CONSISTENCY OF JACKKNIFE ESTIMATORS OF THE
VARIANCES OF SAMPLE QUANTILES

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ABSTRACT

Let \( \sigma^2 \) be the asymptotic variance of the sample \( p \)-quantile \( (0 < p < 1) \). Consistency of the delete-\( d \) jackknife estimators of \( \sigma^2 \) with \( d \) being a fraction of \( n \) is proved under very weak conditions. Some other results, such as the asymptotic orders of the moments of the jackknife histograms and an analog of the generalized Helly's theorem, are also established.
1. INTRODUCTION

Let \( F \) be a distribution function defined on \( \mathbb{R} \). The \( p \)-quantile of \( F \) is defined to be \( \theta = F^{-1}(p) = \inf\{t: F(t) \geq p\} \), \( 0 < p < 1 \). Throughout the paper, we assume that

\[
F \text{ is continuously differentiable in a neighborhood of } \theta \text{ and } F'(\theta) > 0.
\]

Let \( \{X_n\} \) be i.i.d. samples from \( F \) and \( F_n(x) = n^{-1} \sum_{i=1}^{n} I(X_i \leq x) \) be the empirical distribution function, where \( I(A) \) is the indicator function of the set \( A \). An estimator of \( \theta \) is the sample \( p \)-quantile \( \hat{\theta} = F_n^{-1}(p) \). Under (1.1), Ghosh (1971) proved that

\[
\hat{\theta} = \theta + \frac{1}{n} \sum_{i=1}^{n} \phi_n(X_i)R_n,
\]

where \( \phi(x) = [p - I(x \leq \theta)]/F'(\theta) \) and \( R_n = o_p(n^{-1/2}) \). From (1.2) and the central limit theorem,

\[
n^{1/2}(\hat{\theta} - \theta) \rightarrow N(0, \sigma^2) \text{ in distribution},
\]

where

\[
\sigma^2 = p(1-p)/[F'(\theta)]^2.
\]

As \( \sigma^2 \) is not known, making further inferences, such as setting confidence interval for \( \theta \), requires a (weak) consistent estimator of \( \sigma^2 \).

The jackknife method introduced by Quenouille (1956) and Tukey (1958) provides a very convenient way of estimating the accuracy of a point estimator. For the sample \( p \)-quantile, however, the traditional delete-1 jackknife estimator of \( \sigma^2 \) is known to be inconsistent. Shao and Wu (1986) studied the general delete-\( d \) jackknife estimator \( \hat{\sigma}^2_{J(d)} \) (defined in (2.1)) and proved its consistency.
For the sample \( p \)-quantile \( \hat{\theta} \), an application of their general result shows that if \( d \) diverges to infinity at the same rate as \( n \),

\[
\hat{\sigma}^2_{J(d)} \rightarrow_p \sigma^2
\]  

(1.4)

under the conditions (1.1) and

\[
\lim_{n \to \infty} n \text{Var} \hat{\theta} = \sigma^2,
\]

(1.5)

where \( \rightarrow_p \) denotes convergence in probability.

However, the justification of (1.5) may not be trivial in many situations. Usually it requires some moment conditions and some further smoothness conditions on \( F \) (e.g., \( EX_1^2 < \infty \) and \( F'' \) exists in a neighborhood of \( \theta \), see Duttwiler (1973)). Without any moment condition, \( \text{Var} \hat{\theta} \) may not even exist.

In the present paper we show that under (1.1), (1.4) holds for certain choices of \( d \), without any further condition (Theorem 1). For the proof of this result, we state in Section 2 some other results and give their proofs in Section 3. These results are stated as theorems since they are of interest in their own right.

