Resampling Estimators for Generalized L-Statistics
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RESAMPLING ESTIMATORS FOR GENERALIZED L-STATISTICS

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ABSTRACT

A wide class of statistics, the generalized L-statistics, was introduced in Serfling (1984). The generalized L-statistics are asymptotically normal under weak conditions. This report consists of two parts. In part I, we show that the jackknife estimators of the asymptotic variances of generalized L-statistics are consistent. In part II, bootstrap methods for generalized L-statistics are studied. The results provide methods for large sample statistical analysis based on generalized L-statistics.
PART I

JACKKNIFE VARIANCE ESTIMATORS FOR GENERALIZED L-STATISTICS

1. Introduction

The generalized L-statistics was introduced by Serfling (1984). It generalizes the classes of U-statistics and L-statistics and consists of other types of statistics such as trimmed sample variance, trimmed U-statistics and Winsorized U-statistics. See Serfling (1984, 1985) for other examples. Let $X_1,\ldots, X_n$ be independent and identically distributed samples from an unknown population distribution $F$, $m$ be a fixed positive integer and $h(x_1,\ldots, x_m)$ be a given symmetric function. Denote the distribution function of $h(X_1,\ldots, X_m)$ by $H(y)$, i.e.,

$$H(y) = P_F\{ h(X_1,\ldots, X_m) \leq y \}, \quad y \in \mathbb{R}.$$ 

Let

$$H_n(y) = n_{(m)}^{-1} \sum_{c_m} I[h(X_{i_1},\ldots, X_{i_m}) \leq y],$$

where $I[A]$ is the indicator function of the set $A$, $n_{(m)} = n(n-1) \cdots (n-m+1)$ and $\sum_{c_m}$ is the summation taken over the $n_{(m)}$ $m$-tuples $(i_1,\ldots, i_m)$ of distinct elements from $\{1,\ldots, n\}$. We consider a class of smooth generalized L-statistics defined by $T(H_n)$, where $T$ is defined to be

$$T(G) = \int y J[G(y)]dG(y), \quad \text{for any distribution function } G,$$

and $J$ is a function on $[0,1]$ (Serfling, 1984). When $h = x$, $H_n$ reduces to the ordinary empirical distribution and $T(H_n)$ reduces to the ordinary L-statistics. When $J \equiv 1$, $T(H_n)$ is a U-statistic. It was shown in Serfling (1984) that the influence function of $T(H_n)$ is

$$\phi(z) = -m \int [g(y,z) - H(y)]J[H(y)]dy,$$

where

$$g(y,z) = \int \cdots \int I[h(x_1,\ldots, x_{m-1}, z) \leq y].$$
Furthermore, under either condition A or condition B stated below, the generalized L-statistics are asymptotically normal, i.e.,
\[ n^{1/2}(T(H_n) - T(H)) \rightarrow N(0, \sigma^2) \] in distribution,
where \( \sigma^2 = E_F \varphi^2(X_1) \) and is assumed to be finite.

**Condition A.** \( J(t) = 0 \) for \( 0 \leq t < \alpha \) or \( \beta < t \leq 1 \), where \( 0 < \alpha < \beta < 1 \) are constants, \( J \) is continuous on \([\alpha, \beta]\) and \( H \) is continuous.

**Condition B.** \( J \) is continuous on \([0, 1]\) and \( H \) is continuous and satisfies
\[ \int [H(y)(1 - H(y))]^{1/2} \, dy < \infty. \] (1.3)

The statistics with \( J \) functions satisfying condition A are referred to as trimmed statistics in the literature and they usually provide robust estimators (Huber, 1981). Condition (1.3) is equivalent to \( E_F h^2(X_1, \ldots, X_m) < \infty \) if the distribution \( H \) has regularly varying tails (see Feller, 1966, p.268) with a finite exponent. It is implied by \( E_F [h(X_1, \ldots, X_m)]^{2+\delta} < \infty \) for a \( \delta > 0 \).

