WONG-ZAKAI CORRECTIONS, RANDOM EVOLUTIONS, 
AND SIMULATION SCHEMES FOR SDE's

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Technical Report # 91-01

Department of Statistics
Purdue University

January, 1991
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ABSTRACT
A general weak limit theorem for solutions of stochastic differential equations driven by arbitrary semimartingales is applied to give a unified treatment of limit theorems for random evolutions and consistency results for numerical schemes for stochastic differential equations. The asymptotic distribution of the error in an Euler scheme is studied. The Wong-Zakai correction in the random evolution limit arises through an integration by parts.
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Abstract

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In other words, if \( \{Y_n\} \) is a sequence of semimartingales converging in distribution and satisfies C1, then \( \{Y_n\} \) is good. This condition is shown to be necessary for goodness in Kurtz and Protter (1991b).

The following is a consequence of goodness.

Proposition 1.2 Let \( f : \mathbb{R}^k \to \mathbb{M}^{km} \) be bounded and continuous, for each \( n \), let \( (U_n,Y_n) \) be an \( \mathcal{F}_n \)-adapted process in \( D_{\mathbb{R}^k \times \mathbb{R}^{km}}[0,\infty) \), and let \( \{Y_n\} \) be a good sequence of semimartingales with \( (U_n,Y_n) \Rightarrow (U,Y) \).

If for each \( n \), \( X_n \) is a solution of

\[
X_n(t) = U_n(t) + \int_0^t f(X_n(s-))dY_n(s),
\]

then \( \{(X_n,Y_n)\} \) is relatively compact (in the sense of convergence in distribution) and any limit point \( (U,X,Y) \) satisfies

\[
X(t) = U(t) + \int_0^t f(X(s-))dY(s).
\]

(More general results can be found in Slomiński (1989) and Kurtz and Protter (1991a).)

It is well-known from the work of Wong and Zakai (1965) that the conclusion of Proposition 1.2 fails for many natural sequences \( \{Y_n\} \) approximating Brownian motion.

Example 1.3 Let \( \{\xi_k\} \) be independent and identically distributed with mean zero and variance \( \sigma^2 \). Define

\[
W_n^\sigma(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \xi_k.
\]

Then it is easy to check that \( \{W_n^\sigma\} \) is a good sequence which, by the Donsker invariance principle, converges in distribution to \( \sigma W \) where \( W \) is standard Brownian motion.

Example 1.4 Let \( W \) be standard Brownian motion and let \( W_n^b \) satisfy \( W_n^b(0) = 0 \) and

\[
\frac{d}{dt} W_n^b(t) = n\left( W\left(\frac{[nt]}{n} + \frac{1}{n}\right) - W\left(\frac{[nt]}{n}\right)\right).
\]

Then \( W_n^b \Rightarrow W \), but \( \{W_n^b\} \) is not good (e.g., \( \int_0^t W_n^b\,dW_n^b \Rightarrow \int_0^t W\,dW + \frac{1}{2}t \)). Since the sample paths of \( W_n^b \) have finite variation it is a semimartingale, but in the canonical decomposition of \( W_n^b \) into a local martingale plus a finite variation process, the local martingale is zero and \( T_k(W_n^b) = O(\sqrt{n}) \). (The decomposition used in Theorem 1.1 does not need to be the canonical decomposition, but, of course, the counter example ensures that no decomposition will satisfy C1.)

Example 1.5 Let \( \{\xi_k, k \geq 0\} \) be a finite, irreducible Markov chain with transition matrix \( P = ((p_{ij})) \). Let \( \pi = (\pi_1, \ldots, \pi_M) \) give the stationary distribution, and let \( f \) be a function satisfying

\[
\sum_m f(m)\pi_m = 0.
\]

Define

\[
W_n^\pi(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} f(\xi_k).
\]

Letting \( P_g(i) = \sum_j g(j)p_{ij} \), by (1.7) there exists a function \( h \) such that \( Ph - h = f \). Substituting in (1.8), we obtain

\[
W_n^\pi(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} (Ph(\xi_k) - h(\xi_k))
\]

\[
= \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} (Ph(\xi_{k-1}) - h(\xi_k))
\]

\[
+ \frac{1}{\sqrt{n}} \left( Ph(\xi_0) - Ph(\xi_0)\right)
\]

\[
\equiv Y_n(t) + Z_n(t).
\]
\[ sp(\delta x - ((\varepsilon u)^{2})^{(\delta)I}) \int u = (t^{\delta} x) \circ (t)^{\delta} M = (t)^{\delta} M \] (3.2)

and

\[ sp((\varepsilon u)^{2})^{(\delta)I} \int = (t)^{\delta} \Lambda \] (3.2)

