Sequential Estimation Procedure for the
Two-Parameter Exponential Family of Distributions

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A SEQUENTIAL ESTIMATION PROCEDURE FOR A
TWO-PARAMETER EXPONENTIAL FAMILY
OF DISTRIBUTIONS; FIRST ORDER RESULTS

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ABSTRACT

We consider the problem of sequentially estimating one parameter in a class of two-parameter exponential family of distributions. We assume a squared error loss with a fixed cost of estimation error. The stopping rule, based on the maximum likelihood estimate of the nuisance parameter, is shown to be independent of the terminal estimate. The first order asymptotic properties of the risk function are investigated. It is shown that the suggested procedure is an asymptotically risk efficient procedure. This procedure is exemplified for the normal, gamma and the inverse Gaussian densities, which follow as special cases of our general results.
1. Introduction. Consider a model in which the underlying distribution of a sequence of 
(i.i.d.) random variables \( X_1, X_2, \ldots \) depends on two unknown parameters \( \theta_1 \) and \( \theta_2 \). 
For a fixed (and finite) sample size, measures of accuracy of an estimate for the parameter 
of interest \( \theta_2 \), say, typically depend on the unknown value of the nuisance parameter \( \theta_1 \). 
Thus to achieve a given level of accuracy on has to proceed sequentially: to determine the 
final (random) sample size based on an estimate of \( \theta_1 \), and then the terminal estimate of 
\( \theta_2 \) is determined based on this sample.

Procedures of this nature were discussed initially by Stein (1945, 1949), as two-stage 
procedures for estimation and interval estimation of prescribed accuracy, for the mean 
of normally distributed r.v.'s, when the variance \( \sigma^2 \) is unknown. Later, this problem 
(for the normal mean), was tackled by purely sequential procedures proposed by Robbins 
(1959), (for point estimation) and Chow and Robbins (1965), (for fixed width interval 
estimation). Although the normal case has been studied extensively, there are, to the best 
of our knowledge, only few studies (cited below), dealing with other cases of underlying 
distributions.

To illustrate the general problem on hand, consider the following point estimation problem. 
Let \( X_1, X_2, \ldots \), be i.i.d. random variables with unknown mean \( \mu \) (the parameter 
of interest) and variance \( \sigma^2 \). Having recorded the first \( n \) observations \( x_1, \ldots, x_n \), let the 
loss incurred in estimating \( \mu \) by \( \hat{\mu}_n = \sum x_i/n \) be:

\[
L_\rho(\hat{\mu}_n) = \rho(\hat{\mu}_n - \mu)^2 + n
\]

where \( \rho (> 0) \), is the known weight of the estimation error relative to the sampling cost. 
The objective is to minimize the associated risk;

\[
R_\rho(n) = E_{\mu, \sigma}(L_\rho(\hat{\mu}_n)) = \frac{\rho \sigma^2}{n} + n .
\]

with respect to the sample size \( n \). When \( \sigma \) is known, the expected loss, \( R_\rho(n) \) is minimized 
by taking a sample of size (an integer adjacent to) \( n_0 = \rho^{\frac{1}{2}}\sigma \), with corresponding risk, 
\( R_\rho(n_0) = 2n_0 \). However, when \( \sigma \) is an unknown nuisance parameter, no fixed sample 
size minimizes \( R_\rho(n) \) simultaneously for all \( \sigma \). This motivates the following choice of a 
random sample size \( N_\rho \), when \( \sigma^2 \) is unknown,

\[
N_\rho = \inf\{ n \geq m_0, \hat{\sigma}^2_n \leq a_n n^2/\rho \}
\]

where \( m_0 (\geq 2) \) is the initial sample size, \( \hat{\sigma}^2_n \) is a suitable estimate of \( \sigma^2 \), (for example; 
\( \hat{\sigma}^2_n = s^2_n = \sum (x_i - \bar{x})^2/(n - 1) \)), and \( a_n \) is some nonincreasing sequence, \( a_n \to 1 \) as
$n \to \infty$). According to this procedure, the parameter $\mu$ is estimated at termination, by $\hat{\mu}_{N_{\rho}}$. Clearly then, the study of the stopping variable $N_{\rho}$ and the risk associated with it, become important.

For normal random variables, the statistic $s_n^2$ is ancillary to $\mu$ and it can be shown (by using Helmert orthogonal transformation), that the event $\{N_{\rho} = n\}$ and $\hat{\mu}_n$ are independent. This property was heavily exploited by most researcher who worked on the normal problem. Robbins (1959) studied the properties of $N_{\rho}$ and provided a recursive formula for its distribution. Later this problem was studied extensively by Starr (1966), and Starr and Woodroofe (1969). Woodroofe (1977) has used second order approximations to study this procedure and to analyze the regret in the expected loss incurred upon using the sample size $N_{\rho}$ as compared to $n_0$.

Extensions of this procedure to nonnormal cases were considered by several authors. Starr and Woodroofe (1972) deal with the negative exponential distribution and provide results on the regret. Here, even though there is one parameter, the problem becomes interesting since the variance of the m.l.e. depends on the unknown parameter. Ghosh and Mukhopadhyay (1979) with a 'distribution free' approach allowed the initial sample size $m_0$ to be a function of $\rho$ and to $\to \infty$ as $\rho \to \infty$. They proved first order result for the risk, showing that the ratio of the risk associated with $N_{\rho}$ to that associated with the hypothetical fixed sample size $n_0$ converges to 1, as $\rho \to \infty$. Mukhopadhyay (1988) surveyed results concerning sequential estimation procedures for the negative exponential distribution, with and without a truncation parameter. Related studies are those of Aras (1987, 1989), dealing with sequential estimation procedure based on censored data from negative exponential distribution. He provided first and second order results, also by allowing the initial sample size $m_0 \to \infty$.

In the present paper, we consider a sequential point estimation problem in a class of two-parameter exponential family of distributions. The model considered here will be restricted by assumptions on its natural parameters $(\theta_1, \theta_2)$, but is general enough to include the normal, the gamma, and the inverse Gaussian distributions, as special cases. This exponential subfamily was first introduced by Bar-Lev and Rieser (1982) in context of UMPU tests based on single test statistics. A description and the basic properties of such an exponential subfamily are provided in section 2. We also present a new independence result, analogous to the one discussed for the normal case, which provides in the general case discussed, the independence of the event $\{N_{\rho} = n\}$ with the terminal estimator. Finally, in Section 3 we present an appropriate stopping rule along with the estimation procedure for the parameter of interest. We then discuss the properties of the suggested
stopping rule and provide the first order properties of its associated risk as \( p \to \infty \), under a loss function similar to (1.1) (and to (1) of Woodroofe (1985)).