For various purposes in statistical analysis, we need a consistent estimator of the unknown asymptotic variance \( \sigma^2 \). In this paper, we prove that the estimators of \( \sigma^2 \) obtained by using the jackknife method (Quenouille, 1956; Tukey, 1958) are strongly consistent. For \( i = 1, \ldots, n \), let \( H_{ni} \) be defined as in (1.1) corresponding to \( n - 1 \) samples \( X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n \). The jackknife estimator of \( \sigma^2 \) is defined to be
\[ s_f^2 = (n-1) \sum_{i=1}^{n} [T(H_{ni}) - \overline{T}_n]^2, \] (1.4)
where \( \overline{T}_n = n^{-1} \sum_{i=1}^{n} T(H_{ni}) \).

In Section 2, the strong consistency of \( s_f^2 \) is proved for trimmed generalized L-statistics. The case of untrimmed generalized L-statistics is treated in Section 3. Since U- and L-statistics are special cases of generalized L-statistics, our result includes the existing results in jackknifing U- and L-statistics (see
Arvesen, 1969; Parr and Schucany, 1982) as special cases.

2. Trimmed generalized L-statistics

Let \( U_n \) be a U-statistic (see Hoeffding, 1948) defined to be

\[
U_n = n^{-1}(m) \sum_{c_m} k(X_{i_1}, \ldots, X_{i_m}),
\]

where \( k(x_1, \ldots, x_m) \) is a symmetric kernel. For each \( i \), let

\[
U_{ni} = [(n-1)(m)]^{-1} \sum_{c_m} k(X_{i_1}, \ldots, X_{i_m}),
\]

where \( (n-1)(m) = (n-1) \cdots (n-m) \) and \( \sum_{c_m} \) is the summation taken over the \( (n-1)(m) \) \( m \)-tuples \( (i_1, \ldots, i_m) \) of distinct elements from the integers \( \{ 1, \ldots, i-1, i+1, \ldots, n \} \). The jackknife estimator of the asymptotic variance of \( U_n \) is

\[
s_n^2 = (n-1)\sum_{i=1}^n (U_{ni} - U_n)^2.
\]

**Lemma 1.** Assume that \( E_F k^2(X_1, \ldots, X_m) < \infty \). Then

\[
s_n^2 \rightarrow m^2 \int \phi^2(y) dF(y) \text{ a.s.,}
\]

where \( \phi(y) = E_F [k(X_1, \ldots, X_m) \mid X_1 = y] - E_F k(X_1, \ldots, X_m) \).

This result was proved in Arvesen (1969, Theorem 5), although he stated a weaker version of this result (the weak consistency). The following lemmas are also needed for the proof of the main results.

**Lemma 2.** Let \( H, H_n \) and \( H_{ni} \) be defined as in Section 1. Then

(i) \( \sum_{i=1}^n [H_n(y) - H_{ni}(y)] = 0 \) for any \( y \).

(ii) \( \| H_n - H_{ni} \|_\infty \leq m(n-m)^{-1} \), where \( \| \cdot \|_\infty \) is the sup norm.

**Proof.** Let
\[ A_{ni}(y) = [(n-1)_{(m-1)}]^{-1} \sum_{i_{m-1}} I[h(X_i, X_{i_1}, \ldots, X_{i_{m-1}}) \leq y] \] (2.1)

where \((n-1)_{(m-1)}=\ldots(n-1)\cdot(n-m+1)\) and \(\sum_{i_{m-1}}\) is the summation taken over the \((n-1)_{(m-1)} \) \(m-1\)-tuples \((i_1, \ldots, i_{m-1})\) of distinct elements from the integers \(\{1, \ldots, i-1, i+1, \ldots, n\}\). A straightforward calculation shows that

\[ H_n(y) - H_{ni}(y) = m(n-m)^{-1}[A_{ni}(y) - H_n(y)]. \]

Then (i) follows from \(n^{-1} \sum_{i=1}^{n} A_{ni}(y) = H_n(y)\) and (ii) follows from both \(\|A_{ni}\|_{\infty}\) and \(\|H_n\|_{\infty}\) are bounded by one. □

**Lemma 3.** Assume that \(H\) is continuous. Then

\[ \|H_n - H\|_{\infty} \to 0 \quad a.s. \]

**Proof.** For each \(y\), \(H_n(y)\) is a U-statistic. From theory of U-statistic, \(H_n(y) \to H(y)\) a.s. Let \(D = \{\) all rational numbers in \(R\}\). Then almost surely, \(H_n(y) \to H(y)\) for all \(y \in D\). Let \(\omega = (X_1, X_2, \ldots)\) be fixed such that \(H_n(y) \to H(y)\) for all \(y \in D\). Since \(D\) is a dense subset of \(R\) and \(H_n\) is a distribution function, \(H_n\) converges weakly to \(H\). From the continuity of \(H\), we have \(\|H_n - H\|_{\infty} \to 0\). This completes the proof. □

We now establish the strong consistency of \(s_f^2\) given by (1.4) for trimmed generalized L-statistics.