2. THE MAIN RESULTS

We first define the delete-\( d \) jackknife estimator of \( \sigma^2 \). For a given \( n \), let \( d = d(n) \leq n \) be an integer and \( r = n - d \). Let \( S_r \) be the collection of subsets of \( \{ 1, \ldots, n \} \) which have size \( r \). For any \( s \in S_r \), let \( F_{n,s}(x) = r^{-1} \sum_{i \in s} I(X_i \leq x) \) and \( \hat{\theta}_{n,s} = F_{n,s}^{-1}(p) \). The delete-\( d \) jackknife estimator of \( \sigma^2 \) is

\[
\hat{\sigma}^2_{J(d)} = \frac{nr}{dN} \sum_s (\hat{\theta}_s - \bar{\theta})^2, \quad \bar{\theta} = \frac{1}{N} \sum_s \hat{\theta}_s,
\]  

(2.1)
where $N=\binom{d}{d}$ and $\sum_s$ is the summation over all subsets in $S_r$.

Let $s^*$ be a random element satisfying

$$P_*(s^*=s) = \frac{1}{N}, \quad s \in S_r,$$

and $E_*$ and $Var_*$ be the expectation and variance taken under $P_*$. Then $\hat{\sigma}_{J(d)}^2$ defined in (2.1) is actually equal to

$$\hat{\sigma}_{J(d)}^2 = \frac{nr}{d} Var_*(\hat{\theta}_{s^*}) = \frac{nr}{d} Var_*(\hat{\theta}_{s^*} - \hat{\theta}),$$

$$= \frac{nr}{d} \{ \sum_{j=k}^{m} \binom{n}{k-1} \binom{n-k}{j-k} (X_{(j)})^2 - \{ \sum_{j=k}^{m} \binom{n}{k-1} \binom{n-k}{j-k} X_{(j)} \}^2 \}, (2.2)$$

since

$$P_*(\hat{\theta}_{s^*} = X_{(j)}) = \binom{n}{k-1} \binom{n-k}{j-k}, \quad k \leq j \leq m, \quad (2.3)$$

where $X_{(j)}$ is the $j$th order statistic, $m=k+d$, $k=\lfloor rP \rfloor$ and $\lfloor x \rfloor$ is the largest integer $\leq x$. (2.2) provides a convenient way of evaluating $\hat{\sigma}_{J(d)}^2$ without involving computations of $\hat{\theta}_s$ for $\binom{d}{d}$ subsets $s \in S_r$.

Given $X=(X_1, \ldots, X_n)'$, the jackknife histogram is defined to be

$$J(t | X) = P_* \{ (\frac{nr}{d})^{\frac{1}{2}} (\hat{\theta}_{s^*} - \hat{\theta}) \leq t \}, \quad t \in \mathbb{R}, \quad (2.4)$$

which can be used as an estimator of $P \{ n^{\frac{1}{2}} (\hat{\theta} - \theta) \leq t \}$. Note that $\hat{\sigma}_{J(d)}^2$ is the variance of $J(t | X)$ for given $X$. 

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In this paper we assume that $d$ is chosen so that
\[ d = \lceil \lambda n \rceil + 1, \quad \text{for a fixed } \lambda, \ 0 < \lambda < 1. \] (2.5)

**Theorem 1:** Assume (1.1). Then for any $d$ satisfying (2.5),
\[ \hat{\sigma}^2_{J(d)} \to_p \sigma^2, \]
where $\sigma^2$ and $\hat{\sigma}^2_{J(d)}$ are defined in (1.3) and (2.1), respectively.

The following results are needed for the proof of Theorem 1. Their proofs are in the next section. Theorem 2 is a general result (i.e., $\hat{\theta}$ is not necessarily the sample $p$-quantile) for the consistency of the jackknife histogram $J(t \mid X)$.

**Theorem 2:** Let $\hat{\theta}$ be an estimator of $\theta$ and admit an expansion (1.2) with $\phi(x)$ satisfying $E \phi(X) = 0$ and $0 < E \phi^2(X) < \infty$ and $R_n = o_p (n^{-\delta})$. Then,
\[ \sup_{t \in \mathbb{R}} |J(t \mid X) - P [n^{\frac{1}{2}} (\hat{\theta} - \theta) \leq t] | \to_p 0, \]
where $J(t \mid X)$ is defined in (2.4).

The following result gives asymptotic orders of the moments of $J(t \mid X)$.

