BLOCK DESIGNS: GENERAL OPTIMALITY RESULTS WITH APPLICATIONS TO SITUATIONS WHERE BALANCED DESIGNS DO NOT EXIST

by

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<td>156</td>
<td>$12\frac{1}{2}$</td>
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<td>$\mathcal{R}_4$-better</td>
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correction to proof - separate sheet
add to the end of the paragraph:
Let $B_d = b^{-1} \Delta_d \frac{\delta}{\delta_d} \frac{\delta}{\delta_p} \Delta^p$.
Proof. The nonzero eigenvalues of $C_{d^*}$ are $\nu_1 k^{-1} + mp + 1 - 2k^{-1}$ and $\nu_1 k^{-1} + mp + 1$ with multiplicities $(v/2) - 1$ and $v/2$, respectively. We also have without loss of generality

$$
C_{d^*} = \frac{1}{k} \begin{bmatrix}
  a & -(b+1) & -b & -b & -b & -b \\
  -(b+1) & a & -b & -b & -b & -b \\
  -b & -b & a & -(b+1) & -b & -b \\
  -b & -b & -(b+1) & a & -b & -b \\
  \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
  -b & -b & -b & -b & a & -(b+1) \\
  -b & -b & -b & -b & -(b+1) & a 
\end{bmatrix}
$$

with $a = (v-1) \lambda_1 + k(mp+1) - (mp^2 + 2p + 1) = k - \lambda_0 + k(mp+1) - (mp^2 + 2p + 1)$ and $b = \lambda_1 + mp^2 + 2p$.

Lemma 6.2.1 says $d^*$ E-betters all $d$ with $\min_1 < i < v \left( r_{di} \right) < r + mp$.

Therefore any $d$ that is to E-better $d^*$ must have $r_{di} = r + mp + 1 = r_{d+i}$ ($1 < i < v$). Lemma 6.2.1 also says $d^*$ E-betters any $d$ with $r_{di} = r_{dj} = r + mp + 1$ and $\lambda_{dij} < \lambda_1 + mp^2 + 2p - (q-1) = \lambda_1 + mp^2 + 2p - 1$, for $i \neq j$. Therefore the only $d$ that $d^*$ might not E-better are those with all $r_{di} = r + mp + 1$ and all $\lambda_{dij} > \lambda_1 + mp^2 + 2p$ for $i \neq j$. This leaves us with only two possibilities.

$d$ can have maximum trace, or $\text{tr } C_d = \text{tr } C_{d^*}$. In this case each row (and column) of $C_d$ has its diagonal element equal to that of $C_{d^*}$, $v-2$ of the $\lambda_{dij}$'s equal to $\lambda_1 + mp^2 + 2p$ and one $\lambda_{dij}$ equal to $\lambda_1 + mp^2 + 2p + 1$. Then $C_d$ is equivalent to $C_{d^*}$ in the sense that it will have the same eigenvalues.

The other case is where $d$ is not of full trace. Then at least two rows
loss of generality these rows correspond to treatments 1 and 2 then 
\[ c_{d11} = c_{d22} = r + mp + 1 - k^{-1}(\lambda_0 + mp^2 + 2p + 2) \] 
and 
\[ c_{d12} = -k^{-1}(\lambda_1 + mp^2 + 2p). \] 
Using (ii) of Theorem 2.2.1,

\[
    u_{d1} < r + mp + 1 - k^{-1}(\lambda_0 + mp^2 + 2p + 2) + k^{-1}(\lambda_1 + mp^2 + 2p) \\
    = \nu_1 k^{-1} + mp + 1 - 2k^{-1} = u_{d*1}.
\]

This completes the proof.
CHAPTER 1
INTRODUCTION

1.1 The Model and Notation

In the one-way elimination of heterogeneity setting of analysis of variance we are applying $v$ treatments or varieties to $N$ experimental units which are organized into $b$ blocks of size $k_j, j = 1, \ldots, b$. If we define $r_i$ ($i = 1, \ldots, v$) to be the number of times treatment $i$ is applied to an experimental unit in the whole experiment, we see that

$$N = \sum_{j=1}^{b} k_j = \sum_{i=1}^{v} r_i.$$ 

Since for $v$, $b$, and $k_1$ through $k_b$ fixed we can apply the treatments in different ways, we will label particular applications of the treatments with $d$, $d^*$, or a subscripted $d$, calling each application a design. This also means that quantities like the $r_i$ may be subscripted, and for the remainder of this chapter quantities that change from design to design will be subscripted for illustration.

We assume the additive model $E(Y_{dmij}) = \alpha_i + \beta_j (m = 1, \ldots, N; i = 1, \ldots, v; j = 1, \ldots, b)$ with $Y_{dmij}$ being the observation on the $m$-th experimental unit, which is in block $j$ and was treated with treatment $i$. The $N$ observations are assumed to be uncorrelated with common variance $\sigma^2$. Defining the vectors $\alpha' = (\alpha_1, \ldots, \alpha_v), \beta' = (\beta_1, \ldots, \beta_b), Y_d' = (Y_{dlij}, \ldots, Y_{dNij}) = (Y_{d1}, \ldots, Y_{dN}), k' = (k_1, \ldots, k_b)$ and $r_d' = (r_{d1}, \ldots, r_{dv})$ we can write the model in matrix form:
\[ E(Y_d) = x_d^0 = [x_d^{(1)}, x_d^{(2)}] \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) . \]

Here \( X_d \) is the \( N \) by \((v+b)\) matrix whose \( m \)-th row consists of all 0's except for a 1 in the \( i \)-th and \((v+j)\)-th columns, since \( E(Y_{dm}) = E(Y_{dmij}) = \alpha_i + \beta_j \). \( X_d^{(1)} \) and \( X_d^{(2)} \) are \( N \) by \( v \) and \( N \) by \( b \), respectively, and partition \( X_d \) into columns corresponding to treatments and blocks.

The normal equations for the full model are \( X_d'X_d^0 = X_d'Y_d \) where

\[
X_d'X_d = \begin{bmatrix}
X_d^{(1)}'X_d^{(1)} & X_d^{(1)}'X_d^{(2)} \\
X_d^{(2)}'X_d^{(1)} & X_d^{(2)}'X_d^{(2)} \\
\vdots & \vdots & \ddots & \vdots \\
X_d^{(2)}'X_d^{(1)} & X_d^{(2)}'X_d^{(2)} \\
\end{bmatrix} = \begin{bmatrix}
x_d^\delta \\
\gamma_d \\
\gamma_d' \\
k^\delta \\
\end{bmatrix} \quad \text{and} \quad Y_d = \begin{bmatrix}
x_d^\delta \\
\gamma_d \\
\gamma_d' \\
k^\delta \\
\end{bmatrix}.
\]

Here \( x^\delta \) represents the diagonal \( q \) by \( q \) square matrix with \( x_i \) in the \( ii \)-th position, \( x_i \) being from the vector \( x' = (x_1, \ldots, x_q) \). \( \gamma_d = (\gamma_{dij}) \) is the \( v \) by \( b \) incidence matrix whose elements \( \gamma_{dij} \) equal the number of times treatment \( i \) appears in block \( j \). Note that \( \sum_{j=1}^b \gamma_{dij} = r_{di} \) and \( \sum_{i=1}^v \gamma_{dij} = k_j \).

In the one-way elimination of heterogeneity setting we are not interested in the block effects, only the treatment effects. The reduced normal equations for \( \gamma \) are

\[
[r_d^\delta - \gamma_d k^{-\delta} \gamma_d'] = [x_d^{(1)}' - \gamma_d k^{-\delta} x_d^{(2)}'] Y_d,
\]

where \( k^{-\delta} = (k^\delta)^{-1} \). \( C_d = r_d^\delta - \gamma_d k^{-\delta} \gamma_d' \) is the coefficient or information matrix for the \( \alpha_i \) in the design \( d \) and is also known as the \( C \)-matrix for \( d \). \( C_d \) is a symmetric, \( v \) by \( v \), positive semi-definite matrix whose row
(and column) sums equal zero. Hence \( \text{rank}(C_d) \leq v-1 \) and \( C_d \) will always have at least one zero eigenvalue. In fact \( \text{rank}(C_d) = v-1 \) if and only if all treatment contrasts \( \sum_{i=1}^{v} w_i \alpha_i \) (where \( \sum_{i=1}^{v} w_i = 0 \)) are estimable. Designs with \( \text{rank}(C_d) = v-1 \) are known as connected designs.

The eigenvalues of \( C_d \) will be used in determining the optimality of a given design \( d \). Let \( \mu_d = 0 \leq \mu_1 \leq \ldots \leq \mu_{v-1} \) be the ordered eigenvalues of \( C_d \).

It will be useful to name the collection of possible designs for a given \( v, b, k \). So we define \( \Omega(v, b, k) = \{d: d \text{ has parameters } v, b, k\} \). \( \Omega \) will be used instead of \( \Omega(v, b, k) \) when \( v, b, k \) are clear from context, and \( k \) instead of \( k \) will mean \( k_1 = \ldots = k_b = k \).

Another useful matrix of parameters is \( \Lambda_d = (\lambda_{dij}) = \gamma_d \gamma_d' \). This \( v \times v \) matrix gives \( \lambda_{dij} = \sum_{s=1}^{b} \gamma_i n_{djs} \gamma_j \), and for \( i \neq j \) \( \lambda_{dij} \) is the total number of times treatments \( i \) and \( j \) are paired in the design. If all the blocks are of the same size, then \( C_d = r_d^* - \frac{1}{k} \gamma_d \gamma_d' = r_d^* - \frac{1}{k} \Lambda_d \).

At this point we present an example. Here and always a design will be represented as an array of elements of \( \{1,2,\ldots,v\} \) with the \( b \) columns representing the blocks.

**Example 1.1.1**: Hedayat and Federer (1974). \( v = 5, b = 6, \) and \( d \) below is variance balanced (see Definition 1.3.1).
\[ r_d^t = (3, 3, 3, 3, 4), \quad k^t = (4, 4, 2, 2, 2), \]

\[
\lambda_d = \eta_d \eta_d^t = \begin{bmatrix}
3 & 2 & 2 & 2 & 1 \\
3 & 2 & 2 & 1 \\
\text{sym} & 3 & 1 & 4
\end{bmatrix},
\]

where "sym" will indicate the matrix is symmetric, and

\[
C_d = r_d^\delta - \eta_d k^\delta \eta_d^t = \begin{bmatrix}
4 & -1 & -1 & -1 & -1 \\
4 & -1 & -1 & -1 \\
\text{sym} & 4 & -1 & 4
\end{bmatrix} = \frac{1}{2}
\]
1.2 The Problem and Motivation

The problem considered in this thesis will be how we can best allocate the treatments to the experimental units so that $C_d$ has as large a trace as possible and its nonzero eigenvalues are as equal as possible. More rigorously, we will search for a design $d^* \in \mathcal{D}(v,b,k)$ that will minimize the value of a function $\phi(C_d)$ for all $d \in \mathcal{D}$. $d^*$ will be called $\phi$-optimal, and the function $\phi$ is an optimality criterion. Before listing the classes of optimality criteria in the next section, some heuristic motivation for the problem will be given.

Historically experimenters have used designs like complete block designs (CBD's, with $k_1 = \ldots = k_b = v$) and balanced incomplete block designs (BIBD's, with $k_1 = \ldots = k_b < v$). These were chosen because they make the $r_{di}$ equal and the $\lambda_{dij}$ ($i \neq j$) equal.

Example 1.2.1:

a) 
\[
\begin{array}{cccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
\end{array}
\]

This is a CBD.

b) 
\[
\begin{array}{cccccccccc}
1 & 1 & 1 & 2 & 2 & 3 & 1 & 1 & 1 \\
2 & 2 & 2 & 3 & 2 & 3 & 4 & 3 & 4 \\
3 & 4 & 3 & 4 & 4 & 2 & 3 & 4 & 3 \\
4 & 3 & 4 & 4 & 4 & 2 & 3 & 4 & 4 \\
\end{array}
\]

This is variance balanced by Definition 1.3.1. It is a BIBD by some definitions, but not in this thesis.
This is a BIBD according to Definition 1.2.1.

Each of the designs in Example 1.2.1 has \( C_d \) of the form
\[(a+b)I_v-bJ_{v,v} \]
where \( I_v \) is the \( v \) by \( v \) identity matrix, and \( J_{v,v} \) is the \( v \) by \( v \) matrix with all elements equal to \( 1 \). \( C_d \) in this form is called completely symmetric, and has \( v-1 \) equal nonzero eigenvalues. Also note that since the trace of \( C_d \), call it \( \text{tr}(C_d) \), is
\[
\sum_{i=1}^{v} c_{dij} = \sum_{i=1}^{v} \left( r_{di} - b \sum_{j=1}^{k} n_{dij} \right) = \frac{1}{k} \sum_{j=1}^{v} \left( \sum_{i=1}^{v} n_{dij} \right),
\]
it is maximized when for each \( j \) the \( n_{dij} \) are as equal as possible. (This is under the restrictions \( n_{dij} \geq 0, \sum_{i=1}^{v} n_{dij} = k \) for each \( j \).) This means the designs in a) and c) of the example also maximize \( \text{tr}(C_d) = \sum_{i=0}^{v-1} u_{di} = \sum_{i=1}^{v-1} u_{di} \).

In this thesis the BIBD will be required to have the maximum possible trace in its class, so the CBD and BIBD will maximize the sum of their \( v-1 \) nonzero and equal eigenvalues. Kiefer (1958, 1975) generalized these maximum trace and balanced designs with the following definition.

