Travelling Salesman with a
Self-Similar Itinerary

by

Steven P. Lalley*
Purdue University

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Steven P. Lalley
Dept. of Statistics
Math. Sci. Bldg
Purdue University
West Lafayette, IN 47907
(317) 494-6030

Abstract

ABSTRACT. Let $X_1, X_2, \ldots$ be i.i.d. random points in $\mathbb{R}^2$ with distribution $\nu$, and let $L_n$ be the length of the shortest path through $X_1, \ldots, X_n$. The exact almost sure rate of growth of $L_n$ is obtained under the assumption that $\nu$ is self-similar in an appropriate sense. This extends a well known theorem of Beardwood, Halton, and Hammersley.

SHORT TITLE. Travelling Salesman with a Self-Similar Itinerary.
1. The Beardwood-Halton-Hammersley Theorem

**Theorem 0:** If $X_1, X_2, \ldots$ are i.i.d. random vectors in $\mathbb{R}^d$ from a probability distribution with compact support and if $L_n$ is the length of the shortest path through $X_1, X_2, \ldots, X_n$, then

$$\frac{L_n}{n^{(d-1)/d}} \rightarrow c_d \int f(x)^{(d-1)/d} dx \text{ a.s.,}$$

where $c_d > 0$ is a constant and $f(x)$ is the density of the absolutely continuous component of the distribution of $X_n$.

This theorem was discovered by Beardwood, Halton, and Hammersley [1]. It is of interest in the study of probabilistic algorithms for the travelling salesman problem [5], [7].

If the distribution of $X_n$ is singular then Th. 0 implies that $L_n/n^{(d-1)/d} \rightarrow 0$ a.s. It is natural to inquire about the rate of growth of $L_n$ in such cases. The purpose of this note is to exhibit the exact rate of growth of $L_n$ when the distribution of $X_n$ and its support are “self-similar” in an appropriate sense. For simplicity, we shall consider only distributions in $\mathbb{R}^2$.

The Beardwood-Halton-Hammersley theorem is ultimately a consequence of the local self-similarity of the ambient space, because this forces a degree of hierarchical organization upon the shortest path (see Lemmas 4 and 8 below). It is easy to find singular distributions lacking self-similarity for which the growth of $L_n$ is highly erratic. Beardwood-Halton-Hammersley-type theorems undoubtedly hold for distributions not considered in this paper, provided some approximate, local self-similarity is present (e.g., normalized Hausdorff measures on Julia sets).

2. A BHH Theorem for Self-Similar Probability Distributions

Let $K$ be a compact subset of $\mathbb{R}^2$ satisfying $K = \bigcup_{i=1}^{m} \psi_i(K)$, where $\psi_i$ is a contractive similarity transformation of $\mathbb{R}^2$ with similarity ratio $r_i \in (0, 1)$ (i.e., for any points $x, y \in \mathbb{R}^2$, $\|\psi_i(x) - \psi_i(y)\| = r_i \|x - y\|$). Say that $K$ is strongly self-similar if there exists a bounded, connected, open set $V$ whose boundary $\partial V$ is a simple, closed, rectifiable curve and such that $V \cap K \neq \emptyset$, $\psi_i(V) \cap \psi_j(V) = \emptyset \forall i \neq j$, and $\bigcup_{i=1}^{m} \psi_i(V) \subset V$. If $\psi_1(K), \ldots, \psi_m(K)$ are pairwise disjoint, say that $K$ is strongly disconnected.

**Examples:** (1) $K$ is the closed unit square; $V$ is the open unit square; $\psi_1, \psi_2, \psi_3, \psi_4$ are the affine mappings that take $V$ to the four nonoverlapping subsquares of side $r_1 = r_2 = r_3 = r_4 = 1/2$, each with one vertex at a vertex of $V$.

(2) $K = \{(x_1, x_2) : x_i \in C\}$, where $C$ is the usual Cantor set; $V$ is the open unit square; $\psi_1, \psi_2, \psi_3, \psi_4$ are the affine mappings that take $V$ to the four nonoverlapping subsquares...
of side $r_1 = r_2 = r_3 = r_4 = 1/3$, each with one vertex at a vertex of $V$. This set $K$ is strongly disconnected.

(3) $K$ is the Sierpinski gasket ([6], plate 141); $V$ = an open equilateral triangle of side 1; $\psi_1, \psi_2, \psi_3$ are the affine mappings that take $V$ to the three equilateral (sub)triangles of side $r_1 = r_2 = r_3 = 1/2$, each with a vertex at a vertex of $V$.

Assume henceforth that $K$ is a strongly self-similar set and that $r_1, r_2, \ldots, r_n$ are the similarity ratios. It is known [4] that the Hausdorff dimension $\delta$ of $K$ is the unique positive real number such that $\sum_{i=1}^{m} r_i^{\delta} = 1$. Note that $\delta < 1$ iff $\Sigma r_i < 1$.

**Theorem 1:** Assume that $\delta < 1$. Let $X_1, X_2, \ldots$ be i.i.d. with distribution $\nu$, where $\nu(K) = 1$, and let $L_n$ be the length of the shortest path through $X_1, X_2, \ldots, X_n$. Then there is a constant $C < \infty$ such that

$$\lim_{n \to \infty} L_n = C \ a.s.$$ (2.1)

The cases $\delta > 1$ and $\delta = 1$ are more interesting. Let $\nu$ be a probability measure on $K$; say that $\nu$ is self-similar if there exist positive real numbers $p_1, p_2, \ldots, p_m$ satisfying $p_1 + \ldots + p_m = 1$ and

$$\nu(\psi_{i_1} \circ \psi_{i_2} \circ \ldots \circ \psi_{i_n}(K)) = p_{i_1} p_{i_2} \ldots p_{i_n}$$

for all finite sequences $i_1 i_2 \ldots i_n$ of indices from $\{1, 2, \ldots, m\}$. (If $p_i = r_i^{\delta}$ then $\nu$ is the normalized $\delta$-dimensional Hausdorff measure on $K$.) Define the similarity exponent $\theta$ of $\nu$ to be the unique real number such that

$$\sum_{i=1}^{m} r_i p_i^{\theta} = 1.$$ 

Observe that $\theta > 0$ iff $\delta > 1$ and $\theta = 0$ iff $\delta = 1$. In general, $(1 - \theta)^{-1}$ is the Hausdorff dimension of the set of generic points of $\nu$.

**Theorem 2:** Let $X_1, X_2, \ldots$ be i.i.d. with distribution $\nu$, where $\nu$ is a self-similar probability measure on $K$ with similarity exponent $\theta > 0$. Let $L_n$ be the length of the shortest path through $X_1, \ldots, X_n$.

