A Connection Between the Expansion of Filtrations
and Girsanov's Theorem

by

Philip Protter
Purdue University

Technical Report #87-6

Department of Statistics
Purdue University

February 1987
Revised January 1989
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Philip Protter*

Mathematics and Statistics Departments
Purdue University
West Lafayette, IN 47907

Summary

We give sufficient conditions under which a continuous local martingale remains a
continuous local martingale under a simultaneous initial expansion of the filtration of \( \sigma \)-
fields and change to an equivalent probability law. In particular this gives a method for a
Brownian motion to remain a Brownian motion under such a double transformation. The
classical example of K. Itô is treated in detail.

* Supported in part by NSF Grant # 8500997
initial expansion of \( F \). For example, if \( Z \) is an \( \mathcal{F} \)-measurable random variable, one might define \( G \) by

\[
\mathcal{G}_t = \bigcap_{s > t} \{ \mathcal{F}_s \vee \sigma(Z) \}.
\]

It then often happens that if \( M \) is an \((F, P)\) continuous local martingale, \( M \) is a \((G, P)\) semimartingale with the (unique) decomposition

\[
M_t = (M_t - \int_0^t L_s \, d\langle M, M \rangle_s) + \int_0^t L_s \, d\langle M, M \rangle_s
= \tilde{M}_t + \int_0^t L_s \, d\langle M, M \rangle_s
\]

for a certain \( G \)-predictable process \( L = (L_t)_{t \geq 0} \). The now classical example of \( K \) by Itô [2] for a Brownian motion \( B = (B_t)_{0 \leq t \leq 1} \) is a case in point:

\[
\begin{align*}
\mathcal{G}_t &= \bigcap_{s > t} \{ \mathcal{F}_s \vee \sigma(B_t) \} \quad \text{and} \\
\tilde{B}_t &= B_t - \int_0^t \frac{B_1 - B_s}{1 - s} \, ds, \quad 0 \leq t < 1.
\end{align*}
\]

(This formula holds as well for Lévy processes; see [4]). Recently Jacod [3] and Yor [7] have given general sufficient conditions for a decomposition such as (1) to hold.

**THEOREM.** Let \( M \) be an \((F, P)\) continuous local martingale, \( M_0 = 0 \), and let \( G \) be an expansion of the filtration \( F \) such that there exists a \( G \)-predictable process \( L \) making

\[
\tilde{M}_t = M_t - \int_0^t L_s \, d\langle M, M \rangle_s
\]

a \((G, P)\) continuous local martingale. Suppose further \( E \{ \exp(\frac{1}{2} \int_0^{t_0} L_s^2 \, d\langle M, M \rangle_s) \} < \infty \) for some \( t_0 \). Then there exists a probability \( Q \) equivalent to \( P \) such that \( M \) is a \((G, Q)\) continuous local martingale, \( 0 \leq t \leq t_0 \).

**Proof:** Let \( t \) be less than \( t_0 \) (\( t_0 \) can equal \( \infty \)). Define

\[
\tilde{M}_t = \int_0^t -L_s \, d\tilde{M}_s.
\]

Then \( \tilde{M} \) is a \((G, P)\) continuous local martingale; the stochastic integral is well defined for \( t \leq t_0 \) as a consequence of Novikov's condition, since \( \langle \tilde{M}, \tilde{M} \rangle \), computed under \((G, P)\),
is equal to \( \langle M, M \rangle \), computed under \( (F, P) \); this is because the quadratic variation of a continuous local martingale \( M \) can be written

\[
\lim_{n \to \infty} \sum_{t_i \in \tau_n[0, t]} (M_{t_{i+1}} - M_{t_i})^2 = \langle M, M \rangle_t
\]

where \( \lim_{n \to \infty} \text{mesh } (\tau_n) = 0 \) and convergence is in probability (that is, it is computed without involving the filtration). In this case \( E(e^{\frac{1}{2}(\mathcal{M}, \mathcal{M})_{t_0}}) = E(e^{\frac{1}{2} \int_0^{t_0} L_s^2 d\langle \mathcal{M}, \mathcal{M} \rangle_s}) \) and Novikov's condition is satisfied for \( \mathcal{M} \).