Definition 1.2.1: For given \( v, b, \) and \( k \) a design \( d \) is said to be a \textbf{balanced block design} (BBD) if \( (1) \) all the \( r_{di} \) are equal, \( (2) \) all the \( \lambda_{dij} \) are equal \( (i \neq j) \), \( (3) \) all the \( \lambda_{dij} \) are equal, and \( (4) \) for each \( (i,j) \) \( |n_{dij}-k/v|<1 \). The CBD has \( k=v \) and we will define a BIBD to be a BBD with \( k<v \).
Optimality criteria used in this thesis to measure the "goodness" of a design will allow the BBD defined above to be optimal, and reward larger $\sum_{i=1}^{v-1} \mu_{d_i}$ and $\mu_{d_i}$ ($i = 1, \ldots, v-1$) that are nearly equal when a BBD does not exist. Two such criteria will be

$$\phi_A(C_d) = \sum_{i=1}^{v-1} \mu_{d_i}^{-1}$$

and

$$\phi_D(C_d) = \prod_{i=1}^{v-1} \mu_{d_i}^{-1}$$

(defined to be $+\infty$ if rank $(C_d) < v-1$).

Alternately, one can look at a solution to the normal equations for $\hat{a}$. If $C_d \hat{a} = (X_d^{(1)}' - \gamma_d \delta^\top X_d^{(2)})Y_d$, then one solution is

$$\hat{a} = C_d^{-1}(X_d^{(1)}' - \gamma_d \delta^\top X_d^{(2)})Y_d$$

where $C_d^{-1}$ is the Moore-Penrose inverse of $C_d$. The covariance matrix of $\hat{a} = \text{Cov}(\hat{a}) = \sigma^2 C_d^{-1}$ in this case. We want to make this covariance matrix small in some sense, and one sense is through $\text{tr}(\text{Cov}(\hat{a})) = \sum_{i=0}^{v-1} (\text{variances of } a_i) = \sum_{i=0}^{v-1} \sigma^2 \lambda_{d_i}$

where the $\lambda_{d_i}$ are the eigenvalues of $C_d^{-1}$. Now rank $(C_d) = \text{rank}(C_d^{-1})$ and each nonzero eigenvalue of $C_d^{-1}$ is the reciprocal of a nonzero eigenvalue of $C_d$. Without loss of generality let $\lambda_{d_i} = \mu_{d_i}$ if $\mu_{d_i} = 0$ and $\lambda_{d_i} = \mu_{d_i}^{-1}$ if $\mu_{d_i} > 0$ ($i = 0, 1, \ldots, v-1$). Then minimizing $\text{tr}(\text{Cov}(\hat{a}))$ corresponds to minimizing $\sigma^2 \sum_{i=1}^{v-1} \lambda_{d_i} = \sigma^2 \sum_{i=1}^{v-1} \mu_{d_i}^{-1} = \phi_A(C_d)$.

In regression settings the determinant $\text{det}(\text{Cov}(\hat{a}))$ is minimized if $\text{Cov}(\hat{a})$ is nonsingular. So in this model minimization of

$$\prod_{i=1}^{v-1} \lambda_{d_i} = \prod_{i=1}^{v-1} \mu_{d_i}^{-1} = \phi_D(C_d)$$

for $\mu_{d_1} > 0$ is also considered as a means of making $\text{Cov}(\hat{a})$ small.
Finally, we choose to work with $C_d$ and $\mu_d \leq \cdots \leq \mu_{d,v - 1}$ because we are not interested in any particular $v - 1$ linearly independent contrasts. If that were the case, we would use the appropriate reparameterization of the model.

1.3 The Optimality Criteria

In this section we will give the various optimality criteria $\phi(C_d)$ to be minimized by some $d^* \in D(v,b,k)$. Recall that $C_d$ has the eigenvalues $\mu_{d0} = 0 \leq \mu_{d1} \leq \cdots \leq \mu_{d,v - 1}$, and note that the symbol $J$ will be used with subscripts to label large classes of optimality criteria. Let $\mathcal{A}_{v,0}$ be the set of $v$ by $v$ symmetric non-negative definite matrices with zero row sums. Then any optimality criteria $\phi(\cdot)$ will be a function from $\mathcal{A}_{v,0}$ into $(-\infty, \infty]$ satisfying restrictions that will vary from criterion to criterion. Unless otherwise stated, $\phi(C_d) = +\infty$ if $\mu_{d1} = 0$.

1. $\phi_A(C_d) = \sum_{i=1}^{v-1} \mu_{di}^{-1}$. This is the A-optimality criterion from Kiefer (1959, 1975).

2. $\phi_D(C_d) = \prod_{i=1}^{v-1} \mu_{di}^{-1}$ or equivalently $\phi_D(C_d) = -\sum_{i=1}^{v-1} \log \mu_{di}$. This is the D-optimality criterion of Kiefer (1958, 1959, 1975).

3. $\phi_E(C_d) = \max_{1 \leq i \leq v - 1} (\mu_{di}^{-1}) = \mu_{d1}^{-1}$. This is the E-optimality criterion of Kiefer (1958, 1959, 1975). It is sometimes easier to think of this as maximizing $\mu_{d1}$.

4. $J_p = \{ \phi_p(C_d) = \sum_{i=1}^{v-1} \mu_{di}^{-p}: 0 < p < \infty \}$. These criteria are essentially the $\phi_p$ criteria from Kiefer (1975).
(5) \( \mathcal{J}_1 = \{ \phi_f(C_d) = \sum_{i=1}^{v-1} f(\mu_{d_i}) : f \) is real valued on \([0, c_0]\); 
\( f \) is continuous, strictly convex and strictly decreasing 
on \([0, c_0]\); \( f \) is continuously differentiable, and \( f' \) is 
strictly concave on \((0, c_0]\), where \( c_0 = \max_{d \in \Delta} \tau(C_d) \). Also 
we allow \( f(0) = \lim_{x \to 0^+} f(x) = + \infty \). These are the type 1 optimality criteria of Cheng (1978).

(6) \( \mathcal{J}_2 = \{ \phi_f(C_d) = \sum_{i=1}^{v-1} f(\mu_{d_i}) : f \) is real valued on \([0, c_0]\); 
\( f \) is continuous, strictly convex and continuously differentiable on \((0, c_0]\); \( f' \) is strictly concave on \((0, c_0]\); 
\( \lim_{x \to 0^+} f(x) = f(0) = + \infty \). Cheng (1981a).

(7) \( \mathcal{J}_3 = \{ \phi_f(C_d) = \sum_{i=1}^{v-1} f(\mu_{d_i}) : f \) is real valued, nonincreasing, convex and continuous on \([0, c_0]\). This class of criteria was introduced by Cheng (1979).

(8) \( \mathcal{J}_4 = \{ \phi_F(C_d) = F(\mu_d, v-1, \ldots, \mu_d) : F \) is Schur-convex and nonincreasing in each argument on \([0, c_0]^{v-1}\). This was introduced to the author by Cheng via a personal communication, but is also in Constantine (1983).

(9) \( \mathcal{J}_U = \{ \phi : \phi \) is convex on \( \Delta_{v,0} \); \( \phi(b C_d) \) is nonincreasing in the scalar \( b \geq 0 \); \( \phi \) is invariant under each permutation of rows and (the same) columns\). This is the Universal Optimality class of Kiefer (1975).

All these criteria are minimized by a design \( d^* \) if \( C_{d^*} \) is completely symmetric and of maximum trace in \( \Delta(v, b, k) \). A BBD of Kiefer will satisfy these conditions, as also will any variance
balanced design $d^*$ with $C_d^*$ of maximum trace (see Definition 1.3.1, Corollary 1.4.4, and Theorem 2.3.1). If such designs do not exist for a given $\mathfrak{D}(v,b,k)$, then these criteria allow competing designs to be ranked. This is not the case with the following classical criteria, where a design satisfying the conditions does or does not exist, and no optimality function is given.

Definition 1.3.1: A design $d$ is variance balanced if and only if $C_d = \mu(I_v - \frac{1}{v} J_v)$ with $\mu = \text{tr}(C_d)/(v-1)$. Rao (1958) and Puri and Nigam (1977).

Definition 1.3.2: A design $d$ is efficiency balanced if and only if $C_d = \lambda(I_v^\delta - \frac{1}{d^*} c_d^\delta)/N$ with $\lambda$ a constant of proportionality. Williams (1975) and Puri and Nigam (1977).

Definition 1.3.3: A design $d$ is pairwise balanced if and only if $\Lambda = \phi_i^\delta + c_1 l_v l_v^t$ where $\phi_i$ and $c$ are real, $c > 0$, and $l_v$ is the $v$ by 1 vector of ones. See Puri and Nigam (1977).

It should be said that variance balance and efficiency balance are special cases of Calinski's (1977) $X^{-1}$ balance, which is new but still does not allow for ranking competing designs that are not $X^{-1}$-balanced.

Finally we must mention the hierarchical relationships among the optimality criteria. $\phi_A$ is a member of $\mathcal{A}_p, \mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ and $\mathcal{A}_U$. $\phi_D$ is a member of $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ and $\mathcal{A}_U$. $\phi_E$ is a member of $\mathcal{A}_4$ and $\mathcal{A}_U$. $\mathcal{A}_p \subset \mathcal{A}_1 \subset \mathcal{A}_3 \subset \mathcal{A}_4 \subset \mathcal{A}_U$ and $\mathcal{A}_2 \subset \mathcal{A}_3$. And with the definition of generalized criteria (given below) we see that $\phi_D$...
and $\phi_E$ are limiting cases of $J_p$, and $\phi_E$ is also a limiting case of $J_1, J_2$ and $J_3$.

**Definition 1.3.4:** A **wide sense criterion** $\phi^0$ for a class $J$ is the composition $\phi^0 = F \circ \phi$ where $\phi \in J$ and $F$ is non-decreasing.

**Definition 1.3.5:** A **generalized criterion** $\phi^*$ for a class $J$ is the pointwise limit of a sequence of wide sense criteria $\phi_i^0$ for $J$:

$$\phi^*(C_d) = \lim_{i \to \infty} \phi_i^0(C_d).$$

Cheng gave these two definitions for $J_1$ in his paper of (1981b) so $\phi_E$ would be a generalized $J_1$ criterion. Without the nonincreasing $F$, $\phi_E$ is not a limit of type 1 criteria, as he originally stated in his 1978 paper.

Actually $\phi_0^0(C_d) = \lim_{p \to 0} \left( \frac{1}{\sqrt{1 - p}} \sum_{i=1}^{\nu} \mu_{di} \right)^{1/p}$ and $\phi_E(C_d) = \lim_{p \to 0} \left( \frac{1}{\sqrt{1 - p}} \sum_{i=1}^{\nu} \mu_{di} \right)^{1/p}$.

The symbol that will be used to represent a class of criteria and its generalized criteria will be $g\mathcal{J}$, where $\mathcal{J}$ was the original class of criteria. Note that $\phi \in \mathcal{J}$ is a generalized criterion for $\mathcal{J}$ so $\mathcal{J} \subseteq g\mathcal{J}$. In the case of $\mathcal{J}_p, \mathcal{J}_1, \mathcal{J}_2$ and $\mathcal{J}_3$, $\phi_E$ is not included in $\mathcal{J}$ but in $g\mathcal{J}$. $\mathcal{J}_4 = g\mathcal{J}_4$ as $\mathcal{J}_4$ is a closed convex cone of functions.

1.4 Results of Others

We will begin with four theorems that are important to this thesis and to this area of statistics. Then some other results that are called upon in later chapters will be presented.

**Theorem 1.4.1:** (Cheng (1978) Theorem 2.2) Let $\mathcal{C} = \{C_d: \ d \in \mathcal{D}\}$ be a class of matrices in $\mathcal{B}_{v,0}$ with $v > 2$. Suppose $C_{d*} \in \mathcal{C}$ has two
distinct nonzero eigenvalues $\mu_2 > \mu_1$ with multiplicities 1 and $v-2$, respectively, and that

(i) $\text{tr}(C_{d^*}) = \max_{d \in \mathcal{D}} (\text{tr} C_d)$,

(ii) $\text{tr}(C_{d^*}) < (\text{tr} C_{d^*})^2/(v-2)$, and

(iii) $C_{d^*}$ maximizes $\text{tr} C_{d^*} - [(v-1)/(v-2)]^{1/3} \sqrt[3]{[\text{tr}(C_d^2) - (\text{tr} C_d)^2/(v-1)]^{1/3}}$ for all $d \in \mathcal{D}$.

Then $d^*$ is optimal with respect to all generalized type 1 criteria ($g_{\mathcal{D}}$-optimal) over all $d \in \mathcal{D}$.

Cheng applied this theorem to a subclass of the following class of designs.

Definition 1.4.1: A design $d \in \mathcal{D}(v,b,k)$ will be called a regular graph design (RGD) if (1) $r_{d1} = \ldots = r_{dv}$, (2) for each $(i,j)$ we have $|n_{dij} - k/v| < 1$, and (3) for $i \neq j$, $i' \neq j'$ we have $|\lambda_{dij} - \lambda_{dij'}| = 0$ or 1. An extreme RGD of type 1 has $C_d$ with two distinct nonzero eigenvalues $\mu_2 > \mu_1$ with multiplicities 1 and $v-2$ respectively.

Let $\lambda_1(d)$ be the smaller value that the $\lambda_{dij}$ take on, and $\lambda_2(d) = \lambda_1(d) + 1$. Also let $n(d) =$ the number of $\lambda_{dij}$ in each row of $A_d$ equal to $\lambda_1(d)$.

Theorem 1.4.2: (Cheng (1978) Theorem 3.1) Let $d^* \in \mathcal{D}(v,b,k)$ be an extreme RGD of type 1 with $\lambda_1(d^*) > 0$ or $\lambda_1(d^*) = 0$ and $n(d^*) < v/2$. Then $d^*$ is $g_{\mathcal{D}}$-optimal over all $d \in \mathcal{D}(v,b,k)$.
Note: In Cheng's Theorem there is a uniqueness result not stated here.