(a) (Nonarithmetic case) If $\log p_1, \ldots, \log p_m$ are contained in no discrete additive subgroup of $\mathbb{R}$ then there is a constant $0 < \mu < \infty$ such that

$$\lim_{n \to \infty} L_n / n^\theta = \mu \ a.s.$$ (2.2)
(b) (Arithmetic case) If \( \log p_1, \ldots, \log p_m \) are contained in \( h\mathbb{Z} \) but in no proper subgroup of \( h\mathbb{Z} \) then there exists a continuous, positive, \( h \)-periodic function \( C(t) \) such that

\[
\lim_{n \to \infty} \frac{L_n}{n^\theta C(\log n)} = 1 \text{ a.s.}
\]

(2.3)

Notice that \((K, \nu)\) may have more than one self-similarity structure. Consider, for example \( K = [0,1] \times [0,1] \) and \( \nu = \text{Lebesgue measure on } K \). One self-similarity structure is given by \( V = \text{open unit square} \); \( \psi_1, \psi_2, \psi_3, \psi_4 \) are the affine mappings that take \( V \) to the four nonoverlapping subsquares \( \left( \frac{i}{2}, \frac{(i+1)}{2} \right) \times \left( \frac{j}{2}, \frac{(j+1)}{2} \right) \), \( i,j = 0 \) or \( 1 \); and \( p_1 = p_2 = p_3 = p_4 = 1/4 \). Another self-similarity structure is given by \( V = \text{open unit square} \); \( \psi_1, \ldots, \psi_9 \) are the affine mappings taking \( V \) to the nine nonoverlapping subsquares \( \left( \frac{i}{3}, \frac{(i+1)}{3} \right) \times \left( \frac{j}{3}, \frac{(j+1)}{3} \right) \), \( i,j = 0,1, \) or \( 2 \); and \( p_1 = p_2 = \ldots = p_9 = \frac{1}{9} \). Observe that \( \theta = \frac{1}{2} \) for each of the structures, and that the closed additive group generated by \( \log \left( \frac{1}{4} \right) \) and \( \log \left( \frac{1}{9} \right) \) is \( \mathbb{R} \). Consequently, (2.3) applied to each of the structures separately implies that (2.2) must hold, because if \( C(t) \) is \( \log 4 \)-periodic and \( \log 9 \)-periodic it must be constant. In general, if there exist distinct self-similarity structures \((\psi_1, \ldots, \psi_m; p_1, \ldots, p_m)\) and \((\tilde{\psi}_1, \ldots, \tilde{\psi}_m; \tilde{p}_1, \ldots, \tilde{p}_m)\) such that the smallest closed subgroup of \( \mathbb{R} \) containing \( \log p_1, \ldots, \log p_m, \log \tilde{p}_1, \ldots, \log \tilde{p}_m \) is \( \mathbb{R} \), then (2.2) holds.

One might wonder if the periodic function \( C(t) \) in (2.3) is ever non-constant. I am convinced that in general it is, for reasons that I will explain in sec. 9.

The case \( \delta = 1 \) occurs iff \( \sum_{i=1}^{m} r_i = 1 \). One example is \( K = \text{any closed line segment in } \mathbb{R}^2 \). In this instance \( L_n \to \text{constant a.s. as } n \to \infty \); thus, one might expect that, in general, if \( \delta = 1 \) then \( L_n \) is bounded. This is false.

**Theorem 3:** Let \( X_1, X_2, \ldots \) be i.i.d. with distribution \( \nu \), where \( \nu \) is a self-similar probability measure on \( K \), and let \( L_n \) be the length of the shortest path through \( X_1, X_2, \ldots, X_n \). Assume that \( K \) is strongly disconnected and strongly self-similar, with dimension \( \delta = 1 \).

Then there is a constant \( 0 < \mu < \infty \) such that

\[
\lim_{n \to \infty} \frac{L_n}{\log n} = \mu \text{ a.s.}
\]

(2.4)

The proof of Th. 1 will be given in sec. 4; the proofs of Th. 2–3 occupy sec. 5–8. Section 3 is a resumé of some important facts about self-similar sets and self-similar distributions.
3. The Structure of Self-Similar Sets and Distributions

The elements of a strongly self-similar set $K$ may be represented as infinite sequences in the following way. Let $i_1 i_2 \ldots$ be an infinite sequence of indices from $\{1, 2, \ldots, m\}$; define

\[ K_{i_1 i_2 \ldots i_n} = \psi_{i_1} \circ \psi_{i_2} \circ \ldots \circ \psi_{i_n}(K), \]

\[ V_{i_1 i_2 \ldots i_n} = \psi_{i_1} \circ \psi_{i_2} \circ \ldots \circ \psi_{i_n}(V), \] and

\[ k_{i_1 i_2 \ldots} = \bigcap_{n=1}^{\infty} K_{i_1 i_2 \ldots i_n}. \]

Observe that $K \supset K_{i_1} \supset K_{i_1 i_2} \supset \ldots$ is a nested sequence of nonempty, compact sets, so the intersection is nonempty. Moreover,

\[ \text{diameter } (K_{i_1 i_2 \ldots i_n}) = r_{i_1} r_{i_2} \ldots r_{i_n} \text{ diameter } (K), \]

so $k_{i_1 i_2 \ldots}$ is a single point. Conversely, each point $x \in K$ has the form $x = k_{i_1 i_2 \ldots}$ for some sequence of indices, because there is a nested chain $K \supset K_{i_1} \supset K_{i_1 i_2} \supset \ldots$ containing $x$. Notice that some points may have multiple representations.

Recall that $V$ is a bounded open set satisfying $\bigcup_{i=1}^{m} \psi_i(V) \subset V$ and $\psi_i(V) \cap \psi_j(V) = \emptyset$ if $i \neq j$. It follows that for any sequence $i_1 i_2 \ldots, \overline{V} \supset \overline{V}_{i_1} \supset \overline{V}_{i_1 i_2} \supset \ldots$ is a nested sequence of nonempty, compact sets with diameters converging to zero, hence $\bigcap_{n=1}^{\infty} \overline{V}_{i_1 i_2 \ldots i_n}$ is a single point. This point must be $k_{i_1 i_2 \ldots}$, because for each $n, K_{i_1 i_2 \ldots i_n} \cap \overline{V}_{i_1 i_2 \ldots i_n} \neq \emptyset$, since $K \cap V \neq \emptyset$. It follows that each $k_{i_1 i_2 \ldots}$ is an element of $\overline{V}$; thus

\[ (3.1) \quad K \subset \overline{V} \text{ and } K_{i_1 i_2 \ldots i_n} \subset \overline{V}_{i_1 i_2 \ldots i_n} \]

for any $n$ and any sequence $i_1 i_2 \ldots$. Note that for any finite sequences $i_1 i_2 \ldots i_n$ and $i'_1 i'_2 \ldots i'_{n'}$ that do not agree in all of the first $(n \wedge n')$ entries,

\[ (3.2) \quad V_{i_1 i_2 \ldots i_n} \cap V_{i'_1 i'_2 \ldots i'_{n'}} = \emptyset. \]

Lemma 1: There exist a finite sequence $j_1 j_2 \ldots j_{\ell}$ of indices such that $K_{j_1 j_2 \ldots j_{\ell}} \subset V$.

Proof: By hypothesis $K \cap V \neq \emptyset$, so there is a point $k_{j_1 j_2 \ldots} \in V$. But the sets $K_{j_1 j_2 \ldots j_n}$ shrink to $k_{j_1 j_2 \ldots}$ as $n \to \infty$ (the diameters converge to zero) and $V$ is open; hence, for $\ell$ sufficiently large, $K_{j_1 j_2 \ldots j_{\ell}} \subset V$. \qed

Proposition 1: Let $p_1, p_2, \ldots, p_m$ be positive real numbers satisfying $p_1 + \ldots + p_m = 1$.