**Theorem 1.** Assume condition A. Then

\[ s_f^2 \to \sigma^2 \quad a.s. \]

**Proof.** Define

\[ W_{ni}(y) = [H_{ni}(y) - H_n(y)]^{-1} \int_{H_n(y)}^{H_{ni}(y)} J(t) dt - J[H(x)] \] (2.2)

for \(H_{ni}(y) \neq H_n(y)\) and \(W_{ni}(y) = 0\) if \(H_{ni}(y) = H_n(y)\). From Lemma 8.1.1B in Serfling (1980),
\[ T(H_{ni}) - T(H_n) = \int [H_n(y) - H_{ni}(y)] J[H(y)] dy \] 

\[ + \int W_{ni}(y)[H_n(y) - H_{ni}(y)] J[H(y)] dy. \]

Let \( U_{ni} = \int [H_{ni}(y) - H(y)] J[H(y)] dy, \quad U_n = \int [H_n(y) - H(y)] J[H(y)] dy, \)

\( R_{ni} = \int W_{ni}(y)[H_n(y) - H_{ni}(y)] dy \) and \( \overline{R}_n = n^{-1} \sum_{i=1}^{n} R_{ni}. \) From Lemma 2,

\( U_n = n^{-1} \sum_{i=1}^{n} U_{ni}. \) Then

\[ s_j^2 = (n-1) \sum_{i=1}^{n} (U_{ni} - U_n)^2 \]

\[ + (n-1) \sum_{i=1}^{n} (R_{ni} - \overline{R}_n)^2 + 2(n-1) \sum_{i=1}^{n} R_{ni} (U_{ni} - U_n). \]

Note that \( U_n \) is a U-statistic with \( \int [I[h(x_1,\ldots,x_m) \leq y] - H(y)] J[H(y)] dy \) as the kernel. Hence from Lemma 1,

\[ (n-1) \sum_{i=1}^{n} (U_{ni} - U_n)^2 \rightarrow \sigma^2 \text{ a.s.} \]

Using Cauchy-Schwarz inequality, the result follows from

\[ (n-1) \sum_{i=1}^{n} R_{ni}^2 \rightarrow 0 \text{ a.s.} \] (2.5)

Let \( a \) and \( b \) be two constants such that \( H(a) < \alpha \) and \( H(b) > \beta \). From Lemma 2(ii) and Lemma 3, for almost all \( \omega = (X_1, X_2, \ldots) \), there is an \( n_\omega > 0 \) such that

\( H_{ni}(a) < \alpha, H_n(a) < \alpha, H_{ni}(b) > \beta \) and \( H_n(b) > \beta \) hold for all \( i \leq n \) and \( n \geq n_\omega \).

Then \( R_{ni} = \int_a^b W_{ni}(y)[H_n(y) - H_{ni}(y)] dy \), since \( J(t) = 0 \) if \( t < \alpha \) or \( t > \beta \). Thus,

\[ \max_{i \leq n} R_{ni}^2 \leq (b-a)^2 \max_{i \leq n} \left( \| W_{ni} \|_\infty \| H_n - H_{ni} \|_\infty \right) \]

\[ \leq C n^{-2} \max_{i \leq n} \| W_{ni} \|_\infty. \]

where \( C \) is a constant. Since \( J \) is a continuous function on \([\alpha, \beta]\), \( \| H_{ni} - H_n \|_\infty \leq m(n-m)^{-1} \) and \( \| H_n - H \|_\infty \rightarrow 0 \text{ a.s.}, \) \( \max_{i \leq n} \| W_{ni} \|_\infty \rightarrow 0 \text{ a.s.} \)

Hence (2.5) holds and the result follows. \( \square \)