In particular Theorem 1.4.2 can be applied to \( d^* \in \mathcal{D}(v,b,k) \) when \( d^* \) is a most balanced group divisible partially balanced block design of type 1 (MB GD PBBD of type 1). That is \( d \) is a GD PBBD with 2 groups and \( \lambda_2 = \lambda_1 + 1 \) in the notation of GD designs.

Theorem 1.4.1 is important in the following way. If no design exists with maximum trace and all its eigenvalues equal, then Cheng's eigenvalue structure of \( 0 < a = \ldots = a < b \) is a logical guess for the next best form, and as it turns out, this structure with maximum trace and two conditions does give optimality over a large class of \( \phi \). In Chapter 2 an optimality result with the eigenvalue structure \( 0 < a < b = \ldots = b \) will be proved as Theorem 2.1.1.

Theorem 1.4.2 has a proof for the situation where the \( r_{di} \) are all equal. This proof is modified in Chapter 3 to apply Theorem 1.4.1 to two situations where the \( r_{di} \) are not equal.

Constantine (1981) investigated some situations where BIBD's did not exist and proved the E-optimality of some designs. In this thesis we will adopt his terminology for those classes where no BIBD exists, some of the E-optimal designs will be proved optimal over a larger class of criteria, and his techniques for proving E-optimality will be extensively applied. Other authors, including Cheng (1980), Jacroux (1980a, 1980b, 1982), and Constantine (1982), have written papers on E-optimality techniques, but the straightforward techniques of Constantine (1981) are most useful here.
First we explain the terminology used for classes $\mathcal{A}(v,b,k)$ where no BIBD exists. The class $\mathcal{A}(4,5,3)$ does not contain a BIBD, but $\mathcal{A}(4,4,3)$ does. We can refer to $\mathcal{A}(4,5,3)$ as a class defined by a "BIBD plus one block". Another example of such a class is $\mathcal{A}(4,7,2)$ since $\mathcal{A}(4,6,2)$ contains a BIBD. In general the "BIBD plus $m$ blocks" ($1 \leq m \leq v/k$) defines a parameter set $\mathcal{A}(v,b+m,k)$ where no BIBD exists, but one exists in $\mathcal{A}(v,b,k)$. Also the "BIBD minus $m$ blocks" ($1 \leq m \leq v/k$) defines $\mathcal{A}(v,b-m,k)$ which does not contain a BIBD but where $\mathcal{A}(v,b,k)$ does.

In many cases a BIBD plus or minus $m$ blocks will also describe a particular design in $\mathcal{A}(v,b+m,k)$ or $\mathcal{A}(v,b-m,k)$ that is constructed by adding or removing $m$ blocks pairwise disjoint with respect to the treatments that they contain.

**Example 1.4.1:** Here is a design in $\mathcal{A}(4,8,2)$ that is a BIBD plus (the last) two disjoint blocks.

$$
\begin{array}{ccccccc}
1 & 1 & 2 & 2 & 3 & 1 & 3 \\
2 & 3 & 4 & 3 & 4 & 4 & 2 & 4
\end{array}
$$

Secondly Constantine proved the E-optimality of a BIBD plus $m$ binary ($n_{dij} = 0,1$) blocks for $1 \leq m < v/k$ in the classes $\mathcal{A}(v,b+m,k)$ where $\mathcal{A}(v,b,k)$ allowed a BIBD to be constructed. He proved the E-optimality of a BIBD minus $m$ (binary) blocks for $v/k^2 \leq m \leq v/k$ in the classes $\mathcal{A}(v,b-m,k)$ where $\mathcal{A}(v,b,k)$ allowed a BIBD, and he proved the E-optimality of a GD PBIBD plus $s$ binary blocks where $s < (v-m)/k$, the GD PBIBD has $\lambda_2 = \lambda_1 + 1$, $m$ groups, and the blocks are compatible with the partition defined by the $m$ groups. These theorems were Theorems 3.1,
3.3, and 3.4 in his paper. As will be shown in Chapter 3, some of these E-optimal designs are actually optimal for larger classes of criteria.

The third aspect of Constantine's paper that is relevant to this thesis involves his Theorem 1.4.3 below and the techniques used to prove it. As he points out, if \( \{\sigma_i\} \) is a collection of \( n \) permutations on the symbols \( 1,2,\ldots,v \) and \( C_d \) is a \( v \times v \) information matrix, then we can define \( \tilde{C}_d = (1/n) \sum_{i=1}^{n} C_d^{\sigma_i} = (1/n) \sum_{i=1}^{n} P_i C_d P_i' \); \( P_i \) being the \( v \times v \) matrix representation of \( \sigma_i \). If \( \mu_{d0} = 0 \leq \mu_{d1} \leq \cdots \leq \mu_{d,v-1} \) and \( \mu_{d0} = 0 \leq \mu_{d1} \leq \cdots \leq \mu_{d,v-1} \) are the eigenvalues of \( C_d \) and \( \tilde{C}_d \), respectively, then \( \tilde{C}_d \) is nonnegative definite with row sums zero, \( \text{tr} \tilde{C}_d = \text{tr} C_d, \mu_{d1} \leq \mu_{d1} \) and \( \mu_{d,v-1} \geq \mu_{d,v-1} \). This technique can be applied to an information matrix \( C_d \) by selecting subsets of \( \{1,2,\ldots,v\} \) and applying all the permutations of the symbols in each subset to that same subset. This smooths out the matrix \( C_d \) by averaging out the \( c_{dij} \)'s in blocks defined by those subsets. In particular Constantine used the fact that \( \mu_{d1} \geq \mu_{d1} \). Here are two of the three parts of his theorem.

**Theorem 1.4.3:** (Constantine (1981) Theorem 3.2) In any block design \( d \in \mathcal{B}(v,b,k) \) the following inequalities hold:

\[
\begin{align*}
(i) \quad \mu_{d1} &\leq \frac{v}{v-1} \min_{1 \leq i \leq v} \left( r_{di} - \frac{1}{k} \sum_{j=1}^{b} n_{dij}^2 \right) \\
(ii) \quad \mu_{d1} &\leq \min_{2 \leq n < v} \left[ \frac{k-1}{kn} \sum_{i=1}^{n} r_{di} + \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} \lambda_{dij} \right]
\end{align*}
\]
This theorem will be extended to unequal $k_j$ with the same method of proof in Theorem 2.2.1. For (i) above $(v/(v-1))(r^1 - k^{-1} \sum_{j=1}^{b} n^2_{ij})$ is an eigenvalue of $C_d$ if you permute over the set of treatments excluding $i$. Then $\nu_{d1} \leq \nu_{d1}$ which is less than or equal to this particular eigenvalue.

Kiefer (1975) proved the following theorem and applied it to BBD's.

**Theorem 1.4.4:** (Kiefer (1975) Proposition 1) Suppose there exists a $d^* \in \mathcal{S}(v,b,k)$ such that (1) $C_{d^*}$ is completely symmetric, and (2) $\text{tr} C_{d^*} = \max_{d \in \mathcal{S}} \text{tr} C_d$. Then $d^*$ is Universally Optimal ($\mathcal{U}$-optimal) in $\mathcal{S}(v,b,k)$.

**Corollary 1.4.4:** (Kiefer (1974) p. 28) If $d^* \in \mathcal{S}(v,b,k)$ is a BBD then $d^*$ is $\mathcal{U}$-optimal in $\mathcal{S}(v,b,k)$.

Theorem 1.4.4 will be applied in the proof of Theorem 2.3.1, an extension of Corollary 1.4.4 to unequal $k_j$.

We conclude this section by presenting some less pivotal but nonetheless needed results due to others.

**Theorem 1.4.5:** (Cheng (1983)) Let $\mathcal{G} \subset \mathbb{R}^{v-1}$. If there exists a $\nu^{*} = (\nu^{*}_{v-1}, \ldots, \nu^{*}_1) \in \mathcal{G}$ such that

1. $\nu^{*}_{v-1} = \ldots = \nu^{*}_2 > \nu^{*}_1$,
2. $\nu^{*}$ maximizes $\sum_1^{v-1} \nu_i$ over $\mathcal{G}$,
3. $\nu^{*}$ minimizes $\sum_1^{v-1} \nu_i^{-q}$ over $\mathcal{G}$ for some $q > 0$, 


then $\mu^*$ minimizes $\sum_{i=1}^{v-1} \mu_i p$ for all $0 < p < q$ and minimizes $\prod_{i=1}^{v-1} \mu_i^1$.

the generalized limiting case.

Lemma 1.4.1: (See Constantine (1981) Lemma 2.2.) Let $C_d$ be an information matrix of the form

$$
\begin{pmatrix}
(a+b)I_p - bJ_{p,p} & -cJ_{p,v-p} \\
-cJ_{v-p,p} & (d+e)I_{v-p} - eJ_{v-p,v-p}
\end{pmatrix}.
$$

Then $C_d$ has the eigenvalues $(a+b), (d+e), v_c$ and $0$ with multiplicities $p-1, v-p-1, 1$ and $1$ respectively.

Lemma 1.4.2: (a: Rao (1973) p. 68. b: See Remark 1.4.1)
Let the matrices $A = (a+b)I_q - bJ_{q,q}, B = -cJ_{q,q}, D = -dJ_{q,v-mq}$ and
and $E = (e+f)I_{v-mq} - fJ_{v-mq,v-mq}$.

a) The $v$ by $v$ information matrix

$$
\begin{bmatrix}
A & B & \cdots & B & B \\
A & \cdots & B & B \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
sym & \cdots & \cdots & A & B
\end{bmatrix}
$$

with $v = mq, q \geq 3$ has the eigenvalues $0, v_c$, and $(a+b)$ with multiplicities $1, m-1$ and $m(q-1)$ respectively.

b) The $v$ by $v$ information matrix
with \( v-mq \geq 1, q \geq 2 \) has the eigenvalues 0, \( vd, (a-(q-1)b+qc) \), \( (a+b) \) and \( (e+f) \) with multiplicities 1, 1, \( m-1 \), \( m(q-1) \) and \( v-mq-1 \) respectively.

Remark 1.4.1: Part b) might have been used by Constantine (1981). In any case the eigenvectors corresponding to the eigenvalues in the order given are \( l_v \):

\[
\begin{bmatrix}
-1_{m_q} \\
1_{v-mq}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\gamma_1 l_q \\
\vdots \\
\gamma_{m-1} l_q \\
\gamma_{m-1, m_q}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\circ_{v-mq, 1} \\
\circ_{v-mq, 1}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\circ_{q, 1} \\
\circ_{q, 1} \\
\vdots \\
\circ_{q, 1} \\
\circ_{v-mq, 1} \\
\circ_{v-mq, 1}
\end{bmatrix}
\]

and
\[
\begin{bmatrix}
\mathcal{O}_{q,1} \\
\vdots \\
\mathcal{O}_{q,1} \\
\mathcal{O}_{q,1} \\
\beta_1 \\
\vdots \\
\beta_{v-mq-1}
\end{bmatrix}
\]

where the \( q \) by \( 1 \) vectors \( a_1, \ldots, a_{q-1}, l_q \) are pairwise orthogonal;
the \( v-mq \) by \( 1 \) vectors \( \beta_1, \ldots, \beta_{v-mq-1}, l_{v-mq} \) are pairwise orthogonal;
the \( m \) by \( 1 \) vectors \( \gamma_1, \ldots, \gamma_{m-1}, l_m \) are pairwise orthogonal with
\( \gamma_i = (\gamma_{i1}, \ldots, \gamma_{im}) \); and \( \mathcal{O}_{x,y} \) is the \( x \) by \( y \) zero matrix.

Lemma 1.4.3: (Essentially Cheng (1978) Lemma 3.1)

a) For \( \sum_{i=1}^{v} n_{dij} = k_j (j = 1, \ldots, b) \), \( \text{tr}(C_d) \) is maximized if the
integers \( n_{dij} (i = 1, \ldots, v) \) satisfy \( |n_{dij} - k_j/v| < 1 \), for
each \( j \).

b) For \( \text{tr}(C_d) \) fixed, \( \text{tr}(C_d^2) \) is minimized if the \( c_{dii} \) are as
equal as possible and the \( c_{dij} (i \neq j) \) are as equal as
possible.

c) \( P_d = \text{tr}(C_d^2) - (\text{tr } C_d)^2/(v-1) \geq 0 \) with equality if and only if
\( C_d \) is completely symmetric.

Lemma 1.4.4: (Essentially Takeuchi (1961) Lemma 1)

Let \( F = aC_d + bI_v + (c/v)J_{v,v} \) where \( a, b \) and \( c \) are constants. If
\( 0 = \mu_0 \leq \mu_1 \leq \ldots \leq \mu_{v-1} \) are the eigenvalues of \( C_d \) with corresponding
eigenvectors \( \lambda_0 = 1_v, \lambda_1, \ldots, \lambda_{v-1} \) then the eigenvalues of \( F \) are \( b+c, a\mu_1+b, \ldots, a\mu_{v-1}+b \) with the same corresponding eigenvectors.
1.5 Final Assumptions and Definitions with Outline of the Text

We begin with two restrictions on the collections $\mathcal{A}(v,b,k)$ of designs that will be adhered to in the remainder of this thesis. $\mathcal{A}(v,b,k)$ will contain only connected designs and connected designs where every block contains at least two treatments.

The first restriction is chosen because we want to be able to estimate no less than $v-1$ linearly independent contrasts. This is always possible except in the most trivial cases, and in fact most of the optimality criteria take on the value $\infty$ in those cases.