There is a unique probability measure $\nu = \nu_{p_1, p_2, \ldots, p_m}$ satisfying $\nu(K) = 1$ and

\[ (3.3) \quad \nu(K_{i_1 i_2 \ldots i_n}) = p_{i_1} p_{i_2} \ldots p_{i_n}. \]
for every finite sequence $i_1i_2\ldots i_n$ from $\{1, 2, \ldots, m\}$. This probability measure also satisfies

\begin{equation}
\nu(K_{i_1i_2\ldots i_n} \cap K_{i'_1i'_2\ldots i'_n}) = 0
\end{equation}

for all sequences $i_1i_2\ldots i_n$ and $i'_1i'_2\ldots i'_n$, that do not agree in each of the first $(n \land n')$ entries.

**Proof:** It follows from (3.1)–(3.2) and Lemma 1 that

\begin{equation}
K_{i_1i_2\ldots i_nj_1\ldots j_\ell} \cap K_{i'_1i'_2\ldots i'_n} = \emptyset
\end{equation}

unless $i_q = i'_q$ for $q = 1, 2, \ldots, (n \land n')$.

Let $\nu$ be a probability measure satisfying (3.3) for all finite sequences $i_1i_2\ldots i_n$. Since $p_{j_1}p_{j_2}\ldots p_{j_\ell} > 0$, a simple calculation shows that for any sequence $i_1i_2\ldots i_n$

\[\nu(K_{i_1i_2\ldots i_n}) = \nu(\bigcup_X K_{x_1x_2\ldots x_q})\]

where $X$ is the set of all finite sequences $x_1x_2\ldots x_q$ such that $q \geq n + \ell, x_s = i_s$ for $1 \leq s \leq n$, and $x_{q-s} = j_\ell$ for $1 \leq s \leq \ell$. Note that if $x_1x_2\ldots x_q \in X$ then $K_{x_1x_2\ldots x_q} \subseteq K_{i_1i_2\ldots i_n}$.

It now follows from (3.5) that (3.4) must hold unless $i_s = i'_s$ for $s = 1, 2, \ldots, (n \land n')$, because for any sequence $x_1x_2\ldots x_q \in X, K_{x_1x_2\ldots x_q} \cap K_{i'_1i'_2\ldots i'_n} = \emptyset$. Consequently, $\nu$ assigns probability 1 to the set of points in $K$ with unique representations $k_{i_1i_2\ldots i_n}$.

Let $X$ be a random point of $K$ with distribution $\nu$, where $\nu$ is a probability measure satisfying (3.3). By the preceding paragraph, $X$ has a unique representation $k_{I_1I_2\ldots}$ where $I_1, I_2, \ldots$ are random variables valued in $\{1, 2, \ldots, m\}$. But (3.3) requires that $I_1, I_2, \ldots$ be i.i.d. with

\begin{equation}
P\{I_n = i\} = p_i, \ i = 1, 2, \ldots, m.
\end{equation}

This uniquely determines $\nu$.

Finally, let $I_1, I_2, \ldots$ be i.i.d. with distribution (3.6), let $X = k_{I_1I_2\ldots}$, and let $\nu$ be the distribution of $X$. With probability one, the finite sequence $j_1j_2\ldots j_\ell$ occurs somewhere in the infinite sequence $I_1I_2\ldots$, so $k_{I_1I_2\ldots}$ is the unique representation of $X$. Hence, $\nu$ satisfies (3.4). It follows that

\begin{align*}
\nu(K_{i_1i_2\ldots i_n}) &= P\{I_1 = i_1, \ldots, I_n = i_n\} \\
&= p_{i_1}p_{i_2}\ldots p_{i_n}
\end{align*}
for any finite sequence $i_1i_2\ldots i_n$; this proves that there is a probability measure on $K$ satisfying (3.3).

\Box

4. Hausdorff Dimension $< 1$

Recall that the similarity exponent $\theta$ is the unique real number such that $\Sigma r_i p^\theta_i = 1$. If $\theta < 0$ then clearly $\Sigma r_i < 1$, so the Hausdorff dimension of $K$ is less than 1.

Lemma 2: If $\Sigma r_i < 1$ then there exists a rectifiable path that visits every point of $K$.

Proof: Let $d = \text{diameter } (K)$ and let $x \in K$. For each finite sequence $i_1i_2\ldots i_n$ of indices define $x(i_1i_2\ldots i_n) = \psi_{i_1} \circ \psi_{i_2} \circ \ldots \circ \psi_{i_n}(x)$. There is obviously a closed path $\gamma_1$ beginning and ending at $x$ that visits $x(1), \ldots, x(m)$ in that order, such that length $(\gamma_1) \leq (m + 1)d$. Define closed paths $\gamma_2, \gamma_3, \ldots$, each beginning and ending at $x$, recursively, as follows. To obtain $\gamma_{n+1}$, follow $\gamma_1$ from $x$ to $x(1)$; then follow $\psi_1(\gamma_n)$ from $x(1)$ back to $x(1)$; then follow $\gamma_1$ from $x(1)$ to $x(2)$; then follow $\psi_2(\gamma_n)$ from $x(2)$ back to $x(2)$; \ldots; finally, follow $\gamma_1$ from $x(m)$ back to $x$. Note that $\gamma_n$ visits $x$ and each $x(i_1i_2\ldots i_k), k \leq n$. Also

$$\text{length } (\gamma_{n+1}) = \text{length } (\gamma_1) + \sum_{i=1}^{m} r_i \text{ length } (\gamma_n)$$

$$\implies \text{length } (\gamma_n) = \left( \sum_{k=0}^{n} \left( \sum_{i=1}^{m} r_i \right)^k \right) \text{ length } (\gamma_1) \leq (1 - \Sigma r_i)^{-1}(m + 1)d \quad \forall n.$$

Assume that each $\gamma_n$ is parametrized by arclength. It is easy to see that as $n \rightarrow \infty$ the paths $\gamma_n$ converge uniformly to a path $\gamma$ with arclength $\leq (1 - \Sigma r_i)^{-1}(m + 1)d$. (In fact, all we need is a subsequence $\gamma_{n_k}$ converging uniformly to a path $\gamma$, and this follows from the Arzela-Ascoli theorem, since $\gamma_n, n \geq 1$, are uniformly equicontinuous.) The path $\gamma$ must visit every point $x(i_1i_2\ldots i_k)$, since $\gamma_n$ does for every $n \geq k$; consequently, $\gamma$ visits every point of $K$ since $\{x(i_1i_2\ldots i_k) : i_1 \ldots i_k \text{ any finite sequence}\}$ is dense in $K$ (because $x(i_1i_2\ldots i_k) \in K_{i_1, i_2, \ldots, i_k}$).

Proof of Th. 1: By hypothesis, the support of $\nu$ is a closed subset of $K$. Define $C = \inf_{\gamma \in \Gamma} \text{length } \gamma$, where $\Gamma$ is the set of all continuous paths which visit every point of support (\nu). By Lemma 2, $C < \infty$.