Let \( \mathcal{D} \) be a continuous time Markov chain with state space \( \mathcal{M} \) and define \( M \) and \( M \) by an equation for random evolutions are derived. Define the product

Previous section

\[ \text{Previous section} \]

Theorem 1.6. Let \( \mathcal{M} \), \( \mathcal{M} \), \( \mathcal{M} \), \( \mathcal{M} \) be the measure, \( \text{defined in (1.13)} \). Then \( \mathcal{M} \) and \( \mathcal{M} \) are continuous and second order differentials. Define \( \mathcal{M} \) in (1.13) be differentiable on \( \mathcal{M} \) and \( \mathcal{M} \) be the measure and \( \mathcal{M} \) and \( \mathcal{M} \), \( \mathcal{M} \), \( \mathcal{M} \) be the measure.

\[ \text{We now consider sequences of stochastic ordinary differential equations in } \mathcal{M} \] (3.2)

Random evolutions

\[ \text{Random evolutions} \]

F. Previous section

Theorem 2.0. Let \( \mathcal{M} \) and \( \mathcal{M} \) be the measure, \( \text{and}\) \( \text{define}\) \( \mathcal{M} \) in (1.13) be differentiable on \( \mathcal{M} \) and \( \mathcal{M} \) be the measure and \( \mathcal{M} \) be the measure.

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Random evolutions

\[ \text{Random evolutions} \]
Then (2.1) becomes

\[ X_n(t) = X_n(0) + \sum_{\beta=1}^{m} \int_0^t G(X_n(s), \beta) dW_n^\beta(s) + \sum_{\beta=1}^{m} \int_0^t H(X_n(s), \beta) dW_n^\beta(s). \]

(2.4)

Let \( h^\beta \) satisfy

\[ \sum_{k=1}^{m} q_{jk}h^\beta(k) = I^\beta(j) - \pi^\beta \]

(2.5)

\( h^\beta \) exists by the uniqueness of \( \pi \), and note that \( Y_n \) defined by

\[ Y_n^\beta(t) = n \int_0^t (I^\beta(\xi(n^2 s)) - \pi^\beta) ds - \frac{1}{n} h^\beta(\xi(n^2 t)) + \frac{1}{n} h^\beta(\xi(0)) \]

(2.6)

is a martingale. Define \( Z_n \) by

\[ Z_n^\beta(t) = \frac{1}{n} h^\beta(\xi(n^2 t)) - \frac{1}{n} h^\beta(\xi(0)) \]

(2.7)

so that \( W_n = Y_n + Z_n \). Let \( N_{ij}(t) \) denote the number of transitions of \( \xi \) from state \( i \) to state \( j \) up to time \( t \). Then

\[ [Y_n^\beta, Y_n^\gamma]^t = \sum_{ij=1}^{m} \frac{N_{ij}(n^2 t)}{n^2} (h^\beta(j) - h^\beta(i))(h^\gamma(j) - h^\gamma(i)) \]

(2.8)

\[ [Y_n^\beta, Z_n^\gamma]^t = -[Y_n^\beta, Y_n^\gamma]^t \]

(2.9)

and

\[ \int_0^t Z_n^\beta(s) dZ_n^\gamma(s) = \sum_{ij=1}^{m} \frac{N_{ij}(n^2 t)}{n^2} h^\beta(i)(h^\gamma(j) - h^\gamma(i)) - \frac{1}{n} h^\beta(\xi(0)) Y_n^\gamma(t). \]

(2.10)

As \( n \to \infty \) we obtain

\[ [Y_n^\beta, Y_n^\gamma]^t \to C_{\beta \gamma}^t \]

(2.11)

and

\[ \int_0^t Z_n^\beta(s) dZ_n^\gamma(s) \to D_{\beta \gamma}^t. \]

(2.12)