The second restriction comes from the fact that a block with only one treatment applied to its experimental units provides no information in $C_d$. Since $C_d = C_d^{(1)} + \ldots + C_d^{(b)}$ where $C_d^{(j)}$ is the C-matrix for the j-th block, if block b, say, has only treatment 1, then

$$C_d^{(b)} = \begin{bmatrix} k_b & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{bmatrix} - \begin{bmatrix} k_b \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdot k_b^{-1} \cdot [k_b,0,\ldots,0] = \mathcal{O}_{v,v}.$$ 

Therefore $C_d = C_d^{(1)} + \ldots + C_d^{(b-1)}$.

**Definition 1.5.1:** A binary block is a block of size $k_j$ where the $n_{dij}$ ($i = 1,\ldots,v$) are as equal as possible, thus taking on only two consecutive values.

In an incomplete block, $k_j < v$, the $n_{dij}$ take on 0 or 1, and this corresponds to the classical definition. If $k_j = p_j v^q + q_j$, then $q_j$ of
the \( n_{dij} = p_{j+1} \) and \( v-q_j \) of the \( n_{dij} = p_{j} \) \((p_{j} \geq 0)\).

**Definition 1.5.2:** The extra-replicated treatments of a binary block \( j \) are those for which \( n_{dij} = p_{j+1} \).

**Definition 1.5.3:** Binary blocks are called disjoint if the extra-replicated treatments of each block, when considered as sets, are disjoint sets.

Definitions 1.5.1 through 1.5.3 are useful for \( k_{j} \geq v \). Note that our BBD has binary blocks by definition.

If \( k = 2 \), then our second assumption above and Definition 1.5.1 force all blocks to be binary in \( \mathcal{D}(v,b,2) \) for any \( v \) and \( b \). Lemma 1.4.3 says all \( d \in \mathcal{D}(v,b,2) \) will have the maximum trace.

A design \( d \in \mathcal{D} \) with all blocks binary will be of maximum trace in \( \mathcal{D} \) by Lemma 1.4.3.

A BBD in \( \mathcal{D}(v,b,k) \) is sometimes represented as a BBD \((v,b,k,r,\lambda)\) where \( r \) and \( \lambda \) are the common values of the \( r_{di} \) and \( \lambda_{dij} \) \((i \neq j)\) respectively. We will use this notation when it is useful.

**Definition 1.5.4** \( d_1 \) is said to be \( \mathcal{D} \)-better than \( d_2 \) if and only if \( \phi(c_{d_1}) \leq \phi(c_{d_2}) \) for all \( \phi \in \mathcal{D} \). **Strictly \( \mathcal{D} \)-better** is used when the inequality is strict.

The notation \( \text{int}[x] \) will denote the greatest integer less than or equal to \( x \).

Recall that the subscript \( d \) on various quantities may be dropped in later chapters if the context allows it.
Before concluding, a brief outline of the remainder of the thesis follows.

Chapter 2 contains original works inspired by Theorems 1.4.1, 1.4.3 and 1.4.4. Lemma 2.4.1 was inspired by Lemma 1.4.1 and necessity.

Chapter 3 looks at applications of Theorem 1.4.1 and Theorem 2.1.1 to BIBD's plus or minus 1 block.

Chapter 4 looks at applications of Theorem 2.1.1 to the BBD plus one block of size v-1, and Chapter 5 at the BBD minus one observation.

Chapters 6 and 7 discuss the BBD plus or minus m² blocks, respectively, with applications of Theorem 2.1.1.

Chapter 8 gives selected A-, D- and E-efficiencies for various interesting designs discussed in Chapters 3 through 7.

Finally Chapter 9 discusses trend-free block designs of Bradley and Yeh (1980) in the light of applying Theorem 2.1.1 and extending one of their later results.
CHAPTER 2
GENERAL THEORETICAL RESULTS

2.1 Results for Schur-convex Optimality Criteria

Before stating and proving the original results in this section some definitions and theorems from the theory of majorization must be presented. They will be paraphrased from Marshall and Olkin (1979) with their location in that book given parenthetically. For this discussion on majorization, any vector $x' = (x_1, \ldots, x_n) \in \mathbb{R}^n$ will have $x(1) \leq \ldots \leq x(n)$ and $x[1] \geq \ldots \geq x[n]$ be the ordered $x_i$.

**Definition 2.1.1:** (Definition 1.A.1) For $x, y \in \mathbb{R}^n$, $y$ majorizes $x$, written $y \succ x$, if and only if we have

\[
\sum_{i=1}^{k} x[i] \leq \sum_{i=1}^{k} y[i], \quad k = 1, \ldots, n-1
\]

and
\[
\sum_{i=1}^{n} x[i] = \sum_{i=1}^{n} y[i]
\]

or equivalently we have

\[
\sum_{i=1}^{k} x(i) \geq \sum_{i=1}^{k} y(i), \quad k = 1, \ldots, n-1
\]

and
\[
\sum_{i=1}^{n} x(i) = \sum_{i=1}^{n} y(i).
\]
Definition 2.1.2: (Definition 1.A.2) For $x, y \in \mathbb{R}^n$, $y$ weakly majorizes $x$, written $y \preceq_w x$, if and only if we have

$$
\sum_{i=1}^{k} x(i) \geq \sum_{i=1}^{k} y(i), \quad k = 1, \ldots, n.
$$

Remark 2.1.1: This is the weak supermajorization defined by Marshall and Olkin (1979).

Definition 2.1.3: (Definition 3.A.1) A function $F: G \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be Schur-convex on $G$ if and only if $x \prec y$ on $G$ implies $F(x) \leq F(y)$. $F$ is strictly Schur-convex if in addition $F(x) < F(y)$ whenever $x \prec y$ but $x$ is not a permutation of $y$.

Lemma 2.1.1: (Proposition 4.B.2) $x \preceq_w y$ if and only if

$$
\sum_{i=1}^{n} f(x_i) \leq \sum_{i=1}^{n} f(y_i)
$$
holds for all continuous, convex nonincreasing $f: \mathbb{R} \rightarrow \mathbb{R}$.

Lemma 2.1.2: (Theorem 3.A.8 with definition p. 443) A function $F: G \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies

$$
x \preceq_w y \text{ on } G \text{ implies } F(x) \leq F(y)
$$
if and only if $F$ is Schur-convex and nonincreasing in each argument on $G$.

Lemma 2.1.3: (Theorem 3.A.8a) $F$ is a real-valued function on $G \subseteq \mathbb{R}^n$. Then $x \preceq_w y$ on $G$ and $x$ is not a permutation of $y$ imply $F(x) < F(y)$ if and only if $F$ is strictly Schur-convex and strictly decreasing in each argument on $G$. 
Lemmas 2.1.1 and 2.1.2 provide some motivation for $J_3$ and $J_4$ optimality criteria, respectively. Lemma 2.1.3 is presented to illustrate what is needed for strict inequality in $F(x) \leq F(y)$.

Since optimality criteria are thought of as functions of $C_d$ we write $\phi(C_d) = F(\mu_d)$ where $\mu' = (\mu_d, \delta_{v-1}, \ldots, \mu_d)$ is the vector of eigenvalues of $C_d$ and $0 < \mu_d \leq \cdots \leq \mu_d, \delta_{v-1}$.

Now we begin the original work.

**Theorem 2.1.1:** Let $C = \{C_d : d \in \mathcal{D}\}$ be a class of matrices in $\mathcal{B}_{v,0}$ with $v \geq 3$. Suppose $C_d^* \in C$ has

(i) two distinct eigenvalues $\mu^*_2 > \mu^*_1 > 0$ with multiplicities $v-2$ and 1 respectively,

(ii) $\text{tr} C_{d^*} = \max_{d \in \mathcal{D}} \text{tr} C_d$,

(iii) $d^*$ E-optimal in $\mathcal{D}$.

Then $d^*$ (or $C_{d^*}$) is optimal with respect to all $J_4$ criteria over all $d \in \mathcal{D}$.

**Proof:** Let $d$ be an arbitrary design in $\mathcal{D}$ with the eigenvalues of $C_d$ being $0 \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{v-1}$. Let $\text{tr} C_d^* = (v-2)\mu^*_2 + \mu^*_1 = \text{tr} C_d + \sum_{i=1}^{v-1} \mu_i + \epsilon$ with $\epsilon \geq 0$. Now $(\mu_{v-1}, \ldots, \mu_1)^+ \geq (\mu_{v-1} + \epsilon, \mu_{v-2}, \ldots, \mu_1) > (\mu^*_2, \ldots, \mu^*_1, \mu^*_1)$

with the last majorization justified as follows. $\mu^*_1 \geq \mu_1$ by E-optimality implies $(v-2)\mu^*_2 \leq \epsilon + \sum_{i=2}^{v-1} \mu_i$. Using the first definition of majorization we have $k\mu^*_2 \leq k(\epsilon + \sum_{i=2}^{v-1} \mu_i)/(v-2)$ for $k = 1, 2, \ldots, v-2$.
and \((v-2)u_2^* + u_1^* = (v-2)(\varepsilon + \sum_{i=2}^{v-1} u_i)/(v-2) + u_1\).

Noting that \(v > x \Rightarrow (\varepsilon)^w > x\) we have \((\mu_{v-1}, \ldots, u_1) \overset{w}{\succ} (\mu_2^*, \ldots, u_2^*, u_1^*)\)
so for any \(\mathcal{J}_4\) criterion \(F(\mu_2^*, \ldots, u_2^*, u_1^*) \leq F(\mu_{v-1}, \ldots, u_1)\), and that completes the proof.

The following two theorems are similar in statement and proof, but are for the eigenvalue structures \(0 < a = \ldots = a < b\) (of Cheng) and \(0 < a < b = \ldots = b < c\).

**Theorem 2.1.2:** Let \(C = \{C_d: d \in \mathcal{D}\}\) be a class of matrices in \(B_{v,0}\) with \(v \geq 3\). Suppose \(C_{d^*} \in C\) has

(i) two distinct eigenvalues \(\mu_2^* > \mu_1^* > 0\) with multiplicities \(1\) and \(v-2\) respectively,

(ii) \(tr \ C_{d^*} = \max_{d \in \mathcal{D}} (tr \ C_d)\),

(iii) \(\mu_2^* = \mu_{d^*}, v-1 \leq \mu_d, v-1\) for all \(d \in \mathcal{D}\).

Then \(d^*\) is optimal with respect to all \(\mathcal{J}_4\) criteria over all \(d \in \mathcal{D}\).

**Proof:** Let \(d \in \mathcal{D}\) be arbitrary with \(C_d\) having eigenvalues \(0 \leq \mu_1 \leq \ldots \leq \mu_{v-1}\) and \(tr \ C_d = \mu_2^* + (v-2)\mu_1^* = tr \ C_d + \varepsilon = \sum_{i=2}^{v-1} u_i + \varepsilon\), \(\varepsilon > 0\).

\((\mu_{v-1}, \ldots, u_1) \overset{w}{\succ} (\mu_{v-1}, (\sum_{i=1}^{v-2} u_i)/(v-2), \ldots, (\sum_{i=1}^{v-2} u_i)/(v-2), \ldots, (\sum_{i=1}^{v-2} u_i)/(v-2))\)

\((\mu_2^*, u_1^*) \overset{w}{\succ} (\mu_{v-1}, (\sum_{i=1}^{v-2} u_i)/(v-2), \ldots, (\sum_{i=1}^{v-2} u_i)/(v-2))\) with the last majorization following from the fact that \(\mu_2^* = \mu_{v-1}\) implies \((v-2)\mu_1^* \geq \varepsilon + \sum_{i=1}^{v-2} u_i\) so
\[ k \mu_1^* \geq k(\varepsilon + \sum_{i=1}^{v-2} \mu_i)/(v-2) \text{ for } k = 1, \ldots, v-2 \text{ and } (v-2)\mu_1^* + \mu_2^* = \\
(\varepsilon + \sum_{i=1}^{v-2} \mu_i) + \mu_{v-1}. \]

**Theorem 2.1.3:** Let \( \mathcal{C} = \{ C_d : d \in \mathcal{D} \} \) be a class of matrices in \( \mathcal{B}_{v,0} \) with \( v \geq 4 \). Suppose \( C_{d^*} \in \mathcal{C} \) has

(i) three distinct eigenvalues \( \mu_3^* > \mu_2^* > \mu_1^* > 0 \) with multiplicities 1, \( v-3 \) and 1 respectively,

(ii) \( \text{tr } C_{d^*} = \max \{ \text{tr } C_d \} \),

(iii) \( d^* \) is \( \mathcal{E} \)-optimal and \( \mu_3^* = \mu_3^* \), \( v-1 \leq \mu_1^*, v-1 \) for all \( d \in \mathcal{D} \).

Then \( D^* \) is optimal with respect to all \( \mathcal{J}_4 \) criteria over all \( d \in \mathcal{D} \).

