CHARACTERIZING THE WEAK
CONVERGENCE OF STOCHASTIC INTEGRALS

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ABSTRACT

Let \((H^n, X^n)\) be a sequence of càdlàg, adapted processes converging weakly to \((H, X)\). \(X^n\) is a good sequence of semimartingales if the above implies that \(\int H^n_u^- dX^n_u\) converges weakly to \(\int H_u^- dX_u\). We show that the known sufficient conditions for the sequence \(X^n\) to be good are also necessary. We further show that if \(X^n\) is good, then \(\int H^n_u^- dX^n_u\) is also good.

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For \( n = 1, 2, \ldots \), let \( \Xi_n = (\Omega^n, \mathcal{F}^n, (\mathcal{F}^n_t)_{t \geq 0}, P^n) \) be a filtered probability space, let \( H^n \) be càdlàg and adapted, and let \( X^n \) be a càdlàg semimartingale. A fundamental question is: Under what conditions does the convergence in distribution of \((H^n, X^n)\) to \((H, X)\) imply that \( X \) is a semimartingale and that \( \int_0^t H^n_{s-} \, dX^n_s \) converges in distribution to \( \int_0^t H_{s-} \, dX_s \)? A slightly more general formulation would put conditions on the sequence \( X^n \) alone such that the convergence above holds for all such sequences \( H^n \). A sequence with this property will be called good. To be precise, let \( \mathcal{M}^{km} \) denote the real-valued, \( k \times m \) matrices, and let \( \mathcal{D}_E[0, \infty) \) denote the space of càdlàg, \( E \)-valued functions with Skorohod topology.

**Definition:** For \( n = 1, 2, \ldots \), let \( X^n \) be an \( \mathbb{R}^k \)-valued, \((\mathcal{F}^n_t)\)-semimartingale, and let the sequence \((X^n)_{n \geq 1}\) converge in distribution in the Skorohod topology to a process \( X \). The sequence \((X^n)_{n \geq 1}\) is said to be good if for any sequence \((H^n)_{n \geq 1}\) of \( \mathcal{M}^{km} \)-valued, càdlàg processes, \( H^n(\mathcal{F}^n_t) \)-adapted, such that \((H^n, X^n)\) converges in distribution in the Skorohod topology on \( \mathcal{D}_{\mathcal{M}^{km} \times \mathbb{R}^m}[0, \infty) \) to a process \((H, X)\), there exists a filtration \((\mathcal{F}_t)\) such that \( H \) is \((\mathcal{F}_t)\)-adapted, \( X \) is an \((\mathcal{F}_t)\)-semimartingale, and

\[
\int_0^t H^n_{s-} \, dX^n_s \Rightarrow \int_0^t H_{s-} \, dX_s.
\]

Jakubowski, Mémin and Pagès [1] give a sufficient condition for a sequence \((X^n)_{n \geq 1}\) to be good called "uniform tightness" or "UT". This condition uses the characterization of a semimartingale as a good integrator (see e.g., Protter [4]), and requires that it hold uniformly in \( n \). On \( \Xi_n \), let \( \mathcal{H}^n \) denote the set of elementary predictable processes bounded by 1: that is, \( \mathcal{H}^n = \{H^n: H^n \text{ has the representation } H^n_t = H^n_0 \mathbb{1}_{\{0\}}(t) + \sum_{i=1}^{p-1} H^n_{t_i} \mathbb{1}_{(t_i, t_{i+1})}(t), \text{ with } H^n_i \in \mathcal{F}^n_{t_i}, \, p \in \mathbb{N}, \text{ and } 0 = t_0 < t_1 < \ldots < t_p < \infty, \, |H^n_t| \leq 1 \} \).

**Definition:** A sequence of semimartingales \((X^n)_{n \geq 1}\), \( X^n \) defined on \( \Xi_n \), satisfies the condition UT if for each \( t > 0 \) the set \( \{ \int_0^t H^n_s dX^n_s, \, H^n \in \mathcal{H}^n, \, n \in \mathbb{N} \} \) is stochastically bounded.