2. Preliminaries; the exponential family and an independence result.

Let

\[
 f(x; \theta) = a(x) \exp\{\theta_1 U_1(x) + \theta_2 U_2(x) + c(\theta)\}, \quad \theta = (\theta_1, \theta_2),
\]

be a density function (w.r.t. Lebesgue measure on \( R \)), which characterizes a regular two-parameter exponential family of distributions, (see Brown (1986)), i.e.; the natural parameter space \( \Theta \) is defined by;

\[
 \Theta = \{\theta \in \mathbb{R}^2 ; e^{-c(\theta)} = \int a(x) \exp\{\theta_1 U_1(x) + \theta_2 U_2(x)\} \, dx < \infty \},
\]

so that \( \Theta \equiv \text{int} \Theta \neq \emptyset \). It is well known that for any \( \theta \in \Theta \) the r.v. \( U = (U_1, U_2) \) has moments of all orders. In particular, we denote;

\[
(2.1) \quad E_\theta(U) = (\nu_1, \nu_2), \quad \nu_i = -\partial c(\theta)/\partial \theta_i, \quad i = 1, 2.
\]

and

\[
(2.2) \quad V_\theta(U) = (\sigma_{ij}), \quad \sigma_{ij} = -\partial^2 c(\theta)/\partial \theta_i \partial \theta_j \quad i, j = 1, 2,
\]

where \( V_\theta(U) \) is the corresponding covariance matrix, (positive definite).

Let \( X_1, \ldots, X_n, \ n > 1, \) be independent r.v.'s having a common density of the form (2.1). We set \( T_{i:n} = \sum_{j=1}^{n} U_i(X_j) \) and denote by \( T_{i:n}, \quad i = 1, 2 \) the usual average. The joint distribution of \( T = (T_{1:n}, T_{2:n}) \) is a member of the two-parameter exponential family, and

\[
(2.3) \quad E_\theta(T) = (n\nu_1, n\nu_2), \quad V_\theta(T) = (n\sigma_{ij}) \quad i, j = 1, 2.
\]

The results stated in the following theorem were proved by Bar-Lev and Reiser (1982), and are concerned with a characterization of (2.1) which admits a single ancillary statistic for \( \theta_2 \) in the presence of \( \theta_1 \), (that is, its distribution depends only on \( \theta_1 \)). For additional applications of this result see also Brown (1986 pp. 44-48). However, that characterization requires the following two assumptions:
**Assumption A.1.** The parameter $\theta_2$ can be represented as; $\theta_2 = -\theta_1 \psi'(\nu_2)$, where $\psi'(\nu_2) = d\psi(\nu_2)/d\nu_2$, for some function $\psi$.

**Assumption A.2.** $U_2(x) = h(x)$, where $h(x)$ is a 1-1 function on the support of (2.1).

**Theorem 2.1.** (Bar-Lev and Reiser, (1982)) Under the above assumptions, the following hold:

a) $U_1[h^{-1}(\bar{T}_{2:n})] = \psi(\bar{T}_{2:n})$ a.s. for $n \geq 1$.

b) The distribution of the statistics

\[ Z_n = T_{1:n} - n\psi(\bar{T}_{2:n}), \]

belongs to the one parameter exponential family with natural parameter $\theta_1$ and density of the form,

\[ f_{Z_n}(z_n, \theta_1) = q(z_n)\exp\{\theta_1 z_n - H_n(\theta_1)\}, \quad \theta_1 \in \Theta_1. \]

c) For each $n \geq 2$ and for any $\theta \in \Theta$, the r.v.'s $Z_n$ and $T_{2:n}$ are independent.

By Theorem 2.1, the statistic $Z_n$ is ancillary to $\theta_2$ in the presence of $\theta_1$, and therefore may be used in fixed sample estimation procedures. However, in the context of sequential estimation, we need a result stronger than that of part (c) of Theorem 2.1. This is given below.

**Theorem 2.2.** Under the above assumptions, for all $n \geq 2$ and $\theta \in \Theta$, the random variables $(Z_2, \ldots, Z_n)$ are jointly independent of $T_{2:n}$, i.e.;

\[ (Z_2, \ldots, Z_n) \perp T_{2:n}. \]

Since the proof of this theorem is rather technical, it is deferred to the Appendix. As was mentioned in Section 1, the result of Theorem 2.2, will enable us to obtain an independence property analogous to the one discussed for the normal case, and thus is of great importance in context of sequential estimation. In light of this, we assume from now on that the two conditions; A.1 and A.2, hold without further reference.

Using (2.2), one can introduce a parameterization of the exponential family by means of the mapping $(\theta_1, \theta_2) \rightarrow (\theta_1, \nu_2)$, which is a homeomorphism, and has its components $\theta_1, \nu_2$ varying independently, (see Barndoff-Nielsen (1978), Theorem 8.4). Accordingly, $(\theta_1, \nu_2) \in \Theta_1 \times \mathcal{N}_2$ where $\mathcal{N}_2$ is connected and open. With such parameterization, and under the above assumptions, the following results can be easily shown to hold, (see Bar-Lev and Reiser (1982)).
**Lemma 2.1.**

a) \( \psi'(\nu_2) \) is not identically constant.

b) The variance of \( U_2 \) is given by:

\[
\sigma_{22}(\theta) \equiv \frac{\partial \nu_2}{\partial \theta_2} = \frac{-1}{\theta_1 \psi''(\nu_2)}, (> 0),
\]

(2.6)

c) The functions \( c(\theta) \) and \( \nu_1(\theta) \) when expressed by means of the mixed parameters \( \theta_1 \) and \( \nu_2 \), have the following form:

\[
\begin{align*}
\{ & c(\theta_1, \nu_2) = \theta_1[\nu_2 \psi'(\nu_2) - \psi(\nu_2)] - G'(\theta_1) \\
& \nu_1 = \psi(\nu_2) + G'(\theta_1)
\end{align*}
\]

(2.7)

where \( G(\theta_1) \) is an infinitely differentiable function on \( \Theta_1 \) for which \( G''(\theta_1) > 0 \), for all \( \theta_1 \in \Theta_1 \).