**Proof:** Let \( d \in \mathcal{D} \) be arbitrary with \( C_d \) having eigenvalues \( 0 \leq \mu_1 \leq \ldots \leq \mu_{v-1} \). Let \( \text{tr } C_{d^*} = \text{tr } C_d + \varepsilon_2 \) (\( \varepsilon_2 \geq 0 \)), \( \mu_{v-1}^* - \mu_3^* = \varepsilon_3 \geq 0 \), and \( \mu_1^* - \mu_1 = \varepsilon_1 \geq 0 \).

\[
\begin{align*}
(\mu_{v-1}, & \mu_{v-2}, \ldots, \mu_2, \mu_1) \\
> & (\mu_{v-1}, (\sum_{i=1}^{v-2} \mu_i)/(v-3), \ldots, (\sum_{i=1}^{v-2} \mu_i)/(v-3), \mu_1) \\
= & (\mu_{v-1}, (\text{tr } C_d - \mu_1 - \mu_3)/(v-3), \ldots, (\text{tr } C_d - \mu_1 - \mu_3)/(v-3), \mu_1) \\
= & (\mu_3^* + \varepsilon_3, (\text{tr } C_{d^*} - \varepsilon_2 - (\mu_1^* - \varepsilon_1) - (\mu_3^* + \varepsilon_3))/(v-3), \\
& \ldots, (\text{tr } C_{d^*} - \varepsilon_2 - (\mu_1^* - \varepsilon_1) - (\mu_3^* + \varepsilon_3))/(v-3), \mu_1^* - \varepsilon_1) 
\end{align*}
\]
\[
\begin{align*}
&= (\mu_\ast^1 + \epsilon_1^\ast, \mu_2^\ast + \frac{\epsilon_1^\ast - \epsilon_2^\ast - \epsilon_3^3}{v-3}, \ldots, \mu_2^\ast + \frac{\epsilon_1^\ast - \epsilon_2^\ast - \epsilon_3^3}{v-3}, \mu_1^\ast - \epsilon_1^\ast) \\
&\succ_w (\mu_3^\ast, \mu_2^\ast, \ldots, \mu_2^\ast + \mu_1^\ast)
\end{align*}
\]

where the last step follows from the definition of weak majorization as follows:

a) \( \mu_1^\ast - \epsilon_1 \leq \mu_1^\ast \)

b) \( \mu_1^\ast - \epsilon_1 + k(\mu_2^\ast + \frac{\epsilon_1^\ast - \epsilon_2^\ast - \epsilon_3^3}{v-3}) = \mu_1^\ast + k\mu_2^\ast \\
- \epsilon_1(1 - \frac{k}{v-3}) - \frac{k}{v-3}(\epsilon_2 + \epsilon_3) \leq \mu_1^\ast + k\mu_2^\ast \\
for k = 1, 2, \ldots, v-3.

\( \mu_1^\ast - \epsilon_1 + (v-3)(\mu_2^\ast + \frac{\epsilon_1^\ast - \epsilon_2^\ast - \epsilon_3^3}{v-3}) + \mu_3^\ast + \epsilon_3 \\
= \mu_1^\ast + (v-3)\mu_2^\ast + \mu_3^\ast - \epsilon_2 \leq \mu_1^\ast + (v-3)\mu_2^\ast \\
+ \mu_3^\ast.
\]

Hence \((\mu_{v-1}^\ast, \mu_{v-2}^\ast, \ldots, \mu_2^\ast, \mu_1^\ast) \succ_w (\mu_3^\ast, \mu_2^\ast, \ldots, \mu_2^\ast, \mu_1^\ast)\) which implies \( \phi(C_{d^\ast}) \leq \phi(C_d) = F(\mu_d) \) for all Schur-convex F nonincreasing in each argument.

Remark 2.1.2: Theorems 2.1.2 and 2.1.3 are not as useful as Theorem 2.1.1 for the following reason. One can often find a connected design in any \( \mathcal{D} \) that has \( \mu_d \cup v-1 < \mu_{d^\ast} \cup v-1 \) if you let \( \text{tr } C_d \) be much less than \( \text{tr } C_{d^\ast} \). If \( \mathcal{D} \) allows only one value for \( \text{tr } C_d \), however, then the theorems could prove helpful.

If Theorems 1.4.1 or 2.1.1 fail to work, then the following Theorems 2.1.4 and 2.1.5, respectively, might be of use.
Theorem 2.1.4: Let $C = \{C_d: d \in \mathbb{D}\}$ be a class of matrices in $\mathcal{A}_{v,0}$ and $v \geq 3$. Suppose $C_{d0}, C_{d*} \in C$ satisfy
1) $0 < \mu_1^0 = \ldots = \mu_{v-2}^0 < \mu_{v-1}^0$ for eigenvalues of $C_{d0}$,
2) $\text{tr}(C_{d0}) = \text{tr}(C_{d*}) = \max_{d \in \mathbb{D}}(\text{tr} C_d)$,
3) $\mu_1^0 \leq \mu_1^* \leq \mu_2^* \leq \ldots \leq \mu_{v-1}^*$ with the $\mu_i^*$'s being the eigenvalues of $C_{d*}$.

Then $\phi(C_{d*}) \leq \phi(C_{d0})$ for any $\phi(C_d) = F(\mu_d, v-1, \ldots, \mu_d)$ where $F$ is Schur-convex on $[0, \infty)^{v-1}$. If in 3) we also have $\mu_1^0 < \mu_i^*$ for some $i \in \{1, 2, \ldots, v-2\}$ then $\phi(C_{d*}) < \phi(C_{d0})$ for any $\phi(C_d) = F(\mu_d, v-1, \ldots, \mu_d)$ where $F$ is strictly Schur-convex on $[0, \infty)^{v-1}$.

Proof: $\sum_{i=1}^{k} \mu_i^* \geq k \mu_1^0 = \sum_{i=1}^{k} \mu_i^0$, $k = 1, \ldots, v-2$ and $\sum_{i=1}^{v-1} \mu_i^* = \sum_{i=1}^{v-1} \mu_i^0$ imply $(\mu_{v-1}^0, \ldots, \mu_1^0) > (\mu_{v-1}^*, \ldots, \mu_1^*)$ which implies $\phi(C_{d0}) > \phi(C_{d*})$ for the Schur-convex $\phi$. If in addition some $\mu_i^0 < \mu_i^*$ for some $i \in \{1, \ldots, v-2\}$ then $\mu_i^*$ is not a permutation of $\mu_i^0$ and $\phi(C_{d0}) > \phi(C_{d*})$ if $\phi$ is strictly Schur-convex.

Corollary 2.1.4: Let $C = \{C_d: d \in \mathbb{D}\}$ be a class of matrices in $\mathcal{A}_{v,0}$ with $v > 2$. Suppose $C_{d*} \in C$ has
1) $0 < \mu_1^* = \ldots = \mu_2^* < \mu_{v-1}^*$ for its eigenvalues.
2) $\text{tr} C_{d*} = \max_{d \in \mathbb{D}}(\text{tr} C_d)$,
3) \( \phi(C_{d*}) \leq \phi(C_d) = F(\mu_{d,v-1}, \ldots, \mu_{d1}) \) for any \( d \in \mathcal{B} \) and \( F \) strictly Schur-convex.

Then any strictly E-better \( d^0 \in \mathcal{B} \) has \( \text{tr} \ C_d^{0} < \text{tr} \ C_{d^*} \). Hence \( d^* \) is E-optimal among full trace designs.

**Proof:** Suppose \( C_d^{0} \) had eigenvalues \( 0 < \mu_1^0 \leq \ldots \leq \mu_{v-1}^0 \) and \( \mu_1^0 > \mu_1^* \). If \( \text{tr} \ C_d^{0} = \text{tr} \ C_{d^*} \) then by Theorem 2.1.4 we would have \( \phi(C_d^{0}) < \phi(C_{d^*}) \), contradicting Condition 3). Therefore an E-better \( d^0 \) is not of full trace.

**Theorem 2.1.5:** Let \( \mathcal{C} = \{C_d: d \in \mathcal{B}\} \) be a class of matrices in \( \mathcal{B}_{v,0} \) with \( v \geq 3 \). Suppose \( C_d^{0}, C_{d*} \in \mathcal{C} \) satisfy

1) \( 0 < \mu_1^0 < \mu_2^0 = \ldots = \mu_{v-1}^0 \) for the eigenvalues of \( C_d^{0} \),

2) \( \text{tr} \ C_d^{0} = \text{tr} \ C_{d^*} = \max_{d \in \mathcal{B}} (\text{tr} \ C_d) \),

3) \( 0 < \mu_1^* \leq \ldots \leq \mu_{v-1}^* \leq \mu_{v-1}^0 \) with the \( \mu_i^* \)'s being the eigenvalues of \( C_{d^*} \).

Then \( \phi(C_{d*}) \leq \phi(C_d^{0}) \) for any \( F(\mu_{d,v-1}, \ldots, \mu_{d1}) \) where \( F \) is Schur-convex on \([0, \infty)^{v-1}\). If in 3) we also have \( \mu_i^* \leq \mu_i^{0} \) for some \( i \in \{2, \ldots, v-1\} \) then \( \phi(C_{d*}) < \phi(C_d^{0}) \) for any \( \phi(C_d) = F(\mu_{d,v-1}, \ldots, \mu_{d1}) \) where \( F \) is strictly Schur-convex.
Proof: \[ \sum_{k}^{v-1} \mu^0_k = (v-k) \mu^0_v \geq \sum_{k}^{v-1} \mu^*_k, \quad k = 2, \ldots, v-1 \] and \[ \sum_{i}^{v-1} \mu^0_i = \sum_{i}^{v-1} \mu^*_i \] so we have \((\mu^0_v, \ldots, \mu^0_1) > (\mu^*_v, \ldots, \mu^*_1)\) which implies \(\phi(C^0_d) \geq \phi(C^*_d)\). If in addition we have \(\mu^*_i < \mu^0_i\) for some \(i \in \{2, \ldots, v-1\}\) then \(\mu^*_i\) is not a permutation of \(\mu^0_i\) and \(\phi(C^0_d) > \phi(C^*_d)\) if \(\phi\) is strictly Schur-convex.

\textbf{Corollary 2.1.5:} Let \(C = \{C_d: \quad d \in \mathcal{D}\}\) be a class of matrices in \(\mathcal{E}_{v, 0}\) with \(v > 2\). Suppose \(C^*_d \in C\) has

1) \(0 < \mu^*_1 < \mu^*_2 = \ldots = \mu^*_v = 1\) for its eigenvalues,

2) \(\text{tr} \quad C^*_d = \max \{\text{tr} \quad C_d\}, \quad d \in \mathcal{D}\),

3) \(\phi(C^*_d) \leq \phi(C^*_d) = F(\mu^*_v, \mu^*_v, \ldots, \mu^*_v)\) for any \(d \in \mathcal{D}\) and \(F\) strictly Schur-convex.

Then any \(d^0 \in \mathcal{D}\) with \(\mu^0_v = \mu^*_v = 1\) has \(\text{tr} \quad C^0_d < \text{tr} \quad C^*_d\).

\textbf{Proof:} Suppose \(\mu^0_v < \mu^*_v\) for some \(d^0\), \(C^0_d\) having eigenvalues \(0 < \mu^0_v \leq \ldots \leq \mu^0_1\). If \(\text{tr} \quad C^0_d = \text{tr} \quad C^*_d\) then by Theorem 2.1.5 we would have \(\phi(C^0_d) < \phi(C^*_d)\), contradicting condition 3).

Theorem 2.1.1 is a \(J_4\) optimality result for \(d^*\) with a given eigenvalue structure, \(C^*_d\) of maximum trace, and \(d^*\) optimal for some criterion in the class \(\mathcal{J}_4\). The eigenvalue structure is \(0 < a < b = \ldots = b\) and the single criterion is \(\phi(E(C_d))\). A logical
question is whether other global optimality results for $\mathcal{J}_4$ exist when
given 1) an eigenvalue structure, 2) maximum trace and 3) optimality
for one or two of the criteria in $\mathcal{J}_4$.

Here we present four numerical examples where a $d^*$ in a hypotheti-
cal $\mathcal{J}$ has one of four eigenvalue structures and is of maximum trace.
Then for $d^*$ considered optimal for one or two of $A$, $D$ and $E$-optimality
examples are given which are strictly better than $d^*$ for the remaining
criteria, indicating other global results of a similar kind do not
exist.

Before we begin we must discuss $v = 2$ and $v = 3$. For $v = 2$ the
lone eigenvalue is just $\text{tr } C_d$ and the reader is referred to Corollary
2.3.1. For $v = 3$ there are just two eigenvalues and they are equal or
different. The eigenvalue structure corresponding to them being
different is just a special case of Theorems 1.4.1, 2.1.1 or 2.1.2
and the reader is referred to those.

In the tables below the nonzero eigenvalues for the hypothetical
designs are given along with the criteria for which $d^*$ is optimal (in
the column "d*") , the criteria for which $d_i$ is optimal (in the column
"d_i") and how $\text{tr } C_d$ compares to $\text{tr } C_{d^*}$ (in the column "tr").

**Example 2.1.1:** $0 < \mu_{d^*_1} < \mu_{d^*_2} = \ldots = \mu_{d^*_v}, v-1, v > 4$. If $d^*$ is
$A$-optimal we have Theorem 1.4.5. If $d^*$ is $E$-optimal we have Theorem
2.1.1. Otherwise see Table 2.1.1.
Table 2.1.1: Examples for $0 < a < b = \ldots = b$

<table>
<thead>
<tr>
<th>$d_1$</th>
<th>$d_2$</th>
<th>$\mu_{d,v-1}$</th>
<th>$d^*$</th>
<th>$d_i$</th>
<th>tr</th>
</tr>
</thead>
<tbody>
<tr>
<td>d*:</td>
<td>4</td>
<td>5 (5) \ldots (5)</td>
<td>5</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>$d_1$:</td>
<td>4.1</td>
<td>4.5 (5) \ldots (5)</td>
<td>5.4</td>
<td>A,D</td>
<td>E</td>
</tr>
<tr>
<td>$d_2$:</td>
<td>4.1</td>
<td>4.6 (5) \ldots (5)</td>
<td>5.3</td>
<td>D</td>
<td>A,E</td>
</tr>
</tbody>
</table>

Example 2.1.2: $0 < \mu_{d_1}^* = \ldots = \mu_{d,v-2}^* < \mu_{d,v-1}^*$, $v \geq 4$. See Table 2.1.2. Note that if $d^*$ is not E-optimal but A- or D-optimal, Corollary 2.1.4 says the E-better design cannot be of full trace. For $d_1 \epsilon > 0$ and is taken as small as needed. For $d_4 \epsilon, \sigma > 0$ and each is taken as small as needed.