3. Untrimmed generalized L-statistics

For untrimmed generalized L-statistics, we prove the following similar result.
Theorem 2. Assume condition B. Then
\[ s^2_f \rightarrow \sigma^2 \text{ a.s.} \]

Proof. From (2.2)-(2.4), we only need to show (2.5) holds. Using Lemma 2(i), we obtain
\[
(n-1)\sum_{i=1}^{n} r_{ni}^2 = (n-1)m^2(n-m)^{-2}\sum_{i=1}^{n} \left( \int W_{ni}(y)[A_{ni}(y) - H(y)]dy \right)^2
\leq Cn^{-1} \sum_{i=1}^{n} \left( \int |A_{ni}(y) - H(y)|dy \right)^2 \max_{i \leq n} ||W_{ni}||_{\infty},
\]
where \( C \) is a constant. From the proof of Theorem 1, \( \max_{i \leq n} ||W_{ni}||_{\infty} \rightarrow 0 \text{ a.s.} \)

Then (2.5) follows from
\[
n^{-1} \sum_{i=1}^{n} \left( \int |A_{ni}(y) - H_n(y)|dy \right)^2 = O(1) \text{ a.s.} \tag{3.1}
\]

Let \( \xi_n = n^{-1} \sum_{i=1}^{n} \left( \int |A_{ni}(y) - H(y)|dy \right)^2 \). Using the notation in (2.1), we have
\[
\xi_n \leq n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} \left( \int |I[h(X_{i}, X_{j})] - H(y)|dy \right)^2
\leq n^{-1} \sum_{i=1}^{n} \left( \int |I[h(X_{i}) - H(y)]|dy \right)^2,
\] \( \tag{3.2} \)
which is a U-statistic with a kernel \( \{ \int |I[h(x_1, \ldots, x_m) - H(y)]|dy \}^2 \). Under condition (1.3),
\[
E_F \left( \int |I[h(X_1, \ldots, X_m) - H(y)]|dy \right)^2 \tag{3.3}
\]
\[
= \int E_F \left( \left[ \int |I[h(X_1, \ldots, X_m) - H(y)]|dy \right]^2 \right) \leq \left( \int [H(y)(1-H(y))]^hdy \right)^2 < \infty.
\]

From the almost sure convergence of U-statistics, the quantity in (3.2) converges almost surely to the quantity in (3.3). Hence \( \xi_n = O(1) \text{ a.s.} \). Similarly,
\[
\int |H_n(y) - H(y)|dy \text{ is bounded by}
\]
\[
n^{-1} \sum_{i=1}^{n} \int |I[h(X_{i}) - H(y)]|dy,
\]
which converges almost surely to
\[
E_F \int |I[(X_1, \ldots, X_m) - H(y)]|dy < \infty
\]
under condition (1.3). Then (3.1) follows from
\[
n^{-1} \sum_{i=1}^{n} \left( \int |A_{ni}(y) - H_n(y)|dy \right)^2 \leq 2 \xi_n + 2 \left( \int |H_n(y) - H(y)|dy \right)^2.
\]
This completes the proof. □

4. Remarks

A different type of generalized L-statistic (Serfling, 1984) is $T(K_n)$, where $T$ is given by (1.2) and

$$K_n(y) = n^{-m} \sum_{i_1=1}^{n} \cdots \sum_{i_m=1}^{n} \mathbb{I}[h(X_{i_1}, \ldots, X_{i_m}) \leq y].$$

$T(H_n)$ and $T(K_n)$ are closely related and have the same limiting distribution. Note that $K_n(y)$ is a V-statistic. Consistency of jackknife estimators for V-statistics can be established using similar techniques in treating jackknife estimators for U-statistics (e.g., Sen, 1977). Therefore, our results in the previous sections can be extended to the statistics $T(K_n)$ with some modifications.
PART II

BOOTSTRAPPING FOR GENERALIZED L-STATISTICS

1. Introduction

Let $X_1, \ldots, X_n$ be independent and identically distributed (i.i.d.) samples from an unknown population distribution $F$ and $T_n = T_n(X_1, \ldots, X_n)$ be a statistic. The bootstrap (Efron, 1979) is a useful nonparametric method for statistical analysis based on $T_n$. For example, the bootstrap can be used to approximate the sampling distribution of a function $L_n = L_n(T_F, T_n)$ and its other characteristics for various purposes in statistical inferences for $T_F$, where $T_F$ depends on $F$ and is an unknown parameter of interest.