**Theorem 3:** Let $\hat{\theta} = F_n^{-1}(p)$ and $\hat{\theta}_s = F_{n, s}^{-1}(p)$. Assume (1.1). Then for any $d$ satisfying (2.5) and $\delta \geq 0$,
\[ E_* \left[ \left( \frac{n \rho}{d} \right)^{\frac{1}{2}} |\hat{\theta}_s - \theta| \right]^{2 + \delta} = O_p (1). \]
The next result is an analog of the generalized Helly's theorem (Serfling, 1980, Appendix).

**Theorem 4:** Let \( Y_n \) be random \( n \) -vectors, \( n=1,2,\ldots, \) and \( Z \) be a random variable. Suppose that for any fixed \( n \) and given \( Y_n = y \), \( Z_n(y) \) is a random variable and as \( n \to \infty \),

\[
P \left[ Z_n(Y_n) \leq t \mid Y_n \right] = P \left( Z \leq t \right) \to 0,
\]

Then as \( n \to \infty \),

\[
E[h(Z_n(Y_n)) \mid Y_n] \to_p E[h(Z)]
\]

for any real-valued bounded continuous function \( h \).

**Proof of Theorem 1:** Let \( Z_{n,s} = \left( \frac{nr}{d} \right)^{1/2} \hat{\theta} + \hat{\theta} \) and \( Z \) be a random variable distributed as \( N(0,\sigma^2) \). For any \( \varepsilon > 0 \) and \( \tau > 0 \), from Theorem 3, there is an \( A > 0 \) such that

\[
P \left( E_s \left| Z_{n,s} \right| \left| ^{2+\delta} \right. A \right) < \varepsilon,
\]

where \( \delta > 0 \) is a fixed number. Choose an \( M > 0 \) such that

\[
E[Z^2 I(Z^2 > M)] < \tau/2
\]

and

\[
4^{-1} \tau M^{\delta/2} > A.
\]

For this \( M \), from Theorems 2 and 4

\[
\Delta_n = \left| E_s \left[ Z_{n,s}^2 I(Z_{n,s}^2 \leq M) \right] - E[Z^2 I(Z^2 \leq M)] \right| \to 0.
\]
From (2.6)-(2.8) and
\[ |E_* Z_{n,s}^2 - EZ^2| \leq \Delta_n + \mathbb{E}_* [Z_{n,s}^2 I(Z_{n,s}^2 > M)] + \mathbb{E} [Z^2 I(Z^2 > M)], \]
we have
\[ P( |E_* Z_{n,s}^2 - EZ^2| \geq \tau ) \leq P( \Delta_n \geq \tau/4 ) + P( E_* |Z_{n,s}^2|^{2+\delta} \geq \epsilon ) \]
\[ \leq \epsilon + P( \Delta_n \geq \tau/4 ). \]
Hence from (2.9),
\[ E_* Z_{n,s}^2 \rightarrow_p EZ^2 = \sigma^2. \]
A similar argument shows \( E_* Z_{n,s} \rightarrow_p EZ = 0 \). Thus the result follows. \( \square \)

3. SOME TECHNICAL PROOFS

**Proof of Theorem 2:** Without loss of generality, we assume that \( E \phi^2(X_1) = 1 \).
It suffices to show that
\[ \sup_{t \in \mathbb{R}} |J(t |X) - \Phi(t)| \rightarrow_p 0, \]
where \( \Phi \) is the standard normal distribution function. For any fixed \( s \in S_r \),
from (1.2),
\[ \hat{\theta}_s = \theta + \frac{1}{r} \sum_{i \in S} \phi_{F_i}(X_i) + R_{n,s} \]
with \( R_{n,s} = o_p (r^{-\frac{1}{2}}) \). Let
\[ \xi_{n,s} = (\frac{nr}{d})^{\frac{1}{2}} \left[ \frac{1}{r} \sum_{i \in S} \phi_{F_i}(X_i) - \frac{1}{n} \sum_{i=1}^n \phi_{F_i}(X_i) \right] \]
and
\[ \alpha_{n,s^*} = \left( \frac{nr}{d} \right)^{1/2} (R_{n,s^*} - R_n). \]