Let $\zeta_n$ be a minimal path through $X_1,\ldots, X_n$. Then $L_n = \text{length } \zeta_n \leq C$, since $X_1, X_2, \ldots, X_n \in \text{support } (\nu)$. Assume that $\zeta_n$ is parametrized by arc length; then $\{\zeta_n\}_{n \geq 1}$ is a uniformly equicontinuous family of functions, so there exists a subsequence $n_k \uparrow \infty$ such that $\zeta_{n_k}$ converges uniformly to a continuous path $\zeta$. With probability one, $\zeta$ visits every point of support (\nu), because $\zeta$ visits $X_1, X_2, \ldots$ and $\{X_1, X_2, \ldots\}$ is dense in support (\nu).
Hence, length $\zeta \geq C$ a.s. But since $\zeta_{n_k}$ converges uniformly to $\zeta$ and $\zeta_{n_k}$ is parametrized by arc length, length $\zeta_{n_k} \to$ length $\zeta$. Finally, since length $\zeta_n$ is nondecreasing in $n$, we must have $L_n \uparrow C$. \hfill \Box

5. The Circle Freeway Lemma

The proofs of Th. 2 and Th. 3 use a "Poissonization" argument. Let $X_1, X_2, \ldots$ be i.i.d. with distribution $\nu = \nu_{p_1, \ldots, p_m}$ (cf. Prop. 1), and let $L_n$ be the length of the shortest path through $X_1, \ldots, X_n$. Let $N(t), t \geq 0$, be a rate 1 Poisson process independent of $X_1, X_2, \ldots$. Then $N(t)/t \to 1$ a.s. as $t \to \infty$. Since $L_n$ is nondecreasing in $n$, we can easily recover the asymptotic behavior of $L_n$ as $n \to \infty$ from the asymptotic behavior of $L_{N(t)}$ as $t \to \infty$. The advantage of introducing $N(t)$ is that if $U_1, U_2, \ldots, U_m$ are pairwise disjoint then the random sets $U_i \cap \{X_1, \ldots, X_{N(t)}\}, i = 1, 2, \ldots, m$, are independent.

For each $t > 0$ and each finite sequence $i_1 i_2 \ldots i_n$ of indices from $\{1, 2, \ldots, m\}$, define $\lambda(t; i_1 i_2 \ldots i_n)$ to be the length of the shortest path through $\{X_1, X_2, \ldots, X_{N(t)}\} \cap K_{i_1 i_2 \ldots i_n}$, and define $\lambda(t) = L_{N(t)}$.

Lemma 3: Let $i_1^{(s)}, i_2^{(s)} \ldots i_n^{(s)}$, $s = 1, 2, \ldots, q$ be finite sequences of indices from $\{1, 2, \ldots, m\}$ such that no two sequences satisfy $i_j^{(s)} = i_j^{(s')} \text{ for } j = 1, 2, \ldots, (n(s) \wedge n(s'))$. Then for each $t > 0$ the random variables $\lambda(t; i_1^{(s)} i_2^{(s)} \ldots i_n^{(s)})$, $s = 1, \ldots, q$ are independent. Moreover, for any $t > 0$ and any sequence $i_1 i_2 \ldots i_n$, the random variable $\lambda(t; i_1 i_2 \ldots i_n)$ has the same distribution as $r_{i_1} r_{i_2} \ldots r_{i_n} \lambda(p_{i_1} p_{i_2} \ldots p_{i_n} t)$.

Proof: This is a routine consequence of (3.4) and the self-similarity of $K$ and $\nu$. \hfill \Box

Lemma 4: There exists a constant $C < \infty$ such that for every $t > 0$

$$\sum_{i=1}^{m} \lambda(t; i) - C \leq \lambda(t) \leq \sum_{i=1}^{m} \lambda(t; i) + C.$$  

(5.1)

Proof: Construct a path through $X_1, \ldots, X_{N(t)}$ as follows. First construct the shortest path through $\{X_1, \ldots, X_{N(t)}\} \cap K_i$ for each $i = 1, 2, \ldots, m$, then join these $m$ paths together in sequence. The length of this path is $\leq \sum_{i=1}^{m} \lambda(t; i) + C$ for some $C < \infty$ independent of $t$ (by the compactness of $K_1, \ldots, K_m$). This proves the upper bound.

Next, consider the shortest path $\gamma$ through $X_1, X_2, \ldots, X_{N(t)}$. The initial and terminal points of $\gamma$ lie in $V_i, V_{i'}$, say; extend $\gamma$ from its endpoints to a path $\overline{\gamma}$ whose initial and terminal points lie on $\partial V_i, \partial V_{i'}$ for some $i, i'$. This may be done in such a way that length $\overline{\gamma} \leq$ length $\gamma + 2 \max_{1 \leq i \leq m} \text{diameter } (V_i)$. Now for any $i \in \{1, 2, \ldots, m\}$, the intersection of $\overline{\gamma}$ with $\overline{V}_i$ consists of a finite collection of paths $\gamma_{j, j = 1, 2, \ldots, q}$ in $\overline{V}_i$, each having its endpoints on $\partial V_i$. Since $\partial V_i$ is a simple, closed, rectifiable curve,
Lemma 5 below implies that $\gamma_1, \gamma_2, \ldots, \gamma_q$ may be joined together to form a single path $\gamma^{(i)}$ satisfying length $(\gamma^{(i)}) \leq \sum_{j=1}^{q} \text{length} (\gamma_j) + 2 \text{length} (\partial V_i)$. This path $\gamma^{(i)}$ goes through each point in $\{X_1, X_2, \ldots, X_{N(t)}\} \cap K_i$, because $K_i \subset \overline{V}_i$ and the original path $\gamma$ goes through $X_1, \ldots, X_{N(t)}$. Thus

$$\sum_{i=1}^{m} \lambda(t; i) \leq \lambda(t) + 2 \max_{1 \leq i \leq m} \text{diam} (V_i) + 2 \sum_{i=1}^{m} \text{length} (\partial V_i).$$

\[ \square \]

**Corollary 1:** If $C < \infty$ is as in Lemma 4 then for every $t < \infty$ and every finite sequence $i_1i_2\ldots i_n$ of indices,

\[ \sum_{i=1}^{m} \lambda(t; i_1i_2\ldots i_n) - Cr_i r_{i_2} \ldots r_{i_n} \leq \lambda(t; i_1i_2\ldots i_n) \leq \sum_{i=1}^{m} \lambda(t; i_1i_2\ldots i_n) + Cr_i r_{i_2} \ldots r_{i_n} \]

**Proof:** The self-similarity of $K$ and $\nu$ imply that the joint distribution of $\lambda(t; i_1i_2\ldots i_n), \lambda(t; i_1i_2\ldots i_{n+1}), \ldots, \lambda(t; i_1i_2\ldots i_{n+m})$ is the same as that of $\left( \prod_{j=1}^{n} r_{i_j} \right) \lambda(t), \left( \prod_{j=1}^{n} r_{i_j} \right) \lambda(t; 1), \ldots, \left( \prod_{j=1}^{n} r_{i_j} \right) \lambda(t; m)$. Thus, (5.2) follows from (5.1). \[ \square \]