Table 2.1.2: Examples for $0 < a = \ldots = a < b$

<table>
<thead>
<tr>
<th>$d_1$</th>
<th>$\mu_{d,v-2}$</th>
<th>$\mu_{d,v-1}$</th>
<th>$d^*$</th>
<th>$d_i$</th>
<th>tr</th>
</tr>
</thead>
<tbody>
<tr>
<td>d*:</td>
<td>4 (4) \ldots (4)</td>
<td>4</td>
<td>5</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>$d_1$:</td>
<td>4+\epsilon \ldots</td>
<td>4+\epsilon</td>
<td>4+\epsilon</td>
<td>A,D</td>
<td>E</td>
</tr>
<tr>
<td>$d_2$:</td>
<td>3.7 (4) (4)</td>
<td>4.65</td>
<td>4.65</td>
<td>A,E</td>
<td>D</td>
</tr>
<tr>
<td>$d_3$:</td>
<td>3.9 (4) (4)</td>
<td>4.185</td>
<td>4.9</td>
<td>D,E</td>
<td>A</td>
</tr>
<tr>
<td>$d_4$:</td>
<td>4+\epsilon \ldots</td>
<td>4+\epsilon</td>
<td>4+\epsilon \frac{1}{(4+\epsilon)^{v-2}} D</td>
<td>A,E</td>
<td>E</td>
</tr>
<tr>
<td>$d_5$:</td>
<td>3.8 (4) (4)</td>
<td>4.6</td>
<td>4.6</td>
<td>E</td>
<td>A,D</td>
</tr>
</tbody>
</table>

\[
(d^* - \sigma) \sum_{i=1}^{v-1} \frac{1}{d^*_i (d^*_i - \sigma)}
\]
Example 2.1.3: \( 0 < \mu d_i^* = \ldots = \mu d_{i+1}^* < \mu d_{i+2}^* = \ldots = \mu d_{v-1}^* \), \( v \geq 5 \), \( 2 \leq i \leq v-3 \). See Table 2.1.3. Note that \( d_1, d_5, d_6 \) and \( d_7 \) are special cases. \( d_7 \) is for \( 2 < i < v-3 \) only. In \( d_2 \) and \( d_6 \) take \( \varepsilon > 0 \) small enough. In \( d_7 \) take \( \varepsilon, \sigma > 0 \) as needed with \( (i-2)\varepsilon = (v-i-3)\sigma \).

Table 2.1.3: Examples for \( 0 < a = \ldots = a < b = \ldots = b \)

<table>
<thead>
<tr>
<th>( d )</th>
<th>( \mu_1 )</th>
<th>( \ldots )</th>
<th>( \mu_{i-1} )</th>
<th>( \mu_i )</th>
<th>( \mu_{i+1} )</th>
<th>( \ldots )</th>
<th>( \mu_{v-2} )</th>
<th>( \mu_{v-1} )</th>
<th>( d^* )</th>
<th>( d_i )</th>
<th>( \text{tr} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d^* )</td>
<td>4</td>
<td>(4)</td>
<td>\ldots</td>
<td>(4)</td>
<td>4</td>
<td>5</td>
<td>(5)</td>
<td>\ldots</td>
<td>(5)</td>
<td>5</td>
<td>*</td>
</tr>
<tr>
<td>( d_1 )</td>
<td>4.1</td>
<td>(i=2)</td>
<td>4.1</td>
<td>(5)</td>
<td>\ldots</td>
<td>(5)</td>
<td>5.7</td>
<td>( A,D )</td>
<td>( E )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( d_2 )</td>
<td>4+\varepsilon</td>
<td>\ldots</td>
<td>4+\varepsilon</td>
<td>4+\varepsilon</td>
<td>(5)</td>
<td>\ldots</td>
<td>(5)</td>
<td>(6-(i+1)\varepsilon)</td>
<td>( A,D )</td>
<td>( E )</td>
<td></td>
</tr>
<tr>
<td>( d_3 )</td>
<td>3.85</td>
<td>(4)</td>
<td>\ldots</td>
<td>(4)</td>
<td>4.19</td>
<td>4.96</td>
<td>(5)</td>
<td>\ldots</td>
<td>(5)</td>
<td>5</td>
<td>( A,E )</td>
</tr>
<tr>
<td>( d_4 )</td>
<td>3.995</td>
<td>(4)</td>
<td>\ldots</td>
<td>(4)</td>
<td>4.105</td>
<td>4.6</td>
<td>(5)</td>
<td>\ldots</td>
<td>(5)</td>
<td>5.3</td>
<td>( D,E )</td>
</tr>
<tr>
<td>( d_5 )</td>
<td>4.1</td>
<td>(i=2)</td>
<td>4.2</td>
<td>4.3</td>
<td>\ldots</td>
<td>(5)</td>
<td>\ldots</td>
<td>(5)</td>
<td>5.4</td>
<td>( D )</td>
<td>( A,E )</td>
</tr>
<tr>
<td>( d_6 )</td>
<td>4+\varepsilon</td>
<td>\ldots</td>
<td>4.1</td>
<td>4.2</td>
<td>4.3</td>
<td>(i=v-3)</td>
<td>(5.4-(v-5)\varepsilon)</td>
<td>( D )</td>
<td>( A,E )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( d_7 )</td>
<td>4+\varepsilon</td>
<td>\ldots</td>
<td>4.1</td>
<td>4.2</td>
<td>4.3</td>
<td>(5-\sigma)</td>
<td>\ldots</td>
<td>(5-\sigma)</td>
<td>5.4</td>
<td>( D )</td>
<td>( A,E )</td>
</tr>
<tr>
<td>( d_8 )</td>
<td>3.9</td>
<td>(4)</td>
<td>\ldots</td>
<td>(4)</td>
<td>4.5</td>
<td>4.5</td>
<td>(5)</td>
<td>\ldots</td>
<td>(5)</td>
<td>5.1</td>
<td>( E )</td>
</tr>
</tbody>
</table>

Example 2.1.4: \( 0 < \mu d_1^* < \mu d_2^* = \ldots = \mu d_{v-2}^* < \mu d_{v-1}^* \), \( v \geq 4 \).

The examples in Table 2.1.4 are for \( v = 4 \), but for \( v \geq 5 \) just add the eigenvalue 4 with multiplicity \( v-4 \).
Table 2.1.4: Examples for $0 < a < b = \ldots = b < c$

<table>
<thead>
<tr>
<th></th>
<th>$\mu_d$</th>
<th>$\mu_d$</th>
<th>$\mu_d$</th>
<th>$d^*$</th>
<th>$d_1$</th>
<th>tr</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d^*$</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>*</td>
<td>*</td>
<td>*</td>
</tr>
<tr>
<td>$d_1$</td>
<td>3.2</td>
<td>3.4</td>
<td>5.4</td>
<td>A,D</td>
<td>E</td>
<td>=</td>
</tr>
<tr>
<td>$d_2$</td>
<td>2.9</td>
<td>4.5</td>
<td>4.6</td>
<td>A,E</td>
<td>D</td>
<td>=</td>
</tr>
<tr>
<td>$d_3$</td>
<td>2.96</td>
<td>4.5</td>
<td>4.5</td>
<td>D,E</td>
<td>A</td>
<td>&lt;</td>
</tr>
<tr>
<td>$d_4$</td>
<td>3.2</td>
<td>3.6</td>
<td>5.2</td>
<td>D</td>
<td>A,E</td>
<td>=</td>
</tr>
<tr>
<td>$d_5$</td>
<td>2.98</td>
<td>4.51</td>
<td>4.51</td>
<td>E</td>
<td>A,D</td>
<td>=</td>
</tr>
</tbody>
</table>

Remark 2.1.3: In Examples and Tables 2.1.2, 2.1.3 and 2.1.4 no single $d_1$ was found to be D- and E-better than an A-optimal $d^*$. For our purposes, $d_1$ and $d_2$ must be taken together to show an A-optimal $d^*$ is D- and E-bettered. There may exist such a single design, or a result saying such a single design will not exist, but it is unknown to the author.

2.2 Bounding the Extreme Nonzero Eigenvalues

In this section we will extend Theorem 1.4.3 of Constantine to unequal $k_j$ and to give two lower bounds on $\mu_{d,v-1}$.

We begin with two upper bounds on $\mu_{d_1}$.

Theorem 2.2.1: In any block design $d \in \mathcal{B}(v,b,k)$ with replications $r_d$ and eigenvalues of $C_d$ being $0 = \mu_0 \leq \mu_1 \leq \ldots \leq \mu_{d,v-1}$ we have
(i) \[ \mu_{d1} \leq \left( \frac{v}{v-1} \right) \min_{1 \leq i \leq v} \left[ r_{d1} - \frac{b}{\sum_{j=1}^{k_i} \frac{n_{dij}^2}{k_j}} \right] \]

and (ii) \[ \mu_{d1} \leq \min_{2 \leq n \leq v} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left( r_{d1} - \frac{b}{\sum_{j=1}^{k_j} \frac{n_{dij}^2}{k_j}} \right) + \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} \left( \frac{\sum_{\ell=1}^{k_{ij}} \frac{n_{dij} n_{dij}}{k_{ij}}}{n} \right) \right\} \]

Proof: (i) Let \( d \in \mathcal{D} \) be arbitrary. By definition of \( C_d \),

\[ c_{dii} = r_{d1} - \sum_{j=1}^{b} \frac{n_{dij}^2}{k_j} \text{ and } c_{dij}(i \neq j) = -\sum_{\ell=1}^{b} \frac{n_{dij} n_{dij}}{k_{ij}}. \]

We now take all permutations of the rows and (the same) columns except for the \( i \)th row. Averaging the resulting \( (v-1)! \) matrices we get \( \tilde{C}_d \). Without changing the eigenvalues of \( \tilde{C}_d \) we can interchange the first and \( i \)-th rows and columns. Then we have \( \tilde{C}_d \) equivalent to

\[
\begin{bmatrix}
    r_{d1} - \frac{b}{\sum_{j=1}^{k_i} \frac{n_{dij}^2}{k_j}} & -\frac{\beta_i}{v-1} \\
    -\frac{\beta_i}{v-1} & (A_i + B_i)I_{v-1} - B_iJ_{v-1, v-1}
\end{bmatrix}
\]

where \( \beta_i = \frac{1}{v-1} \sum_{j=1}^{b} \left( \frac{\sum_{\ell=1}^{k_{ij}} \frac{n_{dij} n_{dij}}{k_{ij}}}{n} \right). \) By Lemma 1.4.1 we know that one eigenvalue of \( \tilde{C}_d \) is \( v\beta_i \). Since \( \tilde{C}_d \) has zero-row sums along with \( C_d \),

\[ v\beta_i = \frac{v}{v-1} \left( r_{d1} - \frac{b}{\sum_{j=1}^{k_j} \frac{n_{dij}^2}{k_j}} \right). \]

Hence \( \mu_{d1} \leq \mu_{d1} \leq v\beta_i = \frac{v}{v-1} \left( r_{d1} - \frac{b}{\sum_{j=1}^{k_j} \frac{n_{dij}^2}{k_j}} \right). \)
Since $i$ was arbitrary, we get $\mu_{d,1}$ less than $\nu_{d,i}$ for each $i$ and hence the result.

(ii) Now we average over varieties $1, \ldots, n$ and $n+1, \ldots, v$ separately, for some $2 \leq n \leq v$. The averaged matrix $\bar{C}_d$ equals

$$
\begin{bmatrix}
(a_n + \alpha_n)I_n - \alpha_n J_{n,n} & -\beta_n J_{n,v-n} \\
-\beta_n J_{v-n,n} & (d_n + \alpha_n)I_{v-n} - \alpha_n J_{v-n,v-n}
\end{bmatrix}
$$

where $a_n = \frac{1}{n} \sum_{i=1}^{n} \left( r_{di} - \frac{b}{j=1} \frac{n_{dij}^2}{k_j} \right)$ and $\alpha_n = \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} \left( \sum_{k=1}^{b} \frac{n_{dik} n_{djk}}{k_k} \right)$.

By Lemma 1.4.1 one eigenvalue of $\bar{C}_d$ is $a_n + \alpha_n$, so we have $\mu_{d,1} \leq \nu_{d,1} \leq a_n + \alpha_n$. Therefore $\mu_{d,1} \leq \min_{2 \leq n \leq v} (a_n + \alpha_n)$ and we get the result.

This ends the proof.

Remark 2.2.1: Part (i) of the theorem was proved by Kiefer (1958) and Chackraborty (1963) by another method.

Theorem 2.2.2: In any block design $d \in \mathcal{W}(v,b,k)$ with replications $r_d$ and $C_d$ having eigenvalues $0 = \mu_{d,0} \leq \mu_{d,1} \leq \cdots \leq \mu_{d,v-1}$ we have

(i) $\mu_{d,v-1} \geq \frac{v}{v-1} \max_{1 \leq i \leq v} \left( r_{di} - \frac{b}{j=1} \frac{n_{dij}^2}{k_j} \right)$

and (ii) $\mu_{d,v-1} \geq \max_{2 \leq n \leq v} \left\{ \frac{1}{n} \sum_{k=1}^{b} \left( r_{dk} - \frac{b}{j=1} \frac{n_{dkj}^2}{k_j} \right) + \frac{2}{n(n-1)} \sum_{1 \leq i < j \leq n} \left( \sum_{k=1}^{b} \frac{n_{dik} n_{djk}}{k_k} \right) \right\}$. 
Proof: The proof of (i) is analogous to the proof of (i) in Theorem 2.2.1. One eigenvalue of $\tilde{C}_d$ is $\frac{v}{v-1} \left( \frac{r_{d_i}}{n_{d_{ij}}^2} - \sum_{j=1}^{b} \frac{n_{d_{ij}}^2}{k_{j}} \right)$ except that here we use $\mu_{d,v-1} \geq \mu_{d,v-1} \geq \frac{v}{v-1} \left( \frac{r_{d_i}}{n_{d_{ij}}^2} - \sum_{j=1}^{b} \frac{n_{d_{ij}}^2}{k_{j}} \right)$ for each $i$. Hence $\mu_{d,v-1}$ must be bigger than the largest of the $(v/(v-1)\alpha_{d_{ij}}$. Similarly for the proof of (ii) we use $\mu_{d,v-1} \geq \mu_{d,v-1} \geq a_n + \epsilon_n$ of the $\tilde{C}_d$ given in the proof of (ii) of Theorem 2.2.1.

