1}(d_{B1})} E\left[ \frac{B_k(Y)}{k} \right].
\]
Since $R_{k1}(d_{B1})$ is independent of the value $k$, the relative regret Bayes risk $\rho_k(d^*_k)$ depends on $k$ only through the part $E\left[\frac{B_k(Y)}{k}\right]$.

By the law of large numbers, the sample mean $\overline{B}_k = \frac{1}{n} \sum_{\ell=1}^{n} B_k(Y(\ell))$ can be used as an estimator of the regret Bayes risk $R_k(d^*_k) - R_k(d_B)$, where $Y(\ell), \ell = 1, 2, \ldots, n$, are iid random vectors, identically distributed with $Y$. Therefore, we use $\overline{B}_k/k = \frac{1}{n} \sum_{\ell=1}^{n} B_k(Y(\ell))/k$ to estimate the relationship between $\rho_k(d^*_k)$ and $k$.

The simulation scheme used in this paper is described as follows.

1. For each $i = 1, \ldots, k$, generate the independent random vector $Y_i = (Y_{i1}, \ldots, Y_{im})$ by the following:
   
   (a) Generate $\Theta_i$ from a $N(\theta_0, \tau^2)$ distribution.

   (b) Given $\Theta_i = \theta_i$, generate random sample $Y_{i1} \ldots Y_{im}$ from a $N(\theta_i, \sigma^2)$ distribution.

2. Based on the data $Y = (Y_1, \ldots, Y_k)$, construct the Bayes and empirical Bayes selection rules $d_B$ and $d^*_k$, respectively, and compute the $B_k(Y)$ value.

3. For each $k$, steps (1) and (2) were repeated 1000 times. The average $\overline{B}_k$ of $B_k(Y(\ell)), \ell = 1, \ldots, 1000$, based on the 1000 repetitions is used as an estimator of the regret Bayes risk $R_k(d^*_k) - R_k(d_B)$ and $\overline{B}_k/k$ as an estimator of $R_{k1}(d_{B1})\rho_k(d^*_k) = [R_k(d^*_k) - R_k(d_B)]/k$. Also, $SE(\overline{B}_k/k)$, the estimated standard error of $\overline{B}_k/k$, is computed.

Table 1 lists a simulation result on the performance of the proposed empirical Bayes selection rule $d^*_k$ for the case where $m = 10, \sigma^2 = 2, \tau^2 = 1.5, \theta_0 = 0$ and $c = 0.3$.

From Table 1, we learn that the values of $\overline{B}_k/k$ decrease quite rapidly as $k$ increases. Note that for $k \geq 40$, the estimated regret Bayes risk values $\overline{B}_k$ oscillate about the value 0.0170, which indicates that $\overline{B}_k/k$ converges to 0 with a rate of convergence of order $O(k^{-1})$, same as the conclusion in Theorem 4.1.
Table 1. Small Sample Performance of $d_k^*$ for $m = 10, \sigma^2 = 2, \tau^2 = 1.5, \theta_0 = 0$ and $c = 0.3$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\bar{B}_k$</th>
<th>$\bar{B}_k/k$</th>
<th>$SE(\bar{B}_k/k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>0.0238</td>
<td>0.00238</td>
<td>0.00538</td>
</tr>
<tr>
<td>20</td>
<td>0.0242</td>
<td>0.00121</td>
<td>0.00290</td>
</tr>
<tr>
<td>30</td>
<td>0.0236</td>
<td>0.00079</td>
<td>0.00186</td>
</tr>
<tr>
<td>40</td>
<td>0.0171</td>
<td>0.00043</td>
<td>0.00096</td>
</tr>
<tr>
<td>50</td>
<td>0.0159</td>
<td>0.00032</td>
<td>0.00060</td>
</tr>
<tr>
<td>60</td>
<td>0.0194</td>
<td>0.00032</td>
<td>0.00062</td>
</tr>
<tr>
<td>70</td>
<td>0.0186</td>
<td>0.00027</td>
<td>0.00057</td>
</tr>
<tr>
<td>80</td>
<td>0.0185</td>
<td>0.00023</td>
<td>0.00043</td>
</tr>
<tr>
<td>90</td>
<td>0.0187</td>
<td>0.00021</td>
<td>0.00037</td>
</tr>
<tr>
<td>100</td>
<td>0.0165</td>
<td>0.00017</td>
<td>0.00034</td>
</tr>
<tr>
<td>110</td>
<td>0.0171</td>
<td>0.00016</td>
<td>0.00034</td>
</tr>
<tr>
<td>120</td>
<td>0.0196</td>
<td>0.00016</td>
<td>0.00027</td>
</tr>
<tr>
<td>130</td>
<td>0.0168</td>
<td>0.00013</td>
<td>0.00023</td>
</tr>
<tr>
<td>140</td>
<td>0.0201</td>
<td>0.00015</td>
<td>0.00028</td>
</tr>
<tr>
<td>150</td>
<td>0.0194</td>
<td>0.00013</td>
<td>0.00022</td>
</tr>
<tr>
<td>160</td>
<td>0.0168</td>
<td>0.00010</td>
<td>0.00019</td>
</tr>
<tr>
<td>170</td>
<td>0.0164</td>
<td>0.00010</td>
<td>0.00016</td>
</tr>
<tr>
<td>180</td>
<td>0.0159</td>
<td>0.00009</td>
<td>0.00015</td>
</tr>
<tr>
<td>190</td>
<td>0.0175</td>
<td>0.00009</td>
<td>0.00016</td>
</tr>
<tr>
<td>200</td>
<td>0.0177</td>
<td>0.00009</td>
<td>0.00015</td>
</tr>
</tbody>
</table>
References