**Lemma 5 (Circle Freeway Lemma):** Let $\gamma_1, \gamma_2, \ldots, \gamma_q$ be piecewise smooth paths in a domain $U$ whose boundary $\partial U$ is a simple, closed, rectifiable curve. Assume that for each $i$ the endpoints of $\gamma_i$ are on $\partial U$. Then there is a path $\gamma$ in $(\partial U) \cup \bigcup_{i=1}^{q} \gamma_i$ that goes through every point of $\bigcup_{i=1}^{q} \gamma_i$ such that

\[ \text{length} (\gamma) \leq \sum_{i=1}^{q} \text{length} (\gamma_i) + 2 \text{length} (\partial U). \]

**Proof:** The $2q$ endpoints of $\gamma_1, \ldots, \gamma_q$ all lie on $\partial U$; label these points $Q_1, Q_2, \ldots, Q_{2q}$ so that they appear in clockwise order around $\partial U$. The paths $\gamma_i, \gamma_j$ may meet inside $U$ at “traffic lights” and may coincide on road segments between traffic lights. Define a graph $G$ whose vertices are the traffic lights and the points $Q_1, \ldots, Q_{2q}$ and with edges as follows. For vertices $V$ and $V'$ there is one edge between $V$ and $V'$ for each path $\gamma_i$ that goes through $V$ and $V'$ without going through any other vertex in between. Also, there is one edge between $Q_{2i-1}$ and $Q_{2i}$ and two edges between $Q_{2i}$ and $Q_{2i+1}$ for each $i = 1, 2, \ldots, q$ (here $Q_{2q+1} = Q_1$) in addition to any edges between $Q_i, Q_j$ already in place because of
paths $\gamma_s$ that connect $Q_i, Q_j$ without going through any vertex in between. The graph $G$
is connected and each vertex has even degree. Therefore, by Euler's "Konigsburg bridges" theorem ([2], Ch. 1, Th. 10) there is an Eulerian circuit. The Eulerian circuit determines a path $\gamma$ with the desired property.

(Note: It is possible that some of the points $Q_i$ may coincide. If this is the case, they should be listed according to multiplicity, and in the construction of the graph $G$ each should be counted as a separate vertex. Thus, some of the edges in $G$ may correspond to paths of length zero in $(\partial U) \cup \left( \bigcup_{i=1}^q \gamma_i \right)$. This does not affect the validity of (5.3).) \hfill \Box

6. Expectation of $\lambda(t)$ when $\delta > 1$

Recall that the similarity exponent $\theta$ of $\nu = \nu_{p_1, \ldots, p_m}$ is the unique real number such that $\sum_{i=1}^m r_i p_i^{\theta} = 1$. Observe that $\theta > 0$ iff $\Sigma r_i > 1$ iff $\delta > 1$.

Lemma 6: If $\theta > 0$ then $E\lambda(t) \uparrow \infty$ as $t \to \infty$.

Proof: $E\lambda(t)$ is clearly monotone in $t$. Let $i^{(s)} = i_1^{(s)} i_2^{(s)} \ldots i_{n_{(s)}}^{(s)}$, $s = 1, 2, \ldots, m^n$ be all the finite sequences of indices of length $n$ from $\{1, 2, \ldots, m\}$. Then

$$\lim_{t \to \infty} P \{\{X_1, X_2, \ldots, X_{N(t)}\} \cap K_{i^{(s)}} \neq \emptyset \ \forall \ s = 1, 2, \ldots, m^n\} = 1,$$

consequently, it suffices to prove that the length of the shortest path through $K_{i^{(1)}}, \ldots, K_{i^{(m^n)}}$ converges to $\infty$ as $n \to \infty$.

Any path through all the sets $K_{i^{(s)}}, s = 1, 2, \ldots, m^n$ must pass through each set of the subcollection $K_{i^{(s)}}, s = 1, 2, \ldots, m^n - \ell$ in which $i^{(s)}$ has the form $i^{(s)} = i_1^{(s)} i_2^{(s)} \ldots i_{n-\ell}^{(s)} j_1 j_2 \ldots j_{\ell}$, where $i_1, i_2, \ldots, i_{n-\ell}$ are arbitrary and $j_1 j_2 \ldots j_{\ell}$ is the sequence constructed in Lemma 1. By (3.1)-(3.2) and Lemma 1,

$$\text{distance } (K_{i^{(s)}}, K_{i^{(s')}}) \geq r_{i_1} r_{i_2} \ldots r_{i_{n-\ell}} d,$$

where $d = \text{distance } (K_{j_1 \ldots j_{\ell}}, \partial V)$ and $i^{(s)} = i_1 i_2 \ldots i_{n-\ell} j_1 j_2 \ldots j_{\ell}, i^{(s')} = i'_1 i'_2 \ldots i'_{n-\ell} j'_1 \ldots j'_{\ell}$ are distinct sequences. Now any path that goes through all the sets $K_{i^{(s)}}, s = 1, 2, \ldots, m^n$, where $i^{(s)} = i_1 i_2 \ldots i_{n-\ell} j_1 j_2 \ldots j_{\ell}$, must exit each on its way to another, with at most one exception. Therefore, its length must be at least

$$\sum_{i_1 i_2 \ldots i_{n-\ell}} r_{i_1} r_{i_2} \ldots r_{i_{n-\ell}} d - d = \left( \sum_{i=1}^m r_i \right)^{n-\ell} d - d.$$

Since $\Sigma r_i > 1$, this becomes large as $n \to \infty$. \hfill \Box

Proposition 2: Assume that $\theta > 0$. 

10
(a) (Nonarithmetic case) If \( \log p_1, \log p_2, \ldots, \log p_n \) are not contained in any discrete additive subgroup of \( \mathbb{R} \) then there is a constant \( \mu > 0 \) such that

\[
\lim_{t \to \infty} E(\lambda(t)/t^\theta) = \mu.
\]

(b) (Arithmetic case) If \( \log p_1, \log p_2, \ldots, \log p_n \) are contained in \( h\mathbb{Z} \) but not in a proper subgroup of \( h\mathbb{Z} \) then there is a continuous monotone function \( \mu(\beta), 0 \leq \beta \leq h \), such that \( \mu(\beta) > 0 \) and

\[
\lim_{n \to \infty} E(\lambda(\exp\{nh + \beta\})/e^{nh^\theta}) = \mu(\beta)
\]

uniformly for \( 0 \leq \beta \leq h \).