2.3 Uniform Optimality and Unequal Block Sizes

The following result extends that of Kiefer (1975), given earlier as Corollary 1.4.4, to unequal $k_j$. Hedayat (1974) had extended the result to unequal $k_j < v$, but here no such restriction is made. All three proofs are analogous.

Theorem 2.3.1: (Corollary to Theorem 1.4.4)

Suppose there exists a connected variance balanced block design $d^* \in \mathcal{B}(v,b,k)$ with parameters $N, v, b$ and $r^*_{d_j}$ satisfying for each block $j$

1) $n^*_{d_{ij}} = k_j/v$ when $k_j/v$ is an integer $(i = 1, \ldots, v)$, or

2) the $n^*_{d_{ij}}$'s $(i = 1, \ldots, v)$ are as close together as possible when $k_j/v$ is not an integer (that is $k_j - v \cdot \int[k_j/v]$ of the $n^*_{d_{ij}}$'s equal $\int[k_j/v]$ + 1 and the rest of the $n^*_{d_{ij}}$'s equal $\int[k_j/v]$.)

subject to $\sum_{i=1}^{v} n^*_{d_{ij}} = k_j$. Then $d^*$ is $\mathcal{D}$-optimal in $\mathcal{B}(v,b,k)$. 
Proof: By Definition 1.3.1 $C_{d^*}$ is completely symmetric, so all that remains is to prove $C_{d^*}$ is of maximum trace in $\mathcal{D}$. For an arbitrary design $d \in \mathcal{D}$,

$$\text{tr } C_d = \sum_{i=1}^{v} r_{di} - \sum_{i=1}^{v} \sum_{j=1}^{b} \frac{n_{dij}^2}{k_j} = N - \sum_{j=1}^{b} \frac{1}{k_j} \sum_{i=1}^{v} n_{dij}^2.$$  

Maximizing $\text{tr } C_d$ subject to $\sum_{i=1}^{v} n_{dij} = k_j$ for each $j$ is accomplished by making the $n_{dij} (i = 1, \ldots, v)$ as equal as possible for each fixed $j$, since this minimizes $\sum_{i=1}^{v} (n_{dij}^2)$ for each fixed $j$. (See Lemma 1.4.3). But this has been accomplished by $d^*$. Thus ends the proof.

Theorem 2.3.1 can be applied to the case of $v = 2$. This case is trivial but is given here for the sake of completeness.

Corollary 2.3.1: If $v = 2$ and $k_j \geq 2$ for $j = 1, \ldots, b$, then $d^* \in \mathcal{A}(2,b,k)$ is $\mathcal{A}_u$-optimal in $\mathcal{A}(2,b,k)$ if $n_{d^*1j} = c_j$ for $k_j = 2c_j$ ($k_j$ even) and $n_{d^*1j} = c_j$ or $c_j+1$ for $k_j = 2c_j+1$ ($k_j$ odd), where $c_j$ is a positive integer.

Proof: For any $d \in \mathcal{A}(2,b,k)$,

$$C_d = \begin{bmatrix} c_{d11} & c_{d12} \\ c_{d21} & c_{d22} \end{bmatrix} = \begin{bmatrix} c_{d11} & -c_{d11} \\ -c_{d11} & c_{d11} \end{bmatrix}$$

since $C_d$ is an information matrix. So $C_{d^*}$ is completely symmetric making $d^*$ variance balanced. $d^*$ is connected since $k_j \geq 2$ so treatments 1 and 2 are paired at least once in $d^*$.

Finally the $n_{d^*ij}$ are as equal as possible, so by Theorem 2.3.1 we are done.
2.4 An Eigenvalue Lemma

The following result gives the eigenvalues for an information matrix with three blocks down the diagonal. In a sense it extends Lemma 1.4.1 of Constantine for two blocks down the diagonal.

**Lemma 2.4.1:** The eigenvalues of the $v$ by $v$ information matrix

$$C = \begin{bmatrix}
(a_1+b_1)I_{m_1} - b_1 J_{m_1, m_1} & -c_{12} J_{m_1, m_2} & -c_{13} J_{m_1, m_3} \\
(a_2+b_2)I_{m_2} - b_2 J_{m_2, m_2} & (a_3+b_3)I_{m_3} - b_3 J_{m_3, m_3}
\end{bmatrix}
$$

(where $m_1 + m_2 + m_3 = v$ and $m_1, m_2, m_3 \geq 1$) are $0, (a_1+b_1), (a_2+b_2), (a_3+b_3), (A+(B_1^2+B_2)\frac{\lambda}{2})/2$ and $(A-(B_1^2+B_2)\frac{1}{2})/2$ with multiplicities $1, m_1-1, m_2-1, m_3-1, 1$ and $1$, respectively, where $A = (m_1+m_2)c_{12} + (m_1+m_3)c_{13} + (m_2+m_3)c_{23}$, $B_1 = (m_1-m_2)c_{12} - (m_1+m_3)c_{13} + (m_2+m_3)c_{23}$ and $B_2 = 4m_1m_2(c_{12}-c_{13})(c_{12}-c_{23})$.

**Proof:** The eigenvector for 0 is of course $1_v$. Let $\alpha_1, \ldots, \alpha_{m_1-1}, 1_{m_1}$ be pairwise orthogonal, $\beta_1, \ldots, \beta_{m_2-1}, 1_{m_2}$ be pairwise orthogonal and $\gamma_1, \ldots, \gamma_{m_3-1}, 1_{m_3}$ be pairwise orthogonal. Then $(a_i+b_i)$ $i = 1, 2, 3$ have the eigenvectors

$$\begin{pmatrix}
\alpha_j \\
\beta_j \\
\gamma_j
\end{pmatrix}, \quad j = 1, \ldots, m_1-1; \quad \begin{pmatrix}
\alpha_j \\
\beta_j \\
\gamma_j
\end{pmatrix}, \quad \lambda = 1, \ldots, m_2-1; \quad \begin{pmatrix}
\alpha_j \\
\beta_j \\
\gamma_j
\end{pmatrix}, \quad \lambda = 1, \ldots, m_3-1.$$
and 

\( \begin{pmatrix} m_1,1 \\ m_2,1 \\ y_p \end{pmatrix} \) 

\( p = 1, \ldots, m_3 - 1 \) respectively. If \( m_i = 1 \), we do not get the eigenvalue \( (a_i + b_i) \) and of course also do not have the corresponding eigenvectors.

The last two eigenvalues remain the same if \( c_{12}, c_{13}, \) and \( c_{23} \) are distinct \( c_{12} = c_{13} \), or \( c_{12} = c_{13} = c_{23} \). The proofs in the three cases differ, but if \( c_{12}, c_{13} \) and \( c_{23} \) are distinct the proof is as follows.

By Lemma 1.4.4 if \( C \) has eigenvalues \( 0 \leq \mu_1 \leq \ldots \leq \mu_{v-1} \) and corresponding eigenvectors \( \xi_0, \xi_1, \ldots, \xi_{v-1} \) then \( F = aC + bI_v + (c/v)J_{v,v} \) has the eigenvalues \( b + c, a\mu_1 + b, \ldots, a\mu_{v-1} + b \) with the same corresponding eigenvectors. This method was used to get the eigenvalues \( (a_i + b_i) \) and 0, and leaves

\[
\begin{bmatrix}
1 - m_1 \\
y_j - m_2 \\
-\left( \frac{m_1}{m_3} + \frac{m_2}{m_3} y_j \right) - m_3
\end{bmatrix}
\]

\[= \lambda_j \begin{bmatrix} 1 - m_1 \\ y_j - m_2 \\ -\left( \frac{m_1}{m_3} + \frac{m_2}{m_3} y_j \right) - m_3 \end{bmatrix} = \lambda_j g_j (i = 1, 2, \ldots)
\]

to be solved for the eigenvalues of \( C + c_{13}J_{v,v}(a_1 + c_{13})I_v \) with their corresponding eigenvectors \( g_j \). We get the following three equations, with the last a linear combination of the first two.
\[(m_1-1)(c_{13}-b_1) + m_2y_j(c_{13}-c_{12}) = \lambda_j \]  \hspace{1cm} (3.4.1)

\[m_1(c_{13}-c_{12}) + (a_2-a_1)y_j + (m_2-1)(c_{13}-b_2)y_j \]

\[-m_3(c_{13}-c_{23})(\frac{m_1}{m_3} + \frac{m_2}{m_3}y_j) = \lambda_jy_j \]

\[m_2y_j(c_{13}-c_{23})-(a_3-a_1)(\frac{m_1}{m_3} + \frac{m_2}{m_3}y_j) \]

\[-(m_3-1)(c_{13}-b_3)(\frac{m_1}{m_3} + \frac{m_2}{m_3}y_j) = -\lambda_j(\frac{m_1}{m_3} + \frac{m_2}{m_3}y_j). \]

Substituting the value of \(\lambda_j\) from (3.4.1) into the second equation we get

\[y_j^2m_2(c_{12}-c_{13}) + y_j(a_2-a_1-(m_2-1)b_2 + (m_1-1)b_1 + m_2c_{23} - m_1c_{13}) \]

\[+m_1(c_{23}-c_{12}) = 0. \]

Solving for \(y_j\), plugging back into (3.4.1), and solving for \(\lambda_j + (a_1 + c_{13})\) which are the eigenvalues of \(C\) (recall they have the same eigenvectors) gives \((A \pm (B_1^2 + B_2^2)^{\frac{1}{2}})/2\) for the last two eigenvalues of \(C\).

**Remark 2.4.1:** An analogous result for four blocks down the diagonal has three of the eigenvalues as the solutions of a cubic equation and did not prove useful.
CHAPTER 3
THE BIBD PLUS OR MINUS M BLOCKS

3.1 The Eigenvalue Structures of Cheng and this Thesis

In this section we present two lemmas that give the eigenvalues for the C-matrices of designs constructed by adding disjoint binary blocks to a BIBD or removing disjoint (binary) blocks from a BIBD.

Lemma 3.1.1: When adding m disjoint binary blocks of size k to a BIBD d in \( \mathcal{D}(v, b, k) \) with \( 1 \leq m \leq v/k \), the eigenvalues of the C-matrix of the resulting design \( d^* \) in \( \mathcal{D}(v, b+m, k) \) are

\[
\begin{align*}
0 & \text{ with multiplicity 1}, \\
\frac{v\lambda}{k} & \text{ with multiplicity } v-m(k-1)-1, \text{ and} \\
\frac{v\lambda}{k} + 1 & \text{ with multiplicity } m(k-1).
\end{align*}
\]

In particular \( C_{d^*} \) has the eigenvalue structure \( 0 < \frac{(v\lambda)}{k} < \frac{(v\lambda)}{k} + 1 = \ldots = \frac{(v\lambda)}{k} + 1 \) when \( m = 1 \) and \( k = v-1 \) \((v \geq 3)\) or when \( m = 2 \) and \( k = v/2 \) \((v \geq 4)\). \( C_{d^*} \) has the eigenvalue structure \( 0 < \frac{(v\lambda)}{k} = \ldots = \frac{(v\lambda)}{k} < \frac{(v\lambda)}{k} + 1 \) when \( m = 1 \) and \( k = 2 \) \((v \geq 3)\).

Proof: \( C_{d^*} = C_d + C_0 \) where

\[
C_d = \left(\frac{r(k-1)}{k} + \frac{\lambda}{k}\right)I_v - \frac{\lambda}{k}J_{v,v}
\]

and \( C_0 \) is the C-matrix for the blocks added. Without loss of
generality we can relabel the treatments 1 through v so that \( C_0 \) is of the form

\[
\begin{bmatrix}
A_1 & \mathcal{O}_{k,k} & \cdots & \mathcal{O}_{k,k} & \mathcal{O}_{k,v-km} \\
A_2 & \cdots & \mathcal{O}_{k,k} & & \mathcal{O}_{k,v-km} \\
& \cdots & \cdots & \cdots & \cdots \\
& & \cdots & \cdots & \mathcal{O}_{k,v-km} \\
& & & \mathcal{O}_{v-km,v-km} \\
\text{sym} & & & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
& & & & \\
\end{bmatrix}
\]  

(3.1.1)

where \( A_i = I_k - k^{-1}J_{k,k} \) for \( i = 1, \ldots, m \). Then \( C_{d^*} \) is of the form

\[
\begin{bmatrix}
B_1 & C & \cdots & C & D \\
& B_2 & C & \cdots & D \\
& & \cdots & \cdots & \cdots \\
& & & B_m & D \\
\text{sym} & & & & B_{m+1}
\end{bmatrix}
\]

(3.1.2)

where

\[
B_i = \left( \frac{(r+1)(k-1)}{k} + \frac{\lambda+1}{k} \right)I_k - \frac{\lambda+1}{k} J_{k,k} \quad i = 1, \ldots, m,
\]

\[
B_{m+1} = \left( \frac{r(k-1)}{k} + \frac{\lambda}{k} \right)I_{v-km} - \frac{\lambda}{k} J_{v-km,v-km^*}
\]

and

\[
C = (-\frac{\lambda}{k})J_{k,k} \text{ and } D = (-\frac{\lambda}{k})J_{v-km,v-km^*}.
\]

Note here that if \( v = mk \) then \( C_0 \) and \( C_{d^*} \) are of the form, respectively,

\[
\begin{bmatrix}
A_1 & \mathcal{O}_{k,n} & \cdots & \mathcal{O}_{k,k} \\
& A_2 & \cdots & \mathcal{O}_{k,k} \\
& & \cdots & \cdots \\
& & & \mathcal{O}_{k,k} \\
\text{sym} & & & & A_m
\end{bmatrix}
\]

(3.1.3)

and
\[
\begin{bmatrix}
B_1 & C & \cdots & C \\
B_2 & \cdots & C \\
sym & \cdots & B_m
\end{bmatrix}
\]  \hspace{1cm} (3.1.4)

From Lemma 1.4.2 the eigenvalues of \( C_d^* \) are \((v\lambda/k)\pm 1, v\lambda/k\) and 0 with the multiplicities given in the hypothesis. \( m(k-1) = v-2 \) for \( m = 2, v = 2k \) and \( m = 1, k = v-1 \), and \( m(k-1) = 1 \) for \( m = 1, k = 2 \).