Here \( G' \) and \( G'' \) denote the first and second derivatives of \( G \), respectively. In fact, using the above results it can be shown, (see Bar-Lev and Reiser (1982)), that the function \( H_n \) in (2.5) is given by:

\[
H_n(\theta_1) = nG(\theta_1) - G(n\theta_1),
\]

so that \( E_{\theta_1}(Z_n) \equiv H'_n(\theta_1) = n(G'(\theta_1) - G'(n\theta_1)) \) and \( V_{\theta_1}(Z_n) \equiv H''_n(\theta_1) = nG''(\theta_1) - n^2G'''(n\theta_1) \). Moreover, since \( H_n(\theta_1) > 0 \) and \( G''(\theta_1) > 0 \), it follows that \( G'(\theta_1) > G'(n\theta_1) \), for all \( \theta_1 \in \Theta_1 \) and \( n > 1 \). Furthermore, parts (a)-(b) of Lemma 2.1 suggest that either \( \Theta_1 \subset \mathbb{R}^- \) or \( \Theta_1 \subset \mathbb{R}^+ \).

**Lemma 2.2.** If \( \Theta_1 \subset \mathbb{R}^- \) (if \( \Theta_1 \subset \mathbb{R}^+ \), then:

a) \( \psi \) is strictly convex (concave) function on \( \mathcal{N}_2 \),

b) \( Z_1 = 0 \) and \( Z_n > Z_{n-1} \) a.s. , \( (Z_n < Z_{n-1} \text{ a.s.}) \) , \( n \geq 2 \)

c) \( G' \) is positive (negative) on \( \Theta_1 \).

**Proof:** We will prove only the case \( \Theta \subset \mathbb{R}^- \) of the lemma, since the proof of the other case is similar. That \( \psi \) is strictly convex on \( \mathcal{N}_2 \), follows immediately from Lemma 2.1(a-b) and that \( Z_1 = 0 \) a.s., follows from part (a) of Theorem 2.1. In fact, since \( X_1, \ldots, X_n \) are identically distributed, we have by Theorem 2.1 (a) that \( U_1(X_j) = \psi(U_2(X_j)) \), a.s., for all \( j = 1, \ldots, n \). Now, \( \psi \) is convex and thus:

\[
\psi(T_{2:n}) < \frac{n-1}{n} \psi(T_{2:n-1}) + \frac{1}{n} U_1(X_n) \quad \text{a.s.,}
\]
which in turn, implies that:

\[ Z_n = T_{1:n} - n\psi(\bar{T}_{2:n}) > T_{1:n-1} - (n-1)\psi(\bar{T}_{2:n-1}) = Z_{n-1} \quad \text{a.s.} \]

Furthermore, since for \( n > 1, \ Z_n > 0, \ a.s. \), it follows from (2.3), (2.7) and Jensen's inequality that for each \( \theta_1 \in \Theta_1 \):

\[ 0 < E(\tilde{Z}_n) = \nu_1 - E[\psi(\tilde{T}_{2:n})] < \nu_1 - \psi(\nu_2) = G'(\theta_1), \]

**Lemma 2.3.** For each \( \theta_1 \in \Theta_1 \), \( \tilde{Z}_n \equiv Z_n/n \overset{a.s.}{\longrightarrow} G'(\theta_1) \).

**Proof:** Clearly, \( T_{i:n}, i = 1, 2 \) are partial sums of i.i.d. random variables having finite moments (of all orders). So that \( \bar{T}_{i:n} \overset{a.s.}{\longrightarrow} \nu_i, \ i = 1, 2. \) Since \( \psi \) is continuous, we have by (2.4) and (2.7), that

\[ \frac{Z_n}{n} = \frac{\bar{T}_{1:n} - \psi(\bar{T}_{2:n})}{n} \overset{a.s.}{\longrightarrow} \nu_1 - \psi(\nu_2) = G'(\theta_1). \]

The following are some examples illustrating the construction of the statistic \( Z_n \).

**Example 1:** The Normal distribution, \( \mathcal{N}(\mu, \sigma^2) \).

(i) \( \theta_1 = -1/2\sigma^2, \ \theta_2 = \mu/\sigma^2, \ \Theta = \mathbb{R}^+ \times \mathbb{R}^- \)

(ii) \( U_1(X) = X^2, \ U_2(X) = X, \ T_{1:n} = \sum_{i=1}^n X_i^2, \ T_{2:n} = \sum_{i=1}^n X_i \)

(iii) \( \nu_2 = -\theta_2/2\theta_1, \ \theta_2 = -2\theta_1\nu_2 \)

(iv) \( \nu_1 = \nu_2^2 - 1/2\theta_1, \ \psi(\nu_2) = \nu_2^2, \ G'(\theta_1) = -1/2\theta_1 \)

(v) \( Z_n = T_{1:n} - n\psi(\bar{T}_{2:n}) = \sum_{i=1}^n (X_i - \bar{X}_n)^2 > 0 \text{ a.s.} \)

**Example 2:** The Gamma distribution, \( \mathcal{G}(\alpha, \lambda) \).

(i) \( \theta_1 = \alpha, \ \theta_2 = -\lambda, \ \Theta = \mathbb{R}^+ \times \mathbb{R}^- \)

(ii) \( U_1(X) = \log(X), \ U_2(X) = X, \ T_{1:n} = \sum_{i=1}^n \log(X_i), \ T_{2:n} = \sum_{i=1}^n X_i \)

(iii) \( \nu_2 = \alpha/\lambda, \ \psi(\nu_2) = \log(\nu_2), \ G'(\alpha) = \Gamma'(\alpha)/\Gamma(\alpha) - \log(\alpha) \)

(iv) \( Z_n = T_{1:n} - n\psi(\bar{T}_{2:n}) = \sum_{i=1}^n \log(X_i/\bar{X}_n) < 0 \text{ a.s.} \)

**Example 3:** The Inverse Gaussian distribution.

(i) \( f(x : \lambda, \alpha) = (2\pi)^{1/2} x^{-3/2} \lambda^{1/2} \exp\{-\frac{\alpha x}{2} - \frac{\lambda}{2} x + (\lambda x/2)^{1/2}\}, \ x, \lambda \in \mathbb{R}^+, \ \alpha \in \mathbb{R}^{+} \cup \{0\} \).