Let $X^*_1, \ldots, X^*_n$ be i.i.d. samples drawn from the empirical distribution $F_n(x) = n^{-1} \sum_{i=1}^{n} I[X_i \leq x]$, where $I[A]$ is the indicator function of the set $A$. $X^*_i$ are called bootstrap samples. A bootstrap analog for $T_n$ is $T^*_n = T_n(X^*_1, \ldots, X^*_n)$. The sampling distribution of $L_n$, $P_F\{L_n(T_F, T_n) \leq t\}$, is approximated by the bootstrap estimate $P_*\{L_n(T_n, T^*_n) \leq t\}$, where $P_*$ is the probability corresponding to the bootstrap sampling.

In many situations $L_n$ is $n^{1/2}(T_n - T_F)$ and it can be approximated by an average of i.i.d. random variables, i.e.,

$$T_n - T_F = n^{-1} \sum_{i=1}^{n} \phi(X_i) + R_n,$$

where $\phi$ is a function depending on $F$ and $T_n$ and satisfies $E_F \phi(X_i) = 0$ and $0 < E_F \phi^2(X_i) < \infty$. Note that $n^{-1} \sum_{i=1}^{n} \phi(X_i) = O_p(n^{-1/2})$. Hence usually $R_n = o_p(n^{-1/2})$. More generally, we have

$$T_n - T_F = U_n + R_n,$$

where $U_n = U_n(X_1, \ldots, X_n)$ is a U-statistic (see Hoeffding, 1948) satisfying $E_F U_n = 0$ and $R_n = o_p(n^{-1/2})$. Serfling (1984) gives a wide class of statistics,
the generalized L-statistics, which have property (1.2). More details for the
generalized L-statistics is given in the next section.

A bootstrap analog of (1.2) is

$$T_n^* - T_n = U_n^* - U_n + R_n^*,$$  \hspace{1cm} (1.3)

where $U_n^* = U_n(X_1^*,...,X_n^*)$ and $R_n^*$ satisfies

$$R_n^* = o_p(n^{-1/2}).$$  \hspace{1cm} (1.4)

Note that the $o_p$ in (1.4) is with respect to the unconditional probability $P$
defined by $P\{A\} = E_F P^*\{A\}$ for any measurable set $A$. Equation (1.3) can be
called a bootstrap representation for the bootstrap statistic $T_n^* - T_n$. A direct
consequence of (1.3)-(1.4) is that the bootstrap estimator of the sampling distribu-
tion $P_F\{ n^{1/2}(T_n - T_F) \leq t \}$ is weakly consistent, i.e.,

$$\sup_t \left| P^*\{ n^{1/2}(T_n^* - T_n) \leq t \} - P_F\{ n^{1/2}(T_n - T_F) \leq t \} \right| = o_p(1).$$  \hspace{1cm} (1.5)

This follows from (1.4) and a well established bootstrap theory for U-statistics
(see Bickel and Freedman, 1981).

For several classes of statistics such as (ordinary) L-statistics and different-
tiable statistical functionals, (1.1) holds and the bootstrap representation holds
with $U_n^* = n^{-1} \sum_{i=1}^n \phi(X_i^*)$ (see Babu and Singh, 1984; Gill, 1987). The purpose
of this paper is to show the bootstrap representation (1.3) holds for a wide class
of statistics, the generalized L-statistics. The result includes that for ordinary
L-statistics since $n^{-1} \sum_{i=1}^n \phi(X_i)$ is a special case of U-statistics.