**Proof:** Consider first the nonarithmetic case. Let \( C \) be the constant in the inequalities (5.3)–(5.4). Fix \( t_* \) (large). Then Lemma 4 and Corollary 1 imply that

\[
\Sigma_t \lambda(t; i_1 i_2 \ldots i_n) - \Sigma_t^* C r_{i_1} r_{i_2} \ldots r_{i_n} \\
\leq \lambda(t) \\
\Sigma_t \lambda(t; i_1 i_2 \ldots i_n) + \Sigma_t^* C r_{i_1} r_{i_2} \ldots r_{i_n}
\]

where the sum \( \Sigma_t \) ranges over all finite sequences \( i_1 i_2 \ldots i_n, n \geq 1 \), satisfying

\[
- \sum_{j=1}^{n-1} \log p_{i_j} < \log(t/t_*) \leq - \sum_{j=1}^{n} \log p_{i_j}
\]

and the sum \( \Sigma_t^* \) ranges over all finite sequences \( i_1 i_2 \ldots i_n, n \geq 0 (n = 0 \text{ is the empty sequence}), \) satisfying

\[
- \sum_{j=1}^{n} \log p_{i_j} < \log(t/t_*).
\]

It follows from Lemma 3 that

\[
\Sigma_t r_{i_1} r_{i_2} \ldots r_{i_n} E\lambda(p_{i_1} \ldots p_{i_n} t) - \Sigma_t^* C r_{i_1} r_{i_2} \ldots r_{i_n} \\
\leq E\lambda(t) \\
\leq \Sigma_t r_{i_1} r_{i_2} \ldots r_{i_n} E\lambda(p_{i_1} \ldots p_{i_n} t) + \Sigma_t^* C r_{i_1} r_{i_2} \ldots r_{i_n}.
\]

Define \( G(t) \) and \( H(t) \) by

\[ G(t) = \Sigma_t^* C r_{i_1} r_{i_2} \ldots r_{i_n} \text{ and } \]
\[ H(t) = \Sigma_t r_{i_1} r_{i_2} \ldots r_{i_n} E\lambda(p_{i_1} p_{i_2} \ldots p_{i_n} t) \]

11
For each of these there is a functional equation, which may be obtained by conditioning on the first coordinate $i_1$:

$$G(t) = \sum_{i=1}^{m} r_i G(p_i t) + C \{ t > t_* \}$$

and

$$H(t) = \sum_{i=1}^{m} r_i H(p_i t) + R(t),$$

where

$$R(t) = \sum_{i=1}^{m} r_i E \lambda(p_i t) 1\{ t_* \geq p_i t > p_i t_* \}.$$ 

Each of these may be rewritten as a renewal equation, using the fact that $\Sigma r_i p_i^\theta = 1$:

$$e^{-\theta s} G(e^s) = \sum_{i=1}^{m} (r_i p_i^\theta)(p_i e^{\theta s} G(p_i e^s)) + C e^{-\theta s} 1\{ s > \log t_* \}$$

and

$$e^{-\theta s} H(e^s) = e^{-\theta s} R(e^s) + \sum_{i=1}^{m} (r_i p_i^\theta)(p_i e^{-\theta s} H(p_i e^s)).$$

Therefore, the renewal theorem for $R$ implies that

$$\lim_{t \to \infty} G(t)/t^\theta = C \int_{\log t_*}^\infty e^{-\theta s} ds/\gamma = C_1(t_*)$$

$$\lim_{t \to \infty} H(t)/t^\theta = \int e^{-\theta s} R(e^s) ds/\gamma = C_2(t_*)$$

where $\gamma = \sum_{i=1}^{m} (r_i p_i^\theta)(- \log p_i) > 0$.

Now observe that for any $\epsilon > 0$, if $t_*$ is chosen sufficiently large then $C_1(t_*) < \epsilon C_2(t_*)$. This is because $E \lambda(t) \uparrow \infty$ (Lemma 6) and

$$\frac{C_1(t_*)}{C_2(t_*)} = \frac{C t_*^{-\theta}}{\theta \sum_{i=1}^{m} r_i \int_{t_0}^{t_*} e^{-\theta s} E \lambda(p_i e^s) ds}.$$ 

We have already shown that

$$-G(t) \leq E \lambda(t) - H(t) \leq G(t)$$

Since $\epsilon > 0$ is arbitrary, (6.1) follows.

The arithmetic case is similar – it uses the renewal theorem for $Z$. The details are omitted.

7. **Expectation of $\lambda(t)$ when $\delta = 1$ and $K$ is StronglyDisconnected**
The set $K$ is strongly disconnected if $K_1, K_2, \ldots, K_m$ are pairwise disjoint. When this is the case, the open set $V$ may be chosen so that the compact sets $\bar{V}_1, \bar{V}_2, \ldots, \bar{V}_m$ are pairwise disjoint. Recall that $\delta = 1$ iff $\sum_{i=1}^{m} r_i = 1$. Assume throughout this section that $\delta = 1$ and that $K$ is strongly disconnected.

**Lemma 7:** For $n \geq 1$, let $\beta_n$ be the length of the shortest path that visits each of the $m^n$ sets $\bar{V}_{i_1 i_2 \ldots i_n}$. Then $\lim_{n \to \infty} \beta_n = \infty$.

**Proof:** Let $\gamma_n$ be the shortest path, and let $\gamma_n$ be the closed curve obtained by connecting the endpoints of $\gamma_n$. Then length $(\gamma_n) \leq$ length $(\gamma_n) +$ diameter $(V)$, so to show that $\gamma_n$ is long it suffices to show that $\gamma_n$ is long.

Define $d = \min \{ \text{distance} (\bar{V}_i, \bar{V}_j) \} > 0$. If $i_1 i_2 \ldots i_n$ and $i'_1 i'_2 \ldots i'_n$ are any two distinct sequences of indices from $\{1, 2, \ldots, m\}$ then

$$\text{distance} (\bar{V}_{i_1 i_2 \ldots i_n}, \bar{V}_{i'_1 i'_2 \ldots i'_n}) \geq r_{i_1} r_{i_2} \ldots r_{i_k} d,$$

where $k = \max \{ g : i_j = i'_j \forall 1 \leq j \leq q \}$, by self-similarity and the nesting property $\bar{V} \subset \bar{V}_{i_1} \subset \bar{V}_{i_1 i_2} \subset \ldots$. Let $i^{(1)}, i^{(2)}, \ldots, i^{(m^n)}$ be the distinct sequences of length $n$. Each $\bar{V}_{i^{(s)}}$ has a segment of $\gamma_n$ going out; hence (7.1) implies that

$$\text{length} (\gamma_n) \geq \sum_{i_1, \ldots, i_n=1}^{m} r_{i_1} r_{i_2} \ldots r_{i_{n-1}} d = \left( \sum_{i=1}^{m} r_i \right)^{n-1} md = md.$$

But this is a rather crude estimate, since some $\bar{V}_{i^{(s')}}$ may have an outgoing segment connecting it to some $\bar{V}_{i^{(s')}}$ where $i^{(s')} differs from $i^{(s)}$ in some coordinate before the $n^{th}$. Thus, some of the terms $r_{i_1} r_{i_2} \ldots r_{i_{n-1}} d$ in the above sum may be upgraded to $r_{i_1} r_{i_2} \ldots r_{i_k} d$ for some $k < n - 1$. In fact, for each sequence $i_1 i_2 \ldots i_k, k < n$, there is at least one sequence $i_{k+1} i_{k+2} \ldots i_n$ such that $\bar{V}_{i_1 i_2 \ldots i_n}$ has an outgoing segment to some $\bar{V}_{i'_1 i'_2 \ldots i'_n}$ where $i_1 i_2 \ldots i_k \neq i'_1 i'_2 \ldots i'_k$. For any such sequence $i_1 i_2 \ldots i_n$ the term $r_{i_1} r_{i_2} \ldots r_{i_{n-1}} d$ in the estimate (7.2) may be upgraded to $r_{i_1} r_{i_2} \ldots r_{i_{k-1}} d$. This upgrading may be done one step at a time, first from $n - 1$ to $n - 2$, then $n - 2$ to $n - 3$, etc. At each step the size of the deleted term is no more than $r$ times the upgraded term, where $r = \max(r_1, r_2, \ldots, r_m) < 1$. Thus (7.2) may be improved to

$$\text{length} (\gamma_n) \geq \sum_{k=1}^{n} \left( \sum_{i_1, \ldots, i_k=1}^{m} r_{i_1} r_{i_2} \ldots r_{i_{k-1}} d(1 - r) \right) = \sum_{k=1}^{n} \left( \sum_{i=1}^{m} r_i \right)^{k-1} md(1 - r) = ndm(1 - r).$$