(ii) \( U_1(X) = 1/X, \ U_2(X) = X, \ \theta_1 = -\lambda/2, \ \theta_2 = -\alpha/2, \ \Theta = \mathbb{R}^- \times (\mathbb{R}^- \cup \{0\}) \).

(iii) \( \nu_2 = -(\theta_1/\theta_2)^{1/2}, \ \psi(\nu_2) = 1/\nu_2, \ G'(\theta_1) = -1/2\theta_1 \).

(v) \( Z_n = \sum_{i=1}^n (1/X_i) - (n/\bar{X}) > 0 \text{ a.s.}; \) although this model is **steep**, all results stated above for a regular model, hold for \( \theta \in \text{int}(\Theta) \). For further discussion, see Bar-Lev and Reiser (1982).
3. The sequential estimation procedure. Suppose that on the basis of \( n \) independent observations \( x_1, \ldots, x_n \) from (2.1), we wish to estimate \( \nu_2 \equiv E(\nu_2) \) in the presence of the nuisance parameter \( \theta_1 \). Let \( \hat{\theta}_1 \) and \( \hat{\nu}_2 \) denote the maximum likelihood estimators of \( \theta_1 \) and \( \nu_2 \), respectively. So that by (2.2), \( \hat{\theta}_1 \) and \( \hat{\nu}_2 \) are the simultaneous solutions of the (log-likelihood derivatives) equations:

\[
\begin{align*}
T_{1:n} - n\nu_1 &= 0 \\
T_{2:n} - n\nu_2 &= 0
\end{align*}
\]

(3.1)

Hence, by using (2.7) in (3.1) we immediately obtain that \( \hat{\nu}_2 = \bar{T}_{2:n} \) and that \( \hat{\theta}_1 \) satisfies the equation:

\[
G'(\hat{\theta}_1) = \bar{T}_{1:n} - \psi(\bar{T}_{2:n}) = Z_n/n .
\]

(3.2)

Further suppose that the loss incurred by using \( \bar{T}_{2:n} \) as an estimate for \( \nu_2 \) is:

\[
L_\rho(\bar{T}_{2:n}) = \rho|\psi''(\nu_2)|(\bar{T}_{2:n} - \nu_2)^2 + n ,
\]

where \( \rho > 0 \). The factor \( \rho|\psi''(\nu_2)| \) represents the importance of the estimation error relative to the cost of one observation. From (2.3) and (2.6) it follows that the for a fixed \( \theta_1 \in \Theta_0 \) the corresponding risk is:

\[
R_\rho(n) = E_\theta[L_\rho(\bar{T}_{2:n})] = \frac{\rho}{n|\theta_1|} + n ,
\]

which is minimized (w.r.t. \( n \)) at integer adjacent to \( n_0 = (\rho/|\theta_1|)^{1/2} \), at which \( R_\rho(n_0) = 2n_0 \). However, since \( \theta_1 \) is unknown, the estimation procedure has to be conducted sequentially, and to be terminated according to the stopping rule \( N_\rho \) where:

\[
N_\rho = \inf\{n \geq m_0 ; |\hat{\theta}_1| > \rho/n^2\}
\]

(3.3)

for some initial sample size \( m_0 \), \( (m_0 \geq 2) \). Moreover, since by Lemma 2.1.c, the function \( G'(\theta_1) \) is strictly increasing on \( \Theta_0 \), it follows from (3.2) and Lemma 2.2, that the stopping rule (3.3) has the following forms:

(i) If \( \Theta_1 \subset \mathbb{R}^- \) then; \( N_\rho = \inf\{n \geq m_0 ; Z_n \leq nG'(\frac{-\theta_1}{n})\} \),

(ii) If \( \Theta_1 \subset \mathbb{R}^+ \) then; \( N_\rho = \inf\{n \geq m_0 ; Z_n \geq nG'(\frac{\theta_1}{n})\} \).
Remark: In either case, the event \( \{N_\rho = n\} \) is determined only by \((Z_{m_0}, \ldots, Z_n)\), and therefore by Theorem 2.2 is independent of \( T_{2,n} \).

By Lemma 2.2, the symmetry of the two cases; \( \Theta_1 \subset \mathbb{R}^+ \) and \( \Theta_1 \subset \mathbb{R}^- \) is evident and in view of (2.5) and proposition 1.6 of Brown (1986), there is no loss of generality by assuming (conveniently) that \( \Theta_1 \subset \mathbb{R}^- \). Accordingly, we let \( \Theta_1 \subset \mathbb{R}^- \) (so that \( \theta_1 < 0 \)), and consider the stopping rule \( N_\rho \) as defined in (i) above.

Since the function \( G' \) is strictly increasing (and positive) on \( \Theta_1 \) and \( Z_n \) converges a.s. to the finite limit \( G'(\theta_1) \), it follows that for each fixed \( \rho \), the stopping rule \( N_\rho \) is finite w.p.1. Moreover, since \( G'(\frac{Z}{n\theta}) \) is decreasing as a function of \( \rho \), \( N_\rho \) is stochastically increasing in \( \rho \) w.p.1, i.e.; for \( 0 < \rho_1 < \rho_2 \), \( N_{\rho_1} < N_{\rho_2} \) w.p.1, and hence, \( \lim_{\rho \to \infty} N_\rho = \infty \) w.p.1.

The main results of this section are presented in the following two theorems.

**Theorem 3.1.** Let \( N_\rho \) be the stopping time as defined in (i) above, then for all \( \theta \in \Theta \) the following properties hold:

a) For each fixed \( \rho \), \( E_\theta(N_\rho) < \infty \)
b) \( \lim_{\rho \to \infty} \frac{N_\rho}{n_0} = 1 \) w.p.1
c) \( \lim_{\rho \to \infty} E_\theta(N_\rho) = 1 \)

As was shown by Starr (1966) and by Woodroofe (1977, 1982), the initial sample size \( m_0 \) plays a crucial role in any attempt to analyze the risk (as well as the regret) associated with \( N_\rho \). Moreover, it was shown, (see Woodroofe (1977), pp. 987), that the left tail behavior of the underlying c.d.f. is also crucial in the risk's assessments. For the general case, we have a need to impose the following conditions on the model at hand. The first condition pertains to \( G' \). Notice that \( G' \) determines both; the boundary for the stopping rule \( N_\rho \), as well as the moments of \( Z_n \). The second condition is imposed to ensure an appropriate initial sample size \( m_0 \).