Simultaneously Selecting Normal Populations Close to a Control

TaChen Liang

Purdue University
West Lafayette IN 47907

U.S. Army Research Office
P.O. Box 12211
Research Triangle Park, NC 27709-2211

The views, opinions and/or findings contained in this report are those of the author(s) and should not be construed as an official Department of the Army position, policy or decision, unless so designated by other documentation.

Approved for public release; distribution unlimited.

We study the problem of selecting populations close to a control from among k normal populations using the parametric empirical Bayes approach. A Bayes selection rule is derived, which depends on certain parameters. When those parameters are unknown, using the empirical Bayes idea, we first present estimators, based on information collected from the k populations for the unknown parameters. Then, mimicking the behavior of the Bayes selection rule, an empirical Bayes selection rule is constructed. The relative regret Bayes risk is used as a measure of performance of the empirical Bayes selection rule. It is shown that the relative regret Bayes risk of the proposed empirical Bayes selection rule converges to zero at a rate of order O(k^{-1}). A simulation study is also carried out to investigate the performance of the proposed empirical Bayes selection rule for small to moderate values of k.
GENERAL INSTRUCTIONS FOR COMPLETING SF 298

The Report Documentation Page (RDP) is used in announcing and cataloging reports. It is important that this information be consistent with the rest of the report, particularly the cover and title page. Instructions for filling in each block of the form follow. It is important to stay within the lines to meet optical scanning requirements.

| Block 1. Agency Use Only (Leave blank) |
| Block 2. Report Date. Full publication date including day, month, and year, if available (e.g. 1 Jan 88). Must cite at least year. |
| Block 3. Type of Report and Dates Covered. State whether report is interim, final, etc. If applicable, enter inclusive report dates (e.g. 10 Jun 87 - 30 Jun 88). |
| Block 4. Title and Subtitle. A title is taken from the part of the report that provides the most meaningful and complete information. When a report is prepared in more than one volume, repeat the primary title, add volume number, and include subtitle for the specific volume. On classified documents enter the title classification in parentheses. |
| Block 5. Funding Numbers. To include contract and grant numbers: may include program element number(s), project number(s), task number(s), and work unit number(s). Use the following labels: |
| C - Contract | PR - Project |
| G - Grant | TA - Task |
| PE - Program | WU - Work Unit |
| Element | Accession No. |
| Block 6. Author(s). Name(s) of person(s) responsible for writing the report, performing the research, or credited with the content of the report. If editor or compiler, this should follow the name(s). |
| Block 7. Performing Organization Name(s) and Address(es). Self-explanatory. |
| Block 8. Performing Organization Report Number. Enter the unique alphanumeric report number(s) assigned by the organization performing the report. |
| Block 9. Sponsoring/Monitoring Agency Name(s) and Address(es). Self-explanatory. |
| Block 10. Sponsoring/Monitoring Agency Report Number. (If known) |
| Block 11. Supplementary Notes. Enter information not included elsewhere such as: prepared in cooperation with:...; Trans. of...; To be published in:... When a report is revised, include a statement whether the new report supersedes or supplements the older report. |

| Block 12a. Distribution/Availability Statement. Denotes public availability or limitations. Cite any availability to the public. Enter additional limitations or special markings in all capitals (e.g. NORFOM, REL, ITAR). |
| DOD - See DoDD 4230.25, “Distribution Statements on Technical Documents.” |
| DOE - See authorities. |
| NTIS - Leave blank. |
| Block 12b. Distribution Code. |
| DOD - Leave blank |
| DOE - Enter DOE distribution categories from the Standard Distribution for Unclassified Scientific and Technical Reports |
| NASA - Leave blank. |
| NTIS - Leave blank. |
| Block 13. Abstract. Include a brief (Maximum 200 words) factual summary of the most significant information contained in the report. |
| Block 14. Subject Terms. Keywords or phrases identifying major subjects in the report. |
| Block 15. Number of Pages. Enter the total number of pages. |
| Block 16. Price Code. Enter appropriate price code (NTIS only). |
| Block 20. Limitation of Abstract. This block must be completed to assign a limitation to the abstract. Enter either UL (unlimited) or SAR (same as report). An entry in this block is necessary if the abstract is to be limited. If blank, the abstract is assumed to be unlimited. |

Standard Form 298 Back (Rev. 2-69)