**Theorem 1.** (Jakubowski–Mémin–Pagès). If \((H^n, X^n)\) on \( \Xi_n \) converges in distribution to \((H, X)\) in the Skorohod topology and if \((X^n)_{n \geq 1}\) satisfies UT, then there exists a filtration
such that $X$ is an $(F_t)$-semimartingale and $\int H^u_{s-} \, dX^u_s$ converges in distribution in the Skorohod topology to $\int H_{s-} \, dX_s$. That is, the sequence $(X^n)_{n \geq 1}$ is good.

The condition UT is sometimes difficult to verify in practice. An alternative condition is given in Kurtz and Protter [2]. To “subtract off” the large jumps in a Skorohod continuous manner, define $h_\delta: [0, \infty) \to [0, \infty)$ by $h_\delta(r) = (1 - \delta/r)^+$, and $J_\delta: D_{R^m}[0, \infty) \to D_{R^m}[0, \infty)$ by

$$J_\delta(x)(t) = \sum_{0 \leq s \leq t} h_\delta(|\Delta x_s|) \Delta x_s,$$

where $\Delta x_s = x(s) - x(s-)$. Let $\int_0^t |dA_s|$ denote the total variation of the process $A$ from 0 to $t$ ($\omega$ by $\omega$).

**Theorem 2.** (Kurtz–Protter). Let $(H^n, X^n)$ on $\Xi_n$ converge in distribution to $(H, X)$ on $\Xi$ in the Skorohod topology on $D_{M^{n,m} \times R^m}[0, \infty)$. Fix $\delta > 0$ (allowing $\delta = \infty$) and let $X^{n,\delta} = X^n - J_\delta(X^n)$. Then $X^{n,\delta}$ is a semimartingale and let $X^{n,\delta} = M^{n,\delta} + A^{n,\delta}$ be a decomposition of $X^{n,\delta}$ into an $(F^n_t)$-local martingale and an adapted process of finite variation on compacts. Suppose

$$(*) \text{ For each } \alpha > 0, \text{ there exist stopping times } T^{n,\alpha} \text{ such that } P(T^{n,\alpha} \leq \alpha) \leq \frac{1}{\alpha} \text{ and }$$

$$\sup_n E\{[M^{n,\delta}, M^{n,\delta}]_{t \wedge T^{n,\alpha}} + \int_0^{t \wedge T^{n,\alpha}} |dA^{n,\delta}_s|\} < \infty.$$  

Then there exists a filtration $(F_t)$ on $\Xi$ such that $H$ is $(F_t)$-adapted and $X$ is an $(F_t)$-semimartingale, and $(H^n, X^n, \int H^n_{s-} \, dX^n_s)$ converges in distribution to $(H, X, \int H_{s-} \, dX_s)$ in the Skorohod topology on $D_{M^{n,m} \times R^m \times R^s}[0, \infty)$. That is, $(X^n)_{n \geq 1}$ is good.

It is shown in Kurtz and Protter [2] and also in Mémin and Slominski [3] that UT and $(*)$ are equivalent sufficient conditions for the sequence $(X^n)$ of (vector–valued) semimartingales to be good.

The next theorem, which is the principal result of this note, shows that the sufficient conditions of Jakubowski–Mémin–Pagès and Kurtz–Protter are also necessary.

**Theorem 3.** Let $X^n$ be a sequence of vector–valued semimartingales on filtered probability spaces $\Xi_n$. If $X^n$ is a good sequence, then $X^n$ satisfies the condition UT and the condition $(*)$ of Theorem 2.
Proof: Since UT and (*) are equivalent, it suffices to show that UT holds. We treat the case \( k = m = 1 \) for notational simplicity.