**Assumption A.3.** For some \( \gamma > 1/2 \), \( \sup_{x \geq 4|\theta_1|} x^{\gamma} G'(-x) \leq M < \infty \).

**Assumption A.4.** The initial sample size \( m_0 \) is such that \( \forall \theta_1 \in \Theta_1, E_{\theta_1} Z_{m_0}^{-\beta} < \infty \) for some \( \beta > \frac{2}{\gamma-1} \).

**Theorem 3.2.** Let \( R_\rho(N) \) denote the risk associated with the stopping rule \( N = N_\rho \), then under Assumptions A.1-A.4:

\[
\lim_{\rho \to \infty} \frac{R_\rho(N)}{R_\rho(n_0)} = 1.
\]
Theorem 3.2 asserts that the proposed estimation procedure is asymptotically risk efficient. That is, the risk incurred by the sequential estimation procedure based on \( N_\rho \), is asymptotically equivalent to the risk incurred by estimation procedure based on the optimal (and hypothetical) fixed sample size \( n_0 \). Note that for the normal case, Assumption A.3 holds with \( \gamma = 1 \) and Assumption A.4, is satisfied for \( \beta > 2 \) and \( m_0 \geq 6 \), (see in comparison: Woodrooffe (1977)). However, it should be noted that it may be possible to relax the above requirements on \( m_0 \) and \( \beta \) in particular cases.

Proof of Theorem 3.1:

a) Let \( \rho \) be fixed so that \( n_0 < \infty \). Fix \( \epsilon > 1 \), clearly

\[
m_0 \leq E_\theta(N_\rho) \leq n_0 + \sum_{n=n_0+1}^{\infty} P_\theta(N_\rho > n) \\
\leq n_0 + (n_0 + 1)(\epsilon - 1) + \sum_{n=K}^{\infty} P_{\theta_1}(Z_n > nG'(\frac{-\rho}{n^2})) ,
\]

where \( K = [(n_0 + 1)\epsilon] + 1 \) and \([x]\) denotes the integer part of \( x \). But according to Lemma A in Appendix, for all \( n \geq K \)

\[
P_{\theta_1}(Z_n > nG'(\frac{-\rho}{n^2})) \leq e^{-(n-n_0)C} ,
\]

for some constant \( C > 0 \). Hence, the last inequality in (3.4), implies that

\[
E_\theta(N_\rho) \leq n_0 + (n_0 + 1)(\epsilon - 1) + \sum_{n=K}^{\infty} e^{-(n-n_0)C} \\
\leq n_0 + (n_0 + 1)(\epsilon - 1) + \frac{e^{-Cn_0(\epsilon - 1)}}{1 - e^{-C}} < \infty.
\]

b) For this part, we make use of Lemma 2.2(b) along with the definition of \( N_\rho \), to obtain the inequalities

\[
(N_\rho - 1)G'(\frac{-\rho}{(N_\rho - 1)^2}) \leq Z_{N_\rho - 1} < Z_{N_\rho} < N_\rho G'(\frac{-\rho}{N_\rho^2}) ,
\]

which hold w.p.1. Since \( Z_{N_\rho}/N_\rho \xrightarrow{a.s.} G'(\theta_1) \), as \( \rho \to \infty \), it follows that \( \lim_{\rho \to \infty} G'(\frac{-\rho}{N_\rho^2}) = G'(\theta_1) \). Then, by using the relation \(-\rho = \theta_1 n_0^2\), the required result follows.

c) From part (b) and Fatou's Lemma \( \liminf_{\rho \to \infty} E_\theta(N_\rho/n_0) \geq 1 \) Also, by (3.5) above, \( \limsup_{\rho \to \infty} E_\theta(N_\rho/n_0) \leq 1 + (\epsilon - 1) \). Finally, by letting \( \epsilon \to 1 \), the proof of (c) is completed.
Proof of Theorem 3.2: By definition,

\[ R_\rho(N) = E_\theta[\rho|\psi"(\nu_2)|(|T_{2;n} - \nu_2|^2 + N_\rho|. \]

Note that by the definition of \( N_\rho \), the event \{\( N_\rho = n \)\}, depends only on \( Z_1 \ldots Z_n \) and hence by Theorem 2.2 is independent on \( T_{2;n} \). Thus:

\[
R_\rho(N_\rho) = E_\theta\left(\frac{\rho}{N_\rho |\theta_1|}\right) + E_\theta(N_\rho)
\]

\[
= E_\theta\left(\frac{n_0^2}{N_\rho}\right) + E_\theta(N_\rho),
\]

and therefore:

\[
\frac{R_\rho(N_\rho)}{R_\rho(n_0)} = \frac{1}{2} E_\theta\left(\frac{n_0}{N_\rho}\right) + \frac{1}{2} E_\theta\left(\frac{N_\rho}{n_0}\right).
\]

In view of Theorem 3.1 it suffices to show that \( \limsup_{\rho \to \infty} E_\theta\left(\frac{n_\rho}{N_\rho}\right) \leq 1 \).

Fix \( 0 < \varepsilon < 1/2 \), then

\[
E_\theta\left(\frac{n_0}{N_\rho}\right) = E_\theta\left(\frac{n_0}{N_\rho} I(N_\rho \leq n_0/2)\right)
\]

\[
+ E_\theta\left(\frac{n_0}{N_\rho} I\left(\frac{n_0}{2} \leq N_\rho < n_0(1 - \varepsilon)\right)\right)
\]

\[
+ E_\theta\left(\frac{n_0}{N_\rho} I(n_0(1 - \varepsilon) \leq N_\rho < n_0(1 + \varepsilon))\right)
\]

\[
+ E_\theta\left(\frac{n_0}{N_\rho} I(N_\rho \geq n_0(1 + \varepsilon))\right)
\]

\[
= B_1 + B_2 + B_3 + B_4, \text{ say.}
\]

By Lemma B in appendix, \( B_1 \to 0 \), as \( \rho \to \infty \).