Then \( \left( \frac{nr}{d} \right)^{1/2} (\hat{\theta}_{s^*} - \hat{\theta}) = \xi_{n,s^*} + \alpha_{n,s^*} \). For any \( \varepsilon > 0 \), from

\[
P_*(\xi_{n,s^*} \leq t - \varepsilon) - P_*(|\alpha_{n,s^*}| > \varepsilon) \leq P_*(\xi_{n,s^*} \leq t - \varepsilon, |\alpha_{n,s^*}| \leq \varepsilon)
\leq P_*(\xi_{n,s^*} + \alpha_{n,s^*} \leq t, |\alpha_{n,s^*}| \leq \varepsilon) \leq P_*(\xi_{n,s^*} \leq t + \varepsilon),
\]

we have

\[
\sup_{t \in \mathbb{R}} |P_*(\xi_{n,s^*} + \alpha_{n,s^*} \leq t) - \Phi(t)| \leq 2P_*(|\alpha_{n,s^*}| > \varepsilon)
+ \sup_{t \in \mathbb{R}} \left[ \max( |P_*(\xi_{n,s^*} \leq t + \varepsilon) - \Phi(t)|, |P_*(\xi_{n,s^*} \leq t - \varepsilon) - \Phi(t)| \right)
\leq \sup_{t \in \mathbb{R}} |P_*(\xi_{n,s^*} \leq t) - \Phi(t)| + 2P_*(|\alpha_{n,s^*}| > \varepsilon)
+ \sup_{t \in \mathbb{R}} \left[ \max( |\Phi(t + \varepsilon) - \Phi(t)|, |\Phi(t - \varepsilon) - \Phi(t)| \right]
\leq \sup_{t \in \mathbb{R}} |P_*(\xi_{n,s^*} \leq t) - \Phi(t)| + (2\pi)^{-1/2} \varepsilon + 2P_*(|\alpha_{n,s^*}| > \varepsilon).
\]

From Theorem 1 of Wu (1987),

\[
\sup_{t \in \mathbb{R}} |P_*(\xi_{n,s^*} \leq t) - \Phi(t)| \to 0.
\]

It remains to be shown that

\[ P_*(|\alpha_{n,s^*}| > \varepsilon) \to 0, \quad \text{for any } \varepsilon > 0. \]

Let \( s_0 = \{ 1, \ldots, n \} \). Since \( X_1, \ldots, X_n \) are i.i.d.,

\[ E[P_*(|\alpha_{n,s}\varepsilon| > \varepsilon)] = P[\left( \frac{nr}{d} \right)^{1/2}|R_{n,s_0} - R_n| > \varepsilon] \to 0 \]

by (2.5) and \( R_{n,s_0} = o(r^{-1/2}) \). This completes the proof. \( \square \)
Note that under (1.1), there is an interval \([\theta-\tau, \theta+\tau]\) on which \(F'(x) \geq l\), where \(\tau > 0\) and \(l > 0\) are some constants. Hence we have the following result.

**Lemma 1:** Assume (1.1). Let \(s_0 = \{1, \ldots, r\}\). If \(0 < t < r^{\frac{1}{2}}\),

\[
P\left[ \left( \frac{nr}{d} \right)^{\frac{1}{2}} \left| \hat{\theta}_{s_0} - \theta \right| > \frac{t}{2} \right] \leq 2e^{-l^{2}l^{2}t/2},
\]

and

\[
P\left[ \left( \frac{nr}{d} \right)^{\frac{1}{2}} \left| \hat{\theta} - \theta \right| > \frac{t}{2} \right] \leq 2e^{-l^{2}l^{2}t/2}.
\]

Also,

\[
\max\left\{ P\left( \left| \hat{\theta} - \theta \right| > \tau^{\frac{1}{2}}\lambda/2 \right), \quad P\left( \left| \hat{\theta}_{s_0} - \theta \right| > \tau^{\frac{1}{2}}\lambda/2 \right) \right\} \leq 2e^{-2r^{2}p},
\]

where \(p = \min\{ F(\theta + \tau^{\frac{1}{2}}\lambda/2) - F(\theta), \quad F(\theta) - F(\theta - \tau^{\frac{1}{2}}\lambda/2) \} > 0\).