It follows that length $(\gamma_n)$ $\to \infty$ as $n \to \infty$.\[\square\]
Lemma 8: Let \( p_n(t) = P\{X_1, X_2, \ldots, X_{N(t)}\} \cap \overline{V}_{i_1i_2\ldots i_n} = \emptyset \) for some sequence \( i_1i_2\ldots i_n \). Then for any \( t > 0 \) and \( n = 1, 2, \ldots \),

\[
\sum_{i_1i_2\ldots i_n} r_{i_1} r_{i_2} \ldots r_{i_n} E\lambda(tp_{i_1} \ldots p_{i_n}) + \beta_n(1 - p_n(t)) - 2 \text{ length } (\partial V) \\
\leq E\lambda(t) \\
\leq \sum_{i_1i_2\ldots i_n} r_{i_1} r_{i_2} \ldots r_{i_n} E\lambda(tp_{i_1} \ldots p_{i_n}) + \beta_n + 2 \text{ diam } (V)
\]

Proof: This is a refinement of Lemma 4 and may be proved in a similar manner. The difference here is that the sets \( \overline{V}_{i(1)}, \ldots, \overline{V}_{i(m^n)} \), where \( i^{(1)}, \ldots, i^{(m^n)} \) are the distinct sequences of length \( n \), are pairwise disjoint and therefore are separated by positive distances.

Construct the shortest path through \( \{X_1, \ldots, X_{N(t)}\} \cap \overline{V}_{i(s)} \) for each sequence \( i^{(s)} \) of length \( n \), then join these together. This may be done in such a way that the length of the resulting path is \( \leq \sum_{i_1i_2\ldots i_n} \{\lambda(t; i_1 \ldots i_n) + 2 \text{ diam } (\overline{V}_{i_1i_2\ldots i_n})\} + \beta_n \). By Lemma 3, \( E\lambda(t; i_1i_2\ldots i_n) = r_{i_1} r_{i_2} \ldots r_{i_n} E\lambda(tp_{i_1} \ldots p_{i_n}) \). Moreover, \( \sum_{i_1i_2\ldots i_n} \text{ diam } (\overline{V}_{i_1i_2\ldots i_n}) = \sum_{i_1\ldots i_n} r_{i_1} \ldots r_{i_n} \text{ diam } (V) = \text{ diam } (V) \), since \( \Sigma r_i = 1 \). This proves the upper bound.

The proof of the lower bound is virtually the same as the proof of the lower bound in Lemma 4.

\[\square\]

Proposition 3: There exists a constant \( 0 < C < \infty \) depending only on \( K \) (not on \( p_1, p_2, \ldots, p_m \)) such that

\[
(7.4) \quad \lim_{t \to \infty} \frac{E\lambda(t)}{\log t} = C/\sum_{i=1}^{m} r_i \log p_i^{-1}
\]

Proof: Choose \( \epsilon > 0 \) small. By Lemma 7 there is an \( n \geq 1 \) such that \( 2 \text{ length } (\partial V) + 2 \text{ diam } (V) < \epsilon \beta_n \). Fix \( t_* \) so large that \( p_n(t) < \epsilon \) for all \( t \geq pt_* \), where \( p = \min(p_1, \ldots, p_m)^n > 0 \). By Lemma 8, if \( t \geq t_* \) then

\[
\sum_{i_1} E\lambda(tp_{i_1} \ldots p_{i_n}) r_{i_1} \ldots r_{i_n} + (1 - 2\epsilon)\beta_n \sum_{i_1i_2\ldots i_n} r_{i_1} \ldots r_{i_n} \\
\leq E\lambda(t) \\
\leq \sum_{i_1} E\lambda(tp_{i_1} \ldots p_{i_n}) r_{i_1} \ldots r_{i_n} + (1 + \epsilon)\beta_n \sum_{i_1i_2\ldots i_n} r_{i_1} \ldots r_{i_n}
\]

where the sum \( \Sigma_i \) ranges over all sequences \( i_1i_2\ldots i_n, k \geq 1 \), satisfying

\[\sum_{j=1}^{n} \log p_{ij} < \log(t/t_*) \leq \sum_{j=1}^{n} \log p_{ij} \]

and the sum \( \Sigma_i^* \) ranges over all sequences \( i_1i_2\ldots i_n, k \geq 1 \), satisfying

\[-\sum_{j=1}^{n} \log p_{ij} < \log(t/t_*) \].

14
Define \( G(t) \) and \( H(t) \) by
\[
G(t) = \sum_t r_{i_1} r_{i_2} \ldots r_{i_k} \quad \text{and} \\
H(t) = \sum_t r_{i_1} r_{i_2} \ldots r_{i_k} E\lambda(p_{i_1} \ldots p_{i_k} t).
\]

We have shown that
\[
H(t) + (1 - 2\epsilon)\beta_n G(t) \leq E\lambda(t) \leq H(t) + (1 + \epsilon)G(t).
\]

The function \( H(t) \) satisfies a functional equation equivalent to a renewal equation; so the renewal theorem implies that \( H(t) = o(1) \) as \( t \to \infty \). On the other hand, the "elementary renewal theorem" ([3], Th. 5.52) implies that
\[
\lim_{t \to \infty} \frac{G(t)}{\log t} = \frac{1}{n} \sum_{i=1}^{n} \frac{r_i \ldots r_i \log(p_i \ldots p_i)}{\log t}^{-1}
= 1/n \sum_i r_i \log p_i^{-1}.
\]

Therefore
\[
\frac{(1 - 2\epsilon)\beta_n}{n \sum r_i \log p_i^{-1}} \leq \lim_{t \to \infty} \inf \frac{E\lambda(t)}{\log t} \leq \lim_{t \to \infty} \sup \frac{E\lambda(t)}{\log t} \leq \frac{(1 + \epsilon)\beta_n}{n \sum r_i \log p_i^{-1}}.
\]

Since \( \epsilon > 0 \) is arbitrary, (7.4) follows, with \( C = \lim_{n \to \infty} (\beta_n/n) \). \( \square \)

8. Almost Sure Convergence

Theorems 2–3 follow from Props. 2–3 and Lemma 4 by routine arguments. The key is the following.

**Lemma 9:** If \( 0 < \delta < 2 \) then \( \text{var} (\lambda(t)) = o(1) \) as \( t \to \infty \). If \( \delta = 2 \) then \( \text{var} (\lambda(t)) = o(\log t) \) as \( t \to \infty \).

**Note:** In fact, Steele [7] has shown that \( \text{var} (\lambda(t)) \) is \( o(1) \) even when \( \delta = 2 \). This is not needed for the proofs of Th. 2–3, however.