As immediate consequences of Theorem 3.1 and Lemma A in appendix;

\[
B_2 \leq 2P\left[\frac{1}{2} \leq \frac{N_\rho}{n_0} < 1 - \varepsilon\right] \to 0
\]

\[
B_4 \leq \frac{1}{(1 + \varepsilon)} \frac{1}{(1 + \varepsilon)} P\left[\frac{N_\rho}{n_0} > 1 + \varepsilon\right] \to 0.
\]

Finally, by using the dominated convergence theorem it is easy to show that \( B_3 \to 1 \) as \( \rho \to \infty \), which completes the proof. \( \quad \blacksquare \)

Concluding Remark: It is evident that the independence result, presented in Theorem 2.2, is a crucial key in the risk assessments. It will also turn out to be an important tool in other sequential problems concerning the family of distributions we have discussed here.
These problems include the second order properties of the risk, as well as problems similar to those discussed in Siegmund (1985). Currently, we are studying some of these problems and the results will appear in a future paper.

APPENDIX

Proof of Theorem 2.2. The proof will be carried in two steps.

Step 1. Show that \((Z_{j-1}, Z_j) \perp T_{2;j}\) for all \(j > 2\).

Step 2. Show that if \((Z_2, \ldots, Z_i) \perp T_{2;i}\) for all \(i \leq k\) then \((Z_2, \ldots, Z_{k+1}) \perp T_{2;k+1}\).

We will use the following notations in the proof. For \(j > 2\), \(Z_j = (Z_2, \ldots, Z_j)\), 

\[0_j = (0, \ldots, 0), \quad \alpha_j = (\alpha_2, \ldots, \alpha_j), \quad \beta_j = (\beta_2, \ldots, \beta_j)\] where \(\alpha_i\)'s are and \(\beta_i\)'s are \((j-1)\) times complex numbers to be specified later.

Note that the joint density of \((Z_{j-1}, T_{1;j}, T_{2;j})\) is of the form:

\[f(x_{j-1}, t_{1;j}, t_{2;j}) = K_j(x_{j-1}, t_{1;j}, t_{2;j}) \exp(\theta_1 t_{1;j} + \theta_2 t_{2;j} + j c(\theta))\]

for some function \(K_j(\cdot) > 0\).

For each \(j\), \((j > 2)\), define the functions \(\phi_j\) and \(b_j\) as follows, whenever they exist.

\[\phi_j(\alpha_j, t_{2;j}, \theta_1) = E \left\{ \exp \left( i \sum_{k=2}^{j} \alpha_k z_k \right) \bigg| T_{2;j} = t_{2;j} \right\} \]

\[b_j(\beta_j, t_{2;j}) = \int K_j(x_{j-1}, t_{1;j}, t_{2;j}) \exp \left( \sum_{k=2}^{j-1} \beta_k z_k + \beta_j t_{1;j} \right) dx_2 \ldots dx_{j-1} dt_{1;j}\]

Note that the conditional density of \((Z_2, \ldots, Z_{j-1}, T_{1;j})\) given \(T_{2;j} = t_{2;j}\) is given by

\[(A.1) \quad f(x_{j-1}, t_{1;j}|T_{2;j} = t_{2;j}) = \frac{K_j(x_{j-1}, t_{1;j}, t_{2;j}) \exp(\theta_1 t_{1;j})}{b_j(0_{j-1}, \theta_1, t_{2;j})}\]

To prove Step 1, we need to show that \(\phi_j(0_{j-2}, \alpha_j, t_{2;j}, \theta_1) = \phi(\alpha_{j-1}, \alpha_j, t_{2;j}, \theta_1)\), which is the conditional characteristic function of \((Z_{j-1}, Z_j)\) given \(T_{2;j} = t_{2;j}\), does not involve \(t_{2;j}\) for \((\alpha_{j-1}, \alpha_j)\) in a neighborhood of \(0\) in \(\mathbb{R}^2\).

Using (A.1),

\[\phi(\alpha_{j-1}, \alpha_j, t_{2;j}, \theta_1) = \frac{e^{-i\alpha_j \psi(t_{2;j})} b_j(0_{j-2}, i\alpha_{j-1}, i\alpha_j + \theta_1, t_{2;j})}{b_j(0_{j-1}, \theta_1, t_{2;j})}\]

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where we have used the fact that \( Z_j = T_{1;j} - j\psi(T_{2;j}) \).

Thus

\[
(A.2) \quad e^{-i\alpha_j j\psi(T_{2;j})} b_j(0_{j-2}, i\alpha_{j-1}, i\alpha_j + \theta_1, t_{2;j}) = b_j(0_{j-1}, \theta_1, t_{2;j}) \phi(\alpha_{j-1}, \alpha_j, t_{2;j}, \theta_1)
\]

Note that \( Z_{j-1} \perp T_{2;j-1} \) and hence \( Z_{j-1} \perp T_{2;j-1} + U_2(X_j) = T_{2;j} \). Thus \( \phi(\alpha_{j-1}, 0, t_{2;j}, \theta_1) \) does not involve \( t_{2;j} \). Call this function \( \phi(\alpha_{j-1}, 0, \theta_1) \).

Thus, by substituting \( \alpha_j = 0 \) in (A.2), we get:

\[
(A.3) \quad b_j(0_{j-2}, i\alpha_{j-1}, \theta_1, t_{2;j}) = b_j(0_{j-1}, \theta_1, t_{2;j}) \phi(\alpha_{j-1}, 0, \theta_1).
\]

By extending the parameter space to the complex plane, it is easy to extend the definition of \( \phi(\alpha_{j-1}, 0, \theta_1) \) to \( \phi(\alpha_{j-1}, 0, i\alpha_j + \theta_1) \). Using analytic continuation, equation (A.3) continues to hold when \( \theta_1 \) is replaced by \( i\alpha_j + \theta_1 \). Accordingly

\[
(A.4) \quad b_j(0_{j-2}, i\alpha_{j-1}, i\alpha_j + \theta_1, t_{2;j}) = b_j(0_{j-1}, i\alpha_j + \theta_1, t_{2;j}) \phi(\alpha_{j-1}, 0, i\alpha_j + \theta_1).
\]

Hence, using (A.2)–(A.4),

\[
(A.5) \quad \phi(\alpha_{j-1}, \alpha_j, t_{2;j}, \theta_1) = e^{-i\alpha_j j\psi(T_{2;j})} \frac{b_j(0_{j-2}, i\alpha_{j-1}, i\alpha_j + \theta_1, t_{2;j})}{b_j(0_{j-1}, \theta_1, t_{2;j})} \\
= \phi(\alpha_{j-1}, 0, i\alpha_j + \theta_1) e^{-i\alpha_j j\psi(T_{2;j})} \frac{b_j(0_{j-1}, i\alpha_j + \theta_1, t_{2;j})}{b_j(0_{j-1}, \theta_1, t_{2;j})}.
\]