**Proof:** From Theorem 2.3.2 of Serfling (1980),

\[
P\left[ \left( \frac{nr}{d} \right)^{\frac{1}{2}} \left| \hat{\theta}_{s_0} - \theta \right| > \frac{t}{2} \right] \leq 2e^{-2r^{2}\delta_n^{2}} \quad \text{for all} \quad n,
\]

where \(\delta_n = \min\{ F(\theta + \frac{t}{2} \left( \frac{d}{nr} \right)^{\frac{1}{2}}) - F(\theta), \quad F(\theta) - F(\theta - \frac{t}{2} \left( \frac{d}{nr} \right)^{\frac{1}{2}}) \}\). Since \(0 < t < r^{\frac{1}{2}}\),

\[
\delta_n \geq \frac{t}{2} \left( \frac{d}{nr} \right)^{\frac{1}{2}}l.
\]

Hence the first and the second inequalities follow from \(d/n \geq \lambda\).

The last inequality follows directly from Theorem 2.3.2 of Serfling (1980).
Proof of Theorem 3: Let

\[ A_n = \int_0^{r^{1/\delta}} t^{1+\delta} P_\theta \left[ \frac{nr}{d} \right]^{1/2} \left| \hat{\theta} - \hat{\theta} \right| > t \right] dt, \]

\[ B_n = \int_{r^{1/\delta}}^{\infty} t^{1+\delta} P_\theta \left[ \frac{nr}{d} \right]^{1/2} \left| \hat{\theta} - \hat{\theta} \right| > t \right] dt, \]

and

\[ C_n = \int_{r^{1/\delta}}^{\infty} t^{1+\delta} P_\theta \left[ \frac{nr}{d} \right]^{1/2} \left| \hat{\theta} - \hat{\theta} \right| > t \right] dt. \]

Then

\[ E_\theta \left[ \left( \frac{nr}{d} \right)^{1/2} \left| \hat{\theta} - \hat{\theta} \right| \right]^{2+\delta} = (2+\delta)(A_n + B_n + C_n). \]

From

\[ P \left[ \left( \frac{nr}{d} \right)^{1/2} \left| \hat{\theta} - \hat{\theta} \right| > t \right] \leq P \left[ \left( \frac{nr}{d} \right)^{1/2} \left| \hat{\theta} - \hat{\theta} \right| > \frac{t}{2} \right] + P \left[ \left( \frac{nr}{d} \right)^{1/2} \left| \hat{\theta} - \hat{\theta} \right| > \frac{t}{2} \right] \]

and Lemma 1,

\[ E(A_n) = \int_0^{r^{1/\delta}} t^{1+\delta} P_\theta \left[ \frac{nr}{d} \right]^{1/2} \left| \hat{\theta} - \hat{\theta} \right| > t \right] dt \]

\[ \leq 4 \int_0^{\infty} t^{1+\delta} e^{-t^2/2} dt < \infty. \]

Hence \( E(A_n) = O(1) \) and therefore \( A_n = O_p(1) \). Also,

\[ E(B_n) = \int_{r^{1/\delta}}^{\infty} t^{1+\delta} P_\theta \left[ \frac{nr}{d} \right]^{1/2} \left| \hat{\theta} - \hat{\theta} \right| > t \right] dt \]
\[ \leq P \left( \left| \hat{\theta}_{s_e} - \theta \right| > \tau \lambda^{1/2} \right) \int_{\tau s_e}^{r} t^{1+\delta} dt \]

\[ \leq (2+\delta)^{-1} r^{2+\delta} \left[ P \left( \left| \hat{\theta}_{s_e} - \theta \right| > \tau \lambda^{1/2} / 2 \right) + P \left( \left| \hat{\theta} - \theta \right| > \tau \lambda^{1/2} / 2 \right) \right] \]

\[ \leq 4(2+\delta)^{-1} r^{2+\delta} e^{-2r p}, \]

where the last inequality follows from Lemma 1. Hence \( E(B_n) = o(1) \) and therefore \( B_n \rightarrow_p (1) \). From (2.3),

\[ \max_{s \in S_r} \left| \hat{\theta}_s - \theta \right| \leq \left| X_{(\lfloor r p \rfloor)} - X_{(\lfloor r p \rfloor + d)} \right| = W_n, \]

and it is easy to see that \( W_n / r^{1/2} \rightarrow 0 \ a.s. \). Then

\[ \left| W_n \right| \leq (\lambda r)^{1/2} \ a.s., \]

and therefore \( C_n = o(1) \ a.s. \). Thus the result follows. □

For the proof of Theorem 4, we need the following lemma.