2. Bootstrap representations

Let $h(x_1,...,x_m)$ be a symmetric function on $\mathbb{R}^m$ and $H_F(x)$ be the distri-
bution function of $h(X_1,...,X_m)$, i.e.,

$$H_F(x) = P_F\{ h(X_1,...,X_m) \leq x \}, \quad x \in \mathbb{R}.$$
An empirical version of $H_F(x)$ is

$$H_n(x) = \binom{n}{m}^{-1} \sum_c [h(X_{i_1}, \ldots, X_{i_m}) \leq x],$$

(2.1)

where $\sum_c$ is the summation taken over all combinations of $m$ integers $(i_1, \ldots, i_m)$ chosen from the integers $1, \ldots, n$. Note that $H_n(x)$ is a U-statistics. Let $J$ be a function defined on the interval $[0,1]$, $G$ be a distribution function and

$$T(G) = \int x J[G(x)] dG(x).$$

A class of generalized L-statistics is defined to be $T_n = T(H_n)$ (Serfling, 1984). The corresponding $T_F$ is $T(H_F)$. Examples of generalized L-statistics include U-statistics, (ordinary) L-statistics, trimmed variances, trimmed U-statistics and Winsorized U-statistics (see more examples in Serfling, 1984).

It was shown in Serfling (1984) that $T_n$ satisfies (1.2) with $R_n = o_p(n^{-1/2})$ and

$$U_n = \int [H_F(x) - H_n(x)] J[H_F(x)] dx$$

(2.2)

under the following condition.

**Condition A.** (1) The functions $J$ and $H_F$ are continuous.

(2) The distribution $H_F$ satisfies $\int [H_F(x)(1-H_F(x))]^{1/2} dx < \infty$.

For any integers $1 \leq i_1 \leq i_2 \leq \cdots \leq i_m \leq n$, let $H^{i_1, \ldots, i_m}_F$ be the distribution of $h(X_{i_1}, \ldots, X_{i_m})$. To establish the bootstrap representation (1.3)-(1.4), we need to assume

**Condition B.** $\int [H^{i_1, \ldots, i_m}_F(x)(1-H^{i_1, \ldots, i_m}_F(x))]^{1/2} dx < \infty$ for any integers $i_1 \leq \cdots \leq i_m$.

Note that for a random variable $Y$ with distribution $G$, the condition $\int [G(x)(1-G(x))]^{1/2} dx < \infty$ is almost equivalent to the condition $E_G Y^2 < \infty$ (see Serfling, 1980, p.276) and is implied by $E_G |Y|^{2+\delta} < \infty$ for a $\delta > 0$. Hence condition B is almost the same as $E_F h^2(X_{i_1}, \ldots, X_{i_m}) < \infty$ and implied by
$E_F | h(X_{i_1}, \ldots, X_{i_m}) |^{2+\delta} < \infty$ for any integers $i_1 \leq \cdots \leq i_m$.

**Lemma 1.** Let $H_n^*$ be the bootstrap analog of $H_n$, i.e., $H_n^*$ is defined by (2.1) with $X_{ij}$ replaced by the bootstrap samples $X_{ij}^*$. If $H_F$ is continuous, then

$$\|H_n^* - H_F\| \to 0 \text{ a.s. and } \|H_n^* - H_n\| \to 0 \text{ a.s.},$$

where $\| \|$ is the sup norm.

**Proof.** For any fixed $x$, since $H_n(x)$ is a bounded U-statistic and $H_n^*(x)$ is its bootstrap analog, $H_n^*(x) \to H_F(x)$ a.s. (Athreya et al., 1984). Then almost surely, $H_n^*(x) \to H_F(x)$ for all rational $x$, which implies $H_n^*$ converges weakly to $H_F$ a.s. since all rational numbers form a dense set in $\mathbb{R}$ and $H_n^*$ is a distribution function. Then $\|H_n^* - H_F\| \to 0$ a.s. follows from the continuity of $H_F$. A similar argument shows that $\|H_n - H_F\| \to 0$ a.s. Hence the results hold. \(\square\)

**Theorem 1.** Assume conditions A and B. For the generalized L-statistics $T_n = T(H_n)$, the bootstrap representation (1.3)-(1.4) holds with $U_n$ given by (2.2).