Comparing with equations (3.18)–(3.19) of Bar–Lev and Rieser (1982), (note that their \( b(s, t_2) \equiv b_j(0_{j-1}, s, t_{2;j}) \) and their \( U_1 g^{-1}(T_2) \equiv \psi(T_{2;j}) \)), we get that for \( s = \theta_1 + i\alpha, \alpha \in \mathbb{R} \),

\[
\log b_j(0_{j-1}, s, t_{2;j}) = js\psi(T_{2;j}) + R_j(T_{2;j}) + H_j(s).
\]

Using this, we immediately obtain that

\[
\frac{b_j(0_{j-1}, i\alpha_j + \theta_1, t_{2;j})}{b_j(0_{j-1}, \theta_1, t_{2;j})} = e^{i\alpha_j j\psi(T_{2;j}) - [H_j(i\alpha_j+\theta_1) - H_j(\theta_1)]}.
\]

This shows that (A.5) does not involve \( t_{2;j} \), completing the proof of Step 1.

To prove Step 2, we need to show that \( \phi_{k+1}(\alpha_{k+1}, t_{2;k+1}, \theta_1) \) does not involve \( t_{2;k+1} \) for \( \alpha_{k+1} \) in a neighborhood of 0 in \( \mathbb{R}^k \).

Proceeding as in the proof of Step 1,

\[
(A.6) \quad \phi_{k+1}(\alpha_{k+1}, t_{2;k+1}, \theta_1) = \frac{e^{-i(k+1)\alpha_{k+1} \psi(T_{2;k+1})} b_{k+1}(i\alpha_1, \ldots, i\alpha_k, i\alpha_{k+1} + \theta_1, t_{2;k+1})}{b_{k+1}(0_k, \theta_1, t_{2;k+1})}.
\]
However, by hypothesis, \((Z_1, \ldots, Z_k) \perp T_{2:k}\) and hence \((Z_1, \ldots, Z_k) \perp T_{2:k+1}\). Thus
\[\phi_{k+1}(\alpha_k, 0, t_{2:k+1}, \theta_1) \equiv \phi_{k+1}(\alpha_k, 0, \theta_1),\]
is independent of \(t_{2:k+1}\). Accordingly,
\[(A.7) \quad b_{k+1}(i\alpha_1, \ldots, i\alpha_k, \theta_1, t_{2:k+1}) = b_{k+1}(0_k, \theta_1, t_{2:k+1}) \phi_{k+1}(\alpha_k, 0, \theta_1)\]
Again, arguing as in Step 1,
\[(A.8) \quad b_{k+1}(i\alpha_1, \ldots, i\alpha_k, i\alpha_{k+1} + \theta_1, t_{2:k+1}) = b_{k+1}(0_k, i\alpha_{k+1} + \theta_1, t_{2:k+1}) \times \phi_{k+1}(\alpha_k, 0, i\alpha_{k+1} + \theta_1).\]
Using (A.6), (A.7), (A.8) and equations (3.18)–(3.19) of Bar–Lev and Reiser (1982) the proof of Step 2 can now be completed exactly as in Step 1.

**Lemma A.** Let \(n_0 = (\rho/|\theta_1|)^{1/2}\), \((\theta_1 < 0)\) and \(\epsilon > 1\), be fixed and let \(Z_n\) be as defined in (2.4) with the p.d.f (2.5). Then for all \(n > n_0\epsilon\) there exits a constant \(C_0\) such that:
\[P_{\theta_1}(Z_n > nG'(\frac{-\rho}{n^2})) \leq \exp\{-((n - n_0)C_0(\epsilon - 1)/2\epsilon)\}.\]

**Proof:** By (2.5), the moment generating functions, \(M_{Z_n}(t)\) of \(Z_n\), exists for all \(t < -\theta_1\), and is given by:
\[M_{Z_n}(t) = \exp\{H_n(t + \theta_1) - H_n(\theta_1)\}, \quad t < -\theta_1,\]
with \(H_n(\cdot)\) as defined in (2.8). Let \(\epsilon_n = (\frac{n_0}{n})^2 < 1\) and let \(t_n = \theta_1(\epsilon_n - 1)\). Clearly, \(t_n \in [0, -\theta_1]\). It can be easily verified that
\[p_n \equiv P_{\theta_1}[Z_n > nG'(\theta_1\epsilon_n)] \leq e^{-t_n nG'(\theta_1\epsilon_n)} M_{Z_n}(t_n) \equiv \exp\{\varphi_n(t_n)\},\]
where we have put: \(\varphi_n(t) = H_n(t + \theta_1) - H_n(\theta_1) - t_nG'(\theta_1\epsilon_n), \quad t \geq 0\). However, by the definition of \(H_n(\cdot)\),
\[(A.9) \quad \varphi_n(t_n) = H_n(t_n + \theta_1) - H_n(\theta_1) - t_n nG'(\theta_1\epsilon_n)\]
\[= n[G(\theta_1\epsilon_n) - G(\theta_1)] - [G(n\theta_1\epsilon_n) - G(n\theta_1)] + \theta_1(1 - \epsilon_n)nG'(\theta_1\epsilon_n).\]
Since \(G(n\theta_1\epsilon_n) - G(n\theta_1) > 0\), and \(G''(\cdot) > 0\), the last equality in A.9 implies that
\[\varphi_n(t_n) \leq -n\theta_1^2(1 - \epsilon_n)^2 G''(\theta_1\epsilon_n)/2,\]

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for some $\varepsilon_n^*$ between 1 and $\varepsilon_n$. Note that $G''(x) \geq C_0 > 0$ for all $x \in [\theta_1, 0]$, (see also the discussion following (2.8)). In addition, since $n > n_0 \varepsilon$, we have:

$$(1 - \varepsilon_n)^2 \geq (1 - 1/\varepsilon)(1 - \varepsilon_n) \geq (1 - 1/\varepsilon)(1 - n_0/n).$$

Accordingly:

$$P_n \leq \exp\{\varphi_n(t_n)\} \leq \exp\{-nC_0(1 - \varepsilon_n)^2/2\} \leq \exp\{-(n - n_0)C_0(\varepsilon - 1)/2\varepsilon\},$$

which completes the proof.