Suppose that \((X^n)_{n \geq 1}\) is a good sequence but that UT does not hold. Then there exists a sequence \((H^n)_{n \geq 1}\), \(H^n \in \mathcal{H}_n\), and a sequence \(c_n\) tending to \(\infty\) such that for some \(\varepsilon > 0\),

\[
\liminf_{n \to \infty} P^n\{ \int H^n_s^- \, dX^n_s \geq c_n \} \geq \varepsilon.
\]

But this implies

\[
(1) \quad \liminf_{n \to \infty} P^n\{ \int \frac{1}{c_n} H^n_s^- \, dX^n_s \geq 1 \} \geq \varepsilon
\]

as well. Since \(|H^n| \leq 1\), we have that \(\frac{1}{c_n} H^n\) converges in distribution (uniformly) to the zero process. Since \(X^n\) is good by hypothesis, then \(\int \frac{1}{c_n} H^n_s^- \, dX^n_s\) converges in distribution to \(\int 0 \, dX_s = 0\). This contradicts (1), and we have the result. \(\square\)

We can employ the argument in the previous proof to show that the property of goodness is inherited through stochastic integration.

**Theorem 4.** Let \((X^n)_{n \geq 1}\) be a sequence of \(\mathbb{R}^m\)-valued semimartingales, \(X^n\) defined on \(\Xi_n\), with \((X^n)_{n \geq 1}\) being good. If \(H^n\) defined on \(\Xi_n\) are càdlàg, adapted, \(\mathcal{M}^{km}\)-valued processes, and \((H^n, X^n)\) converges in distribution in the Skorohod topology on \(\mathcal{D}_{\mathcal{M}^{km} \times \mathbb{R}^m}[0, \infty)\), then \(Y^n_t = \int_0^t H^n_s^- \, dX^n_s\) is also a good sequence of semimartingales.

**Proof:** Let \(k = m = 1\). By Theorem 1, it is sufficient to show that \((Y^n)\) satisfies UT. Suppose not. Then, as in the proof of Theorem 3, there exists a sequence \((\tilde{H}^n)\) with \(\tilde{H}^n \in \mathcal{H}^n\), a sequence \(c_n\) tending to \(\infty\), and \(\varepsilon > 0\) such that

\[
\liminf_{n \to \infty} P^n\{ \int \tilde{H}^n_s^- \, dY^n_s \geq c_n \} \geq \varepsilon.
\]

and equivalently

\[
\liminf_{n \to \infty} P^n\{ \int \tilde{H}^n_s^- H^n_s^- \, dX^n_s \geq c_n \} \geq \varepsilon.
\]

which implies
\( \liminf_{n \to \infty} P^n \left\{ \frac{1}{c_n} \tilde{H}_{\pi}^n - H_{\pi}^n \ dX^n \geq 1 \right\} \geq \varepsilon \)

But, as before, the goodness of \( X^n \) implies that the stochastic integral in (2) converges to zero contradicting (2) and verifying UT for \( Y^n \).

As an application of the preceding, we consider stochastic differential equations. Suppose that for each \( n \), \( U^n \) is adapted, càdlàg and \( X^n \) is a semimartingale on \( \Xi_n \), \( (U^n, X^n) \Rightarrow (U, X) \), and \( X^n \) is a good sequence. Let \( F^n, F \) be, for example, Lipschitz continuous such that \( F^n \to F \) uniformly on compacts, and let \( Z^n, Z \) be the unique solutions of
\[
Z^n_t = U^n_t + \int_0^t F^n(Z^n_s) dX^n_s
\]
\[
Z_t = U_t + \int_0^t F(Z_s) dX_s.
\]

Then combining Theorem 4 with Theorem 5.4 of [2] (or similar results in [3] or [5]) yields that \( Z^n \) is also a good sequence converging of course to \( Z \). The preceding holds as well for much more general \( F^n, F \); see [2]. In particular by taking \( F^n(x) = F(x) = x \), we have that \( X^n \) a good sequence implies that \( Z^n = \mathcal{E}(X^n) \) is a good sequence, where \( \mathcal{E}(\mathcal{Y}) \) denotes the stochastic exponential of a semimartingale \( Y \).

REFERENCES