The martingale central limit theorem (see, for example, Ethier and Kurtz (1986), Theorem 7.1.4) gives \( Y_n \Rightarrow Y \) where \( Y \) is a Brownian motion with infinitesimal covariance \( C = ((C_{\beta \gamma})) \). Theorem 1.6 gives the following

Theorem 2.1 Let \( \xi \) in (2.1) be a finite Markov chain with state space \( E = \{1, \ldots, m\} \) and intensity matrix \( Q \), and let \( X_n(0) \) be independent of \( \xi \). Let \( G \) be bounded and continuous, and let \( H \) be bounded and have bounded and continuous first and second derivatives. Assume that \( Q \) is irreducible and that \( \pi, ((C_{\beta \gamma})) \) and \( ((D_{\beta \gamma})) \) are as above. Define \( \bar{G} : \mathbb{R}^k \to \mathbb{R}^k \) by \( \bar{G}(x) = \Sigma_{\beta} G(x, \beta) \pi_{\beta} \) and \( H : \mathbb{R}^k \to \mathbb{M}^{km} \) by \( H(x) = (H(x, 1), \ldots, H(x, m)) \). If \( X_n(0) \Rightarrow X(0) \), then \( \{X_n\} \) is relatively compact and any limit point satisfies

\[ X(t) = X(0) + \int_0^t \bar{G}(X(s)) ds + \int_0^t H(X(s), \beta) dY(s) \]

\[ + \sum_{\alpha, \beta, \gamma} \int_0^t \partial_\alpha H(X(s), \beta, \gamma) dZ^\alpha(s). \]

(2.13)

Remark 2.2 Hersh and Papanicolaou (1972) and Kurtz (1973) prove the above result using functional analytic arguments.
\[ u^\gamma + v^\gamma + w^\gamma = (3.6) \]

We can write
\[ \frac{\partial}{\partial t} u^\gamma + (u^\gamma v^\gamma) + (\frac{\partial}{\partial x} v^\gamma) + (\frac{\partial}{\partial x} v^\gamma) = \frac{\partial}{\partial t} u^\gamma + \lambda \]

Lemma 1. Let \( \lambda \) be an \( \{\text{F}\} \)-semimartingale. For each \( n \), let \( \gamma \) be a

Lemma 2. Note that the Lemma would allow for a mesh determined by the canonical decomposition of \( \lambda \). Use this observation to construct the following equivalent scheme for each \( m \), where \( \gamma \) is the canonical stochastic differential. For each \( m \), let \( \gamma \) be a

Note that if we define \( \gamma \) for \( i < n \), then we

for \( i < n \). For \( i = n \), then

and observe that

and

and

for \( i < n \). For \( i = n \), then

and observe that

Lemma 3. The results discussed above provide an intuitive approach to checking the consistency of these schemes and we will see that they are also

3 Numerical schemes

Wong-Zakai Corrections
and since $E[[M_n]_{t}] \leq E[[M^\delta]_{m}]$ and $E[[T_i(A_n)] \leq E[[T_m(A^\delta)]$, we can apply Theorem 2.7 of Kurtz and Protter (1991a) to conclude that $V_n^m \Rightarrow V^m$.

There is an alternative approach to representing the approximation given by the Euler scheme as a solution of a stochastic differential equation. Define $\eta(t) = t_k$ for $t_k \leq t < t_{k+1}$ (which is the same as $\beta$ defined above, but the general assumptions that will be placed on the sequence $\{\eta_n\}$ below will be different from the assumptions placed on $\{\beta_n\}$). Let $\tilde{X}_0$ satisfy

\begin{equation}
\tilde{X}_0(t) = X(0) + \int_0^t f(\tilde{X}_0 \circ \eta(s-))dy(s).
\end{equation}

Then $\tilde{X}_0(t_k) = X_0(t_k)$. The consistency of the Euler scheme can also be obtained through the analysis of this equation.

**Lemma 3.2** For each $n$, let $Y_n$ be an $\mathbb{R}^{m}$-valued $\{\mathcal{F}^n\}$-semimartingale, $X_n$ a cadlag, $\mathbb{R}^{m}$-valued $\{\mathcal{F}^n\}$-adapted process, and $\eta_n$ a right continuous, nondecreasing $\{\mathcal{F}^n\}$-adapted process. Suppose that $\eta_n(t) \leq t$ and $\eta_n(t) \rightarrow t$ for all $t \geq 0$. Assume that $\{Y_n\}$ is a good sequence and that $(X_n, Y_n) \Rightarrow (X, Y)$ in $D_{\mathbb{R}^{m} \times \mathbb{R}^{m}}[0, \infty)$. Then $\int X_n \circ \eta_n dy_n \Rightarrow \int X dy$.