**Proof.** Let $W_n^*(x) = M[H_n(x), H_n^*(x)] - J[H_F(x)]$ if $H_n(x) \neq H_n^*(x)$ and $= 0$ if $H_n(x) = H_n^*(x)$, where $M(s, t) = \int_s^t J(u)du / (t-s)$. Then from Lemma 8.1.1B of Serfling (1980),

$$T_n^* - T_n = U_n^* - U_n + \int W_n^*(x)[H_n(x) - H_n^*(x)]dx,$$

where $U_n^*$ is the bootstrap analog of $U_n$ given by (2.2) with $H_n$ replaced by $H_n^*$. From Lemma 1 and the continuity of $J$, $\|W_n^*\| \to 0$ a.s. It remains to show that

$$\int |H_n^*(x) - H_n(x)| dx = O_p(n^{-h}). \quad (2.3)$$

Let $E_*$ and $V_*$ be the expectation and variance taken under the bootstrap probability $P_*$, respectively. Since $E_*[H_n^*(x)] = K_n(x)$, where
\[ K_n(x) = n^{-m} \sum_{i=1}^{n} \cdots \sum_{i_m=1}^{n} I[h(X_{i_1}, \ldots, X_{i_m}) \leq x], \quad (2.4) \]

we have
\[
E_F E_*[H_n^*(x) - H_n(x)] \leq \{E_F V_*[H_n^*(x)] + E_F [K_n(x) - H_n(x)]^2\}^{1/2}.
\]

Hence (2.3) follows from
\[
\int (E_F [K_n(x) - H_n(x)]^2)^{1/2} \, dx = O(n^{-1/2}) \quad (2.5)
\]

and
\[
\int (E_F V_*[H_n^*(x)])^{1/2} \, dx = O(n^{-1/2}). \quad (2.6)
\]

Let \( Z_n(x) \) be the average of all terms \( I[h(X_{i_1}, \ldots, X_{i_m}) \leq x] \) with at least one equality \( i_j = i_l, j \neq l \). From Serfling (1980, p.206),
\[
H_n(x) - K_n(x) = [1-n_{(m)}/n^m][H_n(x) - Z_n(x)], \quad (2.7)
\]

where \( n_{(m)} = n(n-1) \cdots (n-m+1) \). Then
\[
E_F [H_n(x) - K_n(x)]^2 \leq C n^{-2} \{E_F [H_n(x) - H_F(x)]^2
\]
\[
+ E_F [Z_n(x) - Z_F(x)]^2 + [Z_F(x) - H_F(x)]^2\},
\]

where \( C \) is a constant and \( Z_F(x) = E_F [Z_n(x)] \). Then (2.5) follows from condition B. Since for given \( X_1, \ldots, X_n, H_n^*(x) \) is a U-statistic, we have
\[
V_*[H_n^*(x)] \leq mn^{-1}K_n(x)[1-K_n(x)]
\]
(see Serfling, 1980, p.183). Then (2.6) follows from
\[
A_n = \int (E_F [K_n(x)(1-K_n(x))])^{1/2} \, dx = O(1).
\]

Note that \( A_n \) is bounded by
\[
\int_{-\infty}^{0} (E_F[K_n(x)])^{1/2} \, dx + \int_{0}^{\infty} (E_F[1-K_n(x)])^{1/2} \, dx.
\]

From (2.7),
\[
E_F[K_n(x)] = [n_{(m)}/n^m]H_F(x) + [1-n_{(m)}/n^m]Z_F(x).
\]

Hence \( A_n \) is bounded by
\[
\int_{-\infty}^{0} [H_F(x)]^{1/2} \, dx + \int_{0}^{\infty} [Z_F(x)]^{1/2} \, dx + \int_{0}^{\infty} [1-H_F(x)]^{1/2} \, dx + \int_{0}^{\infty} [1-Z_F(x)]^{1/2} \, dx.
\]
which is finite under condition B. This completes the proof. □

If the function $J$ is more smooth (condition C), then we can obtain a stronger result than (1.4) under less requirement on the moment of $h(X_{i_1}, \ldots, X_{i_m})$.

**Condition C.** $J$ is Lipschitz continuous of order $\delta$ ($0 < \delta \leq 1$), i.e., there is a constant $C > 0$ such that $|J(t) - J(s)| \leq C|s - t|^\delta$ for any $s, t \in [0, 1]$, and

$$
\int [H_{i_1, \ldots, i_m}(x)(1 - H_{i_1, \ldots, i_m}(x))]^{(1+\delta)^2} dx < \infty
$$

for any integers $i_1, \ldots, i_m$.