**Lemma B.** Suppose that $G'(\cdot)$ and $m_0$ satisfy Assumptions A.3 and A.4, then:

$$E\left(\frac{n_0}{N^\rho} I[N^\rho \leq n_0/2]\right) \to 0 \text{ as } \rho \to \infty.$$

**Proof.** Let $1/2 < \alpha < 1$ be fixed, (to be chosen later), and let $C$ be a generic constant. Then

$$E\left(\frac{n_0}{N^\rho} I[m_0 \leq N^\rho \leq n_0/2]\right) \leq n_0 E\left(\frac{1}{N^\rho} I[m_0 \leq N^\rho \leq n_0^\alpha]\right) + n_0^{1-\alpha} P(n_0^{\alpha} < N^\rho \leq n_0/2)$$

$$= I_1 + I_2 \quad \text{(say)}.$$

Now, for the first term $I_1$,

$$E\left(\frac{1}{N^\rho} I[m_0 \leq N^\rho \leq n_0^\alpha]\right) = \sum_{k=m_0}^{[n_0^\alpha]} \frac{1}{k} P(N^\rho = k)$$

$$\leq \sum_{k=m_0}^{[n_0^\alpha]} \frac{1}{k} P(Z_k \leq kG'(\theta_1(\frac{n_0^2}{k}))$$

$$\leq \sum_{k=m_0}^{[n_0^\alpha]} \frac{1}{k} P(Z_k < \frac{k^{1+2\gamma}}{n_0^{2\gamma} |\theta_1|^{\gamma} M})$$

$$\leq \sum_{k=m_0}^{[n_0^\alpha]} \frac{1}{k} P(Z_{m_0} < \frac{k^{1+2\gamma}}{n_0^{2\gamma} C})$$

$$\leq E(Z_{m_0}^{-\beta}) C n_0^{-2\gamma\beta} \sum_{k=m_0}^{[n_0^\alpha]} k^{(1+2\gamma)\beta - 1},$$

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where the last two inequalities following Assumptions A.3-A.4, and Lemma 2.2 (b). Accordingly
\[
I_1 \leq E \left( Z_{m_0}^{-\beta} \right) C n_0^{(1-2\gamma \beta + \alpha \beta(1+2\gamma))},
\]
which \( \to 0 \) for \( \alpha < (2\gamma \beta - 1)/\beta(1+2\gamma) \).

Clearly, \( I_2 \leq n_0^{(1-\alpha)} P \left[ Z_k < k G'(\theta_1 \left( \frac{n_0}{k} \right)^2) \right] \), for some \( k \in (n_0^{\alpha}, \frac{n_0}{2}] \).

Define: \( L_{1:k} = T_{1:k} - k \nu_1 \), \( L_{2:k} = \varphi(T_{2:k}) - \varphi(\nu_2) \), note that \( Z_k = L_{1:k} - k L_{2:k} \), and by (2.7), \( \nu_1 - \varphi(\nu_2) = G'(\theta_1) \), hence;
\[
I_2 \leq n_0^{(1-\alpha)} P \left[ L_{1:k} - k L_{2:k} < k \Delta_k, \text{ for some } k \in (n_0^{\alpha}, \frac{n_0}{2}] \right]
\]
with \( \Delta_k = G'(\theta_1 \left( \frac{n_0}{k} \right)^2) - G'(\theta_1) \). Since \( G'() \) is increasing and \( k < \frac{n_0}{2} \),
\[
\Delta_k \leq G'(4\theta_1) - G'(\theta_1) \equiv -2\varepsilon(<0), \text{ say.}
\]

Thus,
\[
I_2 \leq n_0^{(1-\alpha)} P \left[ L_{1:k} - k L_{2:k} < -k \varepsilon, \text{ for some } k \in (n_0^{\alpha}, \frac{n_0}{2}] \right]
\]
\[
\leq n_0^{(1-\alpha)} P \left[ |L_{1:k}| > k \varepsilon, \text{ for some } k \in (n_0^{\alpha}, \frac{n_0}{2}] \right] +
\]
\[
+ n_0^{(1-\alpha)} P \left[ k |L_{2:k}| > k \varepsilon, \text{ for some } k \in (n_0^{\alpha}, \frac{n_0}{2}] \right]
\]
\[
= I_{21} + I_{22}, \text{ (say).}
\]

Since \( T_{1:k} \) has moments of all orders, it follows immediately, using submartingale inequality that,
\[
I_{21} \leq n_0^{(1-\alpha)} P \left[ \max_{n_0^{\alpha} < k \leq n_0/2} |L_{1:k}| > n_0^{\alpha} \varepsilon \right]
\]
\[
\leq n_0^{(1-\alpha)} n_0^{-\alpha r} \varepsilon^r \left( (L_{1:k}^{\frac{n_0}{\theta_1}}) \right)^r, \ r > 0
\]
\[
= 0 \left( n_0^{1-\alpha + r(\frac{1}{2} - \alpha)} \right).
\]

As for the second term \( I_{22} \), it follows by the continuity of \( \varphi(\cdot) \) that there is \( \delta(\varepsilon) > 0 \) such that \( |x - \nu_2| < \delta(\varepsilon) \Rightarrow |\varphi(x) - \varphi(\nu_2)| < \varepsilon \). Thus,
\[
I_{22} \leq n_0^{(1-\alpha)} P \left[ |T_{2:k} - \nu_2| > \delta(\varepsilon), \text{ for some } k \in (n_0^{\alpha}, \frac{n_0}{2}] \right]
\]
\[
\leq n_0^{(1-\alpha)} P \left[ |T_{2:k} - \nu_2| > n_0^{\alpha} \delta(\varepsilon), \text{ for some } k \in (n_0^{\alpha}, \frac{n_0}{2}] \right]
\]
\[
= 0 \left( n_0^{1-\alpha + r(\frac{1}{2} - \alpha)} \right),
\]
again, by using submartingale inequality as in (A.10). Finally, by combining (A.10) and (A.11) together, we obtain;
\[
I_2 \leq 0 \left( n_0^{1-\alpha + r(\frac{1}{2} - \alpha)} \right) \to 0,
\]

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for \( r \) large and \( \alpha > 1/2 \).

Hence, upon choosing \( 1/2 < \alpha < \frac{(2\gamma \beta - 1)}{\beta(1+2\gamma)} \) with \( \beta > 2/(2\gamma - 1) \) as required, the proof is completed.

\[ \Box \]

Bibliography