Next define \( N = (N_t)_{0 \leq t \leq t_0} \) to be the solution of

\[
N_t = 1 + \int_0^t N_s \, d\mathcal{M}_s,
\]

the stochastic exponential of \( \mathcal{M} \). By Novikov's theorem (see [6]) we have \( E(N_t) = E(N_{t_0}) = 1, \, 0 \leq t \leq t_0 \). We define \( Q \) by

\[
dQ = N_{t_0} \, dP.
\]

Then \( Q \) is equivalent to \( P \) (that is, \( Q \) has the same null sets as does \( P \)). Under \( Q \) the process \( \mathcal{M} \) is still a \( (\mathcal{G}, Q) \)-semimartingale, and it has a decomposition

\[
\mathcal{M}_t = (\mathcal{M}_t - \int_0^t \frac{1}{N_s} \, d\langle N, \mathcal{M} \rangle_s) + \int_0^t \frac{1}{N_s} \, d\langle N, \mathcal{M} \rangle_s
\]

by the Meyer-Girsanov theorem (cf [1, p. 238]). In particular \( \mathcal{M}_t - \int_0^t \frac{1}{N_s} \, d\langle N, \mathcal{M} \rangle_s \) is a \( (G, Q) \) continuous local martingale. However

\[
\int_0^t \frac{1}{N_s} \, d\langle N, \mathcal{M} \rangle_s = \int_0^t \frac{1}{N_s} N_s d\langle \mathcal{M}, \mathcal{M} \rangle_s
\]

\[
= \int_0^t \frac{N_s}{N_s} (-L_s) d\langle \mathcal{M}, \mathcal{M} \rangle_s
\]

\[
= -\int_0^t L_s \, d\langle M, M \rangle_s.
\]

We conclude that

\[
\mathcal{M}_t - \int_0^t \frac{1}{N_s} \, d\langle N, \mathcal{M} \rangle_s
\]

\[
= (M_t - \int_0^t L_s \, d\langle M, M \rangle_s) - (-\int_0^t L_s \, d\langle M, M \rangle_s)
\]

\[
= M_t
\]
is a \((G, Q)\) continuous local martingale, and therefore so is \(M\). □

For an example we can apply the theorem to Itō's classical example mentioned earlier. Fix \(t_0 < \frac{1}{2}\) and let \(\gamma > 0\) be such that \(E\{e^{\gamma(B^2)}\} < \infty\), where \(B\) is a standard Brownian motion and \(B^*_t = \sup_{0 \leq s \leq t} |B_s|\). Any \(\gamma < \frac{1}{2}\) can be taken. Let \(G\) be the filtration \(\mathcal{G}_t = \bigcap_{s \geq t} \{\mathcal{F}_s \vee \sigma(B_t)\}\). Then \(\tilde{B}_t = B_t - \int_0^t \frac{B_1 - B_s}{1 - s} \, ds = B_t - \int_0^t L_s \, ds\) is a \((G, P)\) martingale. Moreover

\[
E\{\exp\left(\frac{1}{2} \int_0^t L_s^2 \, d\langle B, B \rangle_s\right)\} = E\{\exp\left(\frac{1}{2} \int_0^t \left(\frac{B_1 - B_s}{1 - s}\right)^2 \, ds\right)\}
\leq E\{\exp\left(\frac{t_0}{2(1 - t_0)} \sup_{0 < s \leq t_0} (B_1 - B_s)^2\right)\}.
\]

Since \(\sup_{0 < s \leq t_0} (B_1 - B_s)^2\) has the same distribution as \((B^*_{t_0})^2\), the above is less than

\[
E\{\exp(\gamma (B^*_{t_0})^2)\} < \infty,
\]

by our choice of \(\gamma\). Therefore the hypotheses of the theorem are satisfied and we conclude there exist a law \(Q\) equivalent to \(P\) such that \(B\) is a \((G, Q)\) continuous local martingale. However \(\langle B, B \rangle_t = t\) under \((G, Q)\) as well as under \((F, P)\), hence by Lévy's theorem \(B\) is also a \((G, Q)\) Brownian motion, for \(0 \leq t \leq t_0 < \frac{1}{2}\).

One can apply the theorem as well to diffusions. For example let \(Z\) be an \(\mathcal{F}\)-measurable random variable and let \(B\) be an \((\mathcal{F}_t)_{t \geq 0}\) Brownian motion. Let \(\mathcal{G}_t = \bigcap_{s > t} \{\mathcal{F}_s \vee \sigma(Z)\}\) and suppose \(Z\) is such that \(\tilde{B}_t = B_t - \int_0^t L_s \, ds\) is a \((\mathcal{G}_t, P)\)-local martingale. Suppose also \(E\{\exp(1/2 \int_0^t L_s^2 \, ds)\} < \infty\), some \(\alpha > 0\). Let \(X\) be a solution of

\[
X_t = Z + \int_0^t f(X_s) \, dB_s + \int_0^t g(X_s) \, ds.
\]

Since \(X\) has an anticipating initial condition it cannot be an \((F, P)\)-diffusion. However by the Theorem \(B\) is a \((G, Q)\) Brownian motion and \(Z \in \mathcal{G}_0\). Thus \(X\) is a \((G, Q)\) diffusion. This allows for the consideration of a theory of expansion of filtrations for diffusions.

REFERENCES