**Theorem 2.** Assume condition C. Then (1.3) holds with

$$R^*_n = O_p(n^{-(1+\delta)/2}).$$

**Proof.** Using the same notation as in the proof of Theorem 1, we have

$$|W^*_n(x)| \leq C[|H^*_n(x) - H_n(x)|^{\delta} + |H_n(x) - H_F(x)|^{\delta}]$$

by the Lipschitz continuity of $J$. Then

$$
|R^*_n| \leq C \int |H^*_n(x) - H_n(x)|^{1+\delta} dx
+ \int |H_n(x) - H_F(x)|^{\delta} |H^*_n(x) - H_n(x)|^{\delta} dx.
$$

(2.8)

Since $E_F E_* |H^*_n(x) - H_n(x)|^{1+\delta} \leq \{E_F E_* [H^*_n(x) - H_n(x)]^2\}^{(1+\delta)/2}$, the first integral on the right hand side of (2.8) can be shown to be $O_p(n^{-(1+\delta)/2})$ by using the same argument as in the proof of Theorem 1. Note that

$$
E_F E_* |H_n(x) - H_F(x)|^{\delta} |H^*_n(x) - H_n(x)|
= E_F |H_n(x) - H_F(x)|^{\delta} E_* |H^*_n(x) - H_n(x)|
\leq E_F |H_n(x) - H_F(x)|^{\delta} E_* [H^*_n(x) - H_n(x)]^2
\leq [E_F |H_n(x) - H_F(x)|^{2\delta}]^{1/2} \{E_F E_* [H^*_n(x) - H_n(x)]^2\}^{1/2}
$$

and

$$
[E_F |H_n(x) - H_F(x)|^{2\delta}]^{1/2} \leq (m/n)^{\delta/2} [H_F(x)(1 - H_F(x))]^{\delta/2}.
$$

Using the same argument as in the proof of Theorem 1, we have

$$
\int [H_F(x)(1 - H_F(x))]^{\delta/2} \{E_F E_* [H^*_n(x) - H_n(x)]^2\}^{1/2} dx = O(n^{-(1+\delta)/2})
$$
under condition C. Hence the result follows. □

3. Complements

(1) From (2.1), \( U_n \) defined in (2.2) is a U-statistic with a kernel
\[
k(x_1, \ldots, x_m) = \int \{ H_F(x) - I[ h(x_1, \ldots, x_m) \leq x ] \} J[H_F(x)] dx.
\]
Under condition B, \( E_F k^2(X_{i_1}, \ldots, X_{i_m}) < \infty \) for any integers \( i_1 \leq \cdots \leq i_m \) (see Serfling, 1980, Lemma 8.2.5A). Hence (1.5) holds with \( T_n = U_n \) and \( T_n^* = U_n^* \) (see Bickel and Freedman, 1981). Then Theorem 1 or 2 implies that (1.5) holds for the generalized L-statistics \( T_n = T(H_n) \) satisfying condition A and either condition B or condition C.

(2) Under condition A, Serfling (1984) showed that the distribution of \( n^{1/2}(T_n - T_F) \) converges weakly to \( N(0, \sigma^2) \), where \( \sigma^2 \) is given in (3.3) of Serfling (1984) and is generally unknown. In statistical analysis, we often need a consistent estimator of the asymptotic standard deviation \( \sigma \). Let \( Q_n \) and \( q \) be the interquartile ranges of \( P \{ n^{1/2}(T_n^* - T_n) \leq t \} \) and \( N(0,1) \), respectively. Then from (1.5), \( Q_n/q - \sigma = o_P(1) \).

(3) Serfling (1984) introduced another type of generalized L-statistics \( T(K_n) \), where \( K_n \) is defined in (2.4). With some minor changes in the proofs of Theorems 1 and 2, we can establish the bootstrap representation (1.3)-(1.4) for \( T(K_n) \) with \( U_n \) and \( U_n^* \) replaced by
\[
V_n = \int \{ H_F(x) - K_n(x) \} J[H_F(x)] dx
\]
and the bootstrap analog \( V_n^* \), respectively. Note that \( V_n \) is a V-statistic. Since V-statistics are closely related to U-statistics, result (1.5) can be extended to \( T_n = T(K_n) \) in a straightforward manner.
References