**Proof** First observe that for each fixed $\delta > 0$,\n
\begin{equation}
\int_0^t (J_\delta(X_n) \circ \eta_n(s-) - J_\delta(X_n)(s-))dy_n(s) \Rightarrow 0.
\end{equation}

Consequently, there exist $\delta_n \rightarrow 0$ such that\n
\begin{equation}
\int_0^t (J_{\delta_n}(X_n) \circ \eta_n(s-) - J_{\delta_n}(X_n)(s-))dy_n(s) \Rightarrow 0.
\end{equation}

But the asymptotic continuity of $X_n^{\delta_n}$ implies that\n
\begin{equation}
\int_0^t (X_n^{\delta_n}(s-) - X_n^{\delta_n} \circ \eta_n(s-))dy_n(s) \Rightarrow 0
\end{equation}

and hence

\begin{equation}
\int_0^t (X_n(s-) - X_n \circ \eta_n(s-))dy_n(s) \Rightarrow 0
\end{equation}

which gives the lemma.

\begin{equation}
\int_0^t (X(n)(s-) - X_n \circ \eta_n(s-))dy_n(s) \Rightarrow 0
\end{equation}

which gives the lemma.

**Theorem 3.3** For each $n$, let $Y_n$ be an $\mathbb{R}^{m}$-valued $\{\mathcal{F}^n\}$-semimartingale and $\eta_n$ a right continuous, nondecreasing $\{\mathcal{F}^n\}$-adapted process. Suppose that $\eta_n(t) \leq t$ and $\eta_n(t) \rightarrow t$ for all $t \geq 0$. Assume that $\{Y_n\}$ is a good sequence and that $Y_n \Rightarrow Y$. Let $f : \mathbb{R}^{k} \rightarrow \mathbb{R}^{m}$ be bounded and continuous, and let $\tilde{X}_n$ satisfy

\begin{equation}
\tilde{X}_n(t) = X(0) + \int_0^t f(\tilde{X}_n \circ \eta_n(s-))dy_n(s).
\end{equation}

Then $\{(\tilde{X}_n, Y_n)\}$ is relatively compact and any limit point $(X, Y)$ satisfies

\begin{equation}
X(t) = X(0) + \int_0^t f(X(s-))dy(s).
\end{equation}

If the $Y_n$ are defined on the same sample space as $Y$, sup$_{s \leq t}|Y_n(s) - Y(s)| \rightarrow 0$ in probability for each $t > 0$, and sample path uniqueness holds for the solution of (3.15), then sup$_{s \leq t} |\tilde{X}_n(s) - X(s)| \rightarrow 0$ in probability for each $t > 0$.

**Remark 3.4** For $Y_n = Y$, $n = 1, 2, \ldots$, and $\eta_n(t) = \tau^n_k$, $\tau^n_k \leq t < \tau^n_{k+1}$, for a sequence of stopping times $\{\tau^n_k\}$, this result is a special case of Theorem V.16 of Protter (1990).

**Proof** The relative compactness follows from Lemma 4.1 and Proposition 4.3 of Kurtz and Protter (1991a). The fact that any limit point satisfies (3.15) then follows from Lemma 3.2. Under the assumptions of the final assertion, we can treat $(\tilde{X}_n, X)$ as a solution of a single system. Using the uniqueness assumption, it follows that $(\tilde{X}_n, X) \Rightarrow (X, X)$ and hence that $\tilde{X}_n - X \Rightarrow 0$ which gives the desired conclusion.

Note that if $Y$ is a semimartingale and $Y_n = Y$ for all $n$, then $\{Y_n\}$ is good.
\[ \Omega \iff a \iff \sqrt{a} \]

Theorem 3.5. Let \( X \) be an \( \mathbb{R} \)-valued, non-negative martingale, and suppose that for each \( t \) in the interval \((0, T]\), the following inequality holds for all \( \delta > 0 \): 

\[ f(x) = \exp(-\delta x) \]
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