**Lemma 2:** Let \( G_{ny} \) be a distribution function for given \( Y_n = y \). If for any continuity point \( x \) of a distribution function \( G \),

\[ G_{nY_n}(x) - G(x) \rightarrow_p 0, \] \hspace{1cm} (3.1)

then there is a countable set \( \Omega \subset [0, 1] \) such that for \( t \in [0, 1] - \Omega \),

\[ P \left( \left| G_{nY_n}^{-1}(t) - G^{-1}(t) \right| > \varepsilon \right) \rightarrow 0 \quad \text{for any } \varepsilon > 0. \] \hspace{1cm} (3.2)

**Proof:** Suppose that \( t \in [0, 1] \) such that (3.2) does not hold for an \( \varepsilon_o > 0 \). Then there are \( \delta > 0 \) and \( \varepsilon_1 > 0 \) such that \( G^{-1}(t) + \varepsilon_1 \) and \( G^{-1}(t) - \varepsilon_1 \) are continuity points of \( G \), and
\[ P \left( |G_{nY_n}^{-1}(t) - G_n^{-1}(t)| > \varepsilon \right) > \delta \] for infinitely many \( n \).

This implies
\[ P \left( G_{nY_n}^{-1}(t) \geq G_n^{-1}(t) + \varepsilon \right) > \delta/2 \] for infinitely many \( n \),

(3.3)
since from (3.1),
\[ P \left( G_{nY_n}^{-1}(t) \leq G_n^{-1}(t) - \varepsilon \right) > \delta/2 \] for infinitely many \( n \)

implies \( t \leq G(G_n^{-1}(t) - \varepsilon) \), which implies \( G_n^{-1}(t) \leq G_n^{-1}(t) - \varepsilon < G^{-1}(t) \). Hence from (3.1) and (3.3), \( t \geq G(G_n^{-1}(t) + \varepsilon) \geq G(G^{-1}(t)) \). Thus \( t = G(x) \) for \( x \in [G^{-1}(t), G^{-1}(t) + \varepsilon] \). This shows that there are at most countabley many \( t \) for which (3.2) does not hold. □

**Proof of Theorem 4:** Let \( M = \sup |h(x)| \) and \( G_{nY_n} \) and \( G \) be the distribution functions of \( Z_n(y) \) and \( Z \), respectively. Let \( W_n(t, y) = G_{nY_n}^{-1}(t) \) and \( W(t) = G^{-1}(t) \). From Lemma 2, for any \( \varepsilon > 0 \) and \( t \in [0, 1] \),
\[ P \left( |W_n(t, Y_n) - W(t)| > \varepsilon \right) \rightarrow 0, \]

where \( \Omega \) is a countable set. Let \( T \) be a uniform random variable and independent of \( Y_n \). Then
\[ E[P \left( |W_n(t, Y_n) - W(t)| > \varepsilon | T = t \right)] = \int_0^1 P \left( |W_n(t, Y_n) - W(t)| > \varepsilon \right) dt \rightarrow 0. \]

Hence
\[ P \left( |h(W_n(T, y)) - h(W(T))| > \varepsilon | Y_n = y \right) \rightarrow 0. \]

Note that
\[ E \left[ |h(W_n(T, y)) - h(W(T))| | Y_n = y \right] \geq \delta \]

implies

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\[ P \left( |h(W_n(T, y)) - h(W(T))| > \delta/2 \mid Y_n = y \right) \geq \delta/4M. \]

Hence
\[ E[h(W_n(T, y)\mid Y_n = y) - E[h(W(T))] \to 0. \]

The result follows since for given \( Y_n = y \),
\[ E[h(W_n(T, y))\mid Y_n = y] = E[h(Z_n(y))\mid Y_n = y] \]

and
\[ E[h(W(T))] = E[h(Z)]. \]

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