**Proof:** Lemmas 3–4 imply that
\[
\text{var} (\lambda(t)) \leq \sum_{i=1}^{m} r_i^2 \text{var} (\lambda(tp_i)) + C
\]

for a suitable constant \( C < \infty \). If \( 0 < \delta < 2 \) then \( \sum_{i=1}^{m} r_i^2 < 1 \); consequently, iteration of (8.1) gives
\[
\text{var} (\lambda(t)) \leq C \sum_{k=0}^{\infty} \left( \sum_{i=1}^{m} r_i^2 \right)^k < \infty.
\]
If \( \delta = 2 \) then \( \sum_{i=1}^{m} r_i^2 = 1 \). Let \( \p = \max(p_1, \ldots, p_m) < 1 \), and let \( k \) be the smallest integer larger than \( (\log t)/(\log \p^{-1}) \). Iterating (8.1) \( k \) times gives

\[
\text{var} \left( \lambda(t) \right) \leq \sum_{i_1, \ldots, i_k} r_{i_1}^2 r_{i_2}^2 \ldots r_{i_k}^2 \text{var} \left( \lambda(t p_{i_1} \ldots p_{i_k}) \right) + \sum_{j=0}^{k-1} C \sum_{i_1, i_2, \ldots, i_j} r_{i_1}^2 r_{i_2}^2 \ldots r_{i_j}^2
\]

\[
\leq \left( \sum_{i=1}^{m} r_i^2 \right)^k E\lambda(1)^2 + C \sum_{j=0}^{k-1} \left( \sum_{i=1}^{m} r_i^2 \right)^j
\]

\[= E\lambda(1)^2 + Ck; \]

thus \( \text{var} \left( \lambda(t) \right) = o(\log t) \) as \( t \to \infty \).

**Proof of Th. 2:** Consider first the nonarithmetic case. By Lemma 9, \( \text{var} \left( \lambda(t) \right) = o(\log t) \), and by Prop. 2, \( E\lambda(t) \sim \mu t^\theta \) as \( t \to \infty \). Chebychev's inequality implies that for any \( \alpha > 0, \epsilon > 0 \),

\[
P\left\{ \left| \lambda((1+\alpha)^n)/(1+\alpha)^{n\theta} - \mu \right| > \epsilon \right\} \leq (\text{constant}) \epsilon^2 n/(1+\alpha)^{2n\theta};
\]

consequently, the Borel-Cantelli Lemma implies that \( \lambda((1+\alpha)^n)/(1+\alpha)^{n\theta} \to \mu \) a.s. Since \( \alpha > 0 \) is arbitrary and \( \lambda(t) \) is nondecreasing in \( t \) it follows that \( \lambda(t)/t^\theta \to \mu \) a.s. as \( t \to \infty \). Finally, recall that \( \lambda(t) = L_{N(t)} \) where \( N(t) \) is a rate 1 Poisson process; since \( N(t)/t \to 1 \) a.s., we must have

\[L_n/n^\theta \to \mu \quad \text{a.s.} \quad \text{as } n \to \infty.\]

A similar argument applies in the arithmetic case.

**Proof of Th. 3:** By Lemma 8, \( \text{var} \left( \lambda(t) \right) = o(1) \), and by Prop. 3, \( E\lambda(t) \sim \mu \log t \). Consequently, Chebychev's inequality implies that for any \( \epsilon > 0 \)

\[
P\left\{ \left| \lambda(e^n)/n - \mu \right| > \epsilon \right\} \leq (\text{constant}) \epsilon^2 / n^2
\]

from which it follows, by the Borel-Cantelli Lemma, that \( \lambda(e^n)/n \to \mu \) a.s. Since \( \lambda(t) \) is monotone in \( t \), \( \lambda(t)/\log t \to \mu \) a.s. as \( t \to \infty \). Since \( \lambda(t) = L_{N(t)} \) where \( N(t) \) is a standard Poisson process, it follows that \( L_n/\log n \to \mu \) a.s. as \( n \to \infty \).

9. **Periodicity**

The purpose of this section is to give a heuristic argument explaining why the function \( C(t) \) occurring in (2.3) may, in general, be non-constant. For simplicity we shall restrict our attention to the special case in which \( K = C \times C \), where \( C \) is the usual Cantor set,
and \( \nu = \mu_C \times \mu_C \), where \( \mu_C \) is the Cantor measure. The self-similarity structure is as follows: \( V \) = open unit square; \( \psi_1, \psi_2, \psi_3, \psi_4 \) are the affine mappings taking \( V \) onto the four disjoint squares \( \left( \frac{i}{3}, \frac{i+1}{3} \right) \times \left( \frac{j}{3}, \frac{j+1}{3} \right), i, j \in \{0, 2\}; r_1 = r_2 = r_3 = r_4 = 1/3; \) and \( p_1 = p_2 = p_3 = p_4 = 1/4. \)

The arguments of secs. 5–8 suggest that a nearly minimal path through \( X_1, X_2, \ldots, X_n \) may be constructed as follows. Fix a large (an integer), and let \( k = [a(\log n)/(\log 4)] \). For each of the \( 4^k \) sets \( K_{i_1i_2\ldots i_k} \), assemble the minimal path through \( \{X_1, \ldots, X_n\} \cap K_{i_1i_2\ldots i_k} \), then connect these \( 4^k \) paths to form a single path. The connection of the \( 4^k \) subpaths may be done by joining the sets \( K_{i_1i_2\ldots i_k} \) in lexicographic order, e.g., if \( k = 2, 11 \rightarrow 12 \rightarrow 13 \rightarrow \ldots \rightarrow 43 \rightarrow 44 \). It doesn’t matter if the lexicographic ordering is suboptimal, because most of the length of the complete path comes from the \( 4^k \) subpaths, not from the connections (provided the constant \( a \) is large).

NOTE: This is where the cases \( \delta = 1 \) and \( \delta > 1 \) differ. When \( \delta = 1 \) the hierarchical connections make up most of the total length of the path. This is why the periodicity phenomenon does not occur in Theorem 3.

The lengths of the \( 4^k \) paths through the sets \( \{X_1, \ldots, X_n\} \cap K_{i(s)}, s = 1, \ldots, 4^k \), are nearly i.i.d. random variables whose sum is the primary contribution to \( L_n \). These lengths, after rescaling by a factor of \( 3^k \), are determined by the relative configuration of the points \( \{X_1, \ldots, X_n\} \) in the “squares” \( K_{i(s)} \). As \( n \) goes from \( 4^k/a \) to \( 4^{(k+1)/a} \) the distribution of the configuration of points in a square changes (but notice that this distribution is the same at \( 4^k/a \) and \( 4^{(k+1)/a} \)). When \( n \) is somewhat greater than \( 4^k/a \) the probability of a configuration like that shown in Fig. 1 may be high, whereas when \( n \) is slightly less than \( 4^{(k+1)/a} \) the probability of a configuration like that shown in Fig. 2 may be high. Thus, the distribution of the length of the shortest path through \( \{X_1, \ldots, X_n\} \cap K_{i(s)} \) multiplied by \( 3^k \) should be genuinely periodic in \( \log n / \log 4 \), and therefore \( EL_n/n^\theta \) (where \( \theta = 1 - \log 3 / \log 4 \)) should also be genuinely periodic in \( \log n / \log 4 \).

Figure 1 here

Figure 2 here
References