**Lemma 3.1.2:** When deleting \( m \) disjoint (binary) blocks of size \( k \) from a BIBD \( \mathcal{D}(v,b,k) \) with \( 1 \leq m \leq v/k \), the eigenvalues of the C-matrix of the resulting design \( \mathcal{D}^* \) in \( \mathcal{D}(v,b-m,k) \) are

- 0 with multiplicity 1,
- \( \frac{v\lambda}{k} - 1 \) with multiplicity \( m(k-1) \), and
- \( \frac{v\lambda}{k} \) with multiplicity \( v-m(k-1)-1 \).

In particular \( C_{d^*} \) has eigenvalue structure \( 0 < (v\lambda/k)-1 < (v\lambda/k) = \ldots = (v\lambda/k) \) when \( m = 1 \) and \( k = 2 \) (\( v \geq 3 \)) and \( C_{d^*} \) has eigenvalue structure \( 0 < (v\lambda/k)-1 = \ldots = (v\lambda/k)-1 < (v\lambda/k) \) when \( m = 1, k = v-1 \) (\( v \geq 3 \)) and when \( m = 2, v = 2k \) (\( v \geq 4 \)).

**Proof:** \( C_{d^*} = C_d - C_0 \) where \( C_d \) is as shown in the proof of Lemma 3.1.1. Without loss of generality we can relabel the treatments so that \( C_0 \) is as shown at (3.1.1).

Then \( C_{d^*} \) is of the form of (3.1.2) but now with

\[
B_i = \left( \frac{(r-1)(k-1)}{k} + \frac{\lambda-1}{k} \right) I_k - \frac{\lambda-1}{k} J_{k,k}, \quad i = 1, \ldots, m, \text{ and } B_{m+1},
\]

\( C \) and \( D \) still of the form given in (3.1.2). As before if \( v = mk \) then
$C_0$ and $C_{d^*}$ are as given in (3.1.3) and (3.1.4) respectively, but with $B_1$ through $B_m$ as defined here.

From Lemmas 1.4.2 the eigenvalues of $C_{d^*}$ are $(v\lambda/k)+1$, $v\lambda/k$ and 0 with the multiplicities given in the hypothesis. The rest of the proof follows easily.

The BIBD plus 1 block for $k = 2$, the BIBD minus 1 block for $k = v-1$, and the BIBD minus 2 blocks for $k = v/2$ give Cheng's eigenvalue structure. The first two cases were proved E-optimal by Constantine (1981) and are shown to be $\mathcal{J}_1$-optimal in Section 3.2. The third case is just that of a MB GD PBIBD with $\lambda_1 = \lambda_1+1$. This was proved $\mathcal{J}_1$-optimal by Cheng (1978). See Theorem 1.4.2.

The BIBD plus 1 block for $k = v-1$, the BIBD plus 2 blocks for $k = v/2$, and the BIBD minus 1 block for $k = 2$ yield the eigenvalue structure $0 < a < b = \ldots = b$. The first was shown to be E-optimal by Constantine (1981) and will be shown in Section 3.3 to be $\mathcal{J}_4$-optimal. The second and third cases will also be discussed in that section, though the former is not necessarily optimal and the latter does not yet have a full solution.

3.2 The Eigenvalue Structure $0 < a = \ldots = a < b$ and $\mathcal{J}_1$-optimality

We will begin with the BIBD plus 1 block, $k = 2$.

**Theorem 3.2.1:** Let $d^0$ be a BIBD in $\mathcal{B}(v,b,2)$, $v \geq 3$. Add to $d^0$ one binary block of size 2 and call the resulting design $d^*$. Then $d^*$ is $\mathcal{J}_1$-optimal in $\mathcal{B}(v,b+1,2)$. 
Proof: We will apply Theorem 1.4.1. Since \( k = 2 \), \( d^0 \) is the union of \( e \) copies (\( e \geq 1 \)) of the BIBD in \( \mathcal{B}(v,v(v-1)/2,k) \) that has the \( v(v-1)/2 \) unique pairs of the \( v \) treatments as blocks. Therefore \( C_{d^0} = (e/2)(vI_v - J_v,v) \). Without loss of generality assume we add the block consisting of treatments 1 and 2. Then

\[
C_{d^*} = \frac{1}{2} \begin{bmatrix}
  e(v-1)+1 & -(e+1) & -e & -e & \ldots & -e \\
-(e+1) & e(v-1)+1 & -e & -e & \ldots & -e \\
- & - & \ddots & \ddots & \ddots & \ddots \\
\end{bmatrix}
\]

From Lemma 3.1.2 and the fact that \( \lambda = e \) for the BIBD \( d^0 \), or directly from Lemma 1.4.1, we have for \( C_{d^*} \): \( \mu_{d^*,0} = 0 \), \( \mu_{d^*,1} = \ldots = \mu_{d^*,v-2} = ev/2 \), and \( \mu_{d^*,v-1} = (ev/2)+1 \).

Since \( k = 2 \) then from what was said in Section 1.5, \( d^* \) satisfies condition (i) of Theorem 1.4.1 trivially.

Now \( \text{tr } C_{d^*} = e(v-1)+1 + e(v-1)(v-2)/2 = (ev(v-1)+2)/2 \) and

\[
\text{tr}(C_{d^*}^2) = \sum_{i=1}^{v} \sum_{j=1}^{b} c_{d^*}^2_{ij}
\]

\[
= \frac{2}{4} [(e(v-1)+1)^2+(e+1)^2+(v-2)e^2] + \frac{(v-2)}{4} [e^2(v-1)^2+(v-1)e^2]
\]

\[
= \frac{1}{4} (e^2v^2(v-1)+4ve+4).
\]

For condition (ii) of Theorem 1.4.1 we need to check to see if
\((v-2) \text{tr}(C_d^2) < (\text{tr} C_d^*)^2\), or if

\[
\frac{(v-2)}{4} \left( e^2 v^2 (v-1) + 4 ev + 4 \right) < \frac{1}{4}(ev(v-1)+2)^2 = \frac{1}{4}(e^2 v^2 (v-1)^2 + 4 ev(v-1)+4).
\]

This is true if and only if

\[(v-2)(e^2 v^2 (v-1) + 4 ev + 4) < e^2 v^2 (v-1)^2 + 4 ev(v-1)+4,
\]

if and only if

\[e^2(v^4 - 3v^3 + 2v^2) + e(4v^2 - 8v) + 4v - 8 < e^2(v^4 - 2v^3 + v^2) + e(4v^2 - 4v)+4,
\]

if and only if

\[0 < e^2(v^3 - v^2) + e(4v) + 12 - 4v = e^2(v^3 - v^2) + 4v(e-1) + 12,
\]

which is true as \(e \geq 1\) and \(v \geq 3\).

Now since every design in \(\mathcal{O}(v,b+1,2)\) has the same trace, condition (iii) of Theorem 1.4.1 is equivalent to proving \(C_d^*\) minimizes

\[\text{tr}(C_d^2) - (\text{tr} C_d^*)^2/(v-1)\]

or minimizing \(\text{tr}(C_d^2) = \sum_{i=1}^{v} c_{dii}^2 + 2 \sum_{1 \leq i < j \leq v} c_{dij}^2\).

From Lemma 1.4.3 this is accomplished by making the \(c_{dii}\) as equal as possible and the \(c_{dij}\) as equal as possible. Clearly since \((-2c_{dij})\) must be an integer, making \((-2c_{dij}) = m+1, (-2c_{dij}) = m\) for \(i = 1,2\) and \(j = 3,...,v\) and \((-2c_{dij}) = m\) for \(3 \leq i < j \leq v\) does just that for the \(c_{dij}\)'s. In the case of \(k = 2\), \(2c_{dii}\) is an integer, so making \(2c_{d11} = 2c_{d22} = m(v-1)+1\) and \(2c_{dii} = m(v-1)\) for \(i = 3,4,...,v\) makes the \(c_{dii}\) as equal as possible. Hence \(\text{tr}(C_d^2)\) is minimized by \(d^*\) and we satisfy condition (iii).
Therefore by Theorem 1.4.1 $d^*$ is $g_{\omega_1}$-optimal, and the proof is complete.

For an example of such a design use the design in Example 1.4.1 but without the last block.

**Theorem 3.2.2:** Let $d^0$ be a BIBD in $\mathcal{B}(v,b,v-1)$, $v \geq 3$. Remove from $d^0$ one (binary) block and call the resulting design $d^*$. Then $d^*$ is $g_{\omega_1}$-optimal in $\mathcal{B}(v,b-1,v-1)$.

**Proof:** Again we will apply Theorem 1.4.1. Since $k = v-1$, $d^0$ is the union of $e$ copies ($e \geq 1$) of the BIBD in $\mathcal{B}(v,v,v-1)$ that has the $v$ subsets of size $v-1$ from $\{1, \ldots, v\}$ as blocks. Therefore $b = ev$ and

$$\frac{C_{d^0}}{v-1} = \begin{bmatrix} e(v-1)(v-2) & -e(v-2) & \cdots & -e(v-2) \\ \vdots & \ddots & \ddots & \vdots \\ e(v-1)(v-2) & \cdots & -e(v-2) & \end{bmatrix}$$

That is $r_{d^0} = \ldots = r_{d^0_v} = e(v-1)$, $\lambda_{d^0_{ij}} = e(v-2)$ for $1 \leq i < j \leq v$ and $k-1 = v-2$. We then have $C_{d^*} = C_{d^0} - C_0$ where $C_0$ is the $C$-matrix for the removed block. Without loss of generality

$$C_0 = \frac{1}{v-1} \begin{bmatrix} (v-2) & -1 & \cdots & -1 & 0 \\ (v-2) & \cdots & -1 & 0 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ \vdots & \cdots & (v-2) & 0 \\ \text{sym} & \cdots & \end{bmatrix}$$

and then
\[
C_{d^*} = \frac{1}{v-1} \begin{bmatrix}
(e(v-1)-1)(v-2) & -(e(v-2)-1) & \ldots & -(e(v-2)-1) & -e(v-2) \\
(e(v-1)-1)(v-2) & -(e(v-2)-1) & \ldots & -(e(v-2)-1) & -e(v-2) \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
(e(v-1)-1)(v-2) & -(e(v-2)-1) & \ldots & -(e(v-2)-1) & -e(v-2) \\
\end{bmatrix}
\]

The eigenvalues of \( C_{d^*} \) are 0, \((v\lambda/k)-1\) and \((v\lambda/k)\) with multiplicities 1, \(v-2\) and 1 respectively, where \( \lambda = e(v-2) \) and \( k = v-1 \). So \( d^* \) satisfies the eigenvalue structure of Theorem 1.4.1.

The condition (i), \( C_{d^*} \) being of full trace, is satisfied since \( d^* \) is binary. That is for each \( j \), \( n_{d^*ij} \) is 0 or 1 for \( i = 1, \ldots, v \).

We must now check condition (ii), that \((v-2)\text{tr}(C_{d^*}^{-2}) < (\text{tr} C_{d^*})^2\).

Well

\[
\text{tr} C_{d^*} = \frac{1}{v-1} \left( (v-1)(e(v-1)-1)(v-2) + e(v-1)(v-2) \right)
= (v-2)(ev-1)
\]

\[
\text{tr}(C_{d^*}^{-2}) = \left( \frac{1}{v-1} \right)^2 [(v-1)(e(v-1)-1)^2(v-2)^2 + (v-2)(e(v-2)-1)^2 + e^2(v-2)^2]
+ (v-1)e^2(v-2)^2 + e^2(v-1)^2(v-2)^2
\]

\[
= \frac{1}{v-1} \left[ (v-2)^2(ev-e-1)^2 + (v-2)(ev-2e-1)^2 + 2e^2(v-2)^2 + e^2(v-1)(v-2)^2 \right].
\]

Condition (ii) holds if and only if \( \text{tr}(C_{d^*}^{-2}) < (v-2)(ev-1)^2 \) or

\((v-2)(ev-e-1)^2 + (ev-2e-1)^2 + 2e^2(v-2)^2 + e^2(v-1)(v-2) < (v-1)(ev-1)^2\).

With algebra we get the equivalent statement \( 0 > -ev(ev-2) \). This is true since \( v \geq 3 \) and \( e \geq 1 \).
The proof that condition (iii) holds will be more difficult than in Theorem 3.2.1. We begin by defining \( P_d = \text{tr}(C_d^2) - (\text{tr } C_d)^2/(v-1) \) for a design \( d \). Then condition (iii) asks if \( C_{d*} \) maximizes

\[
\text{tr } C_d - [(v-1)/(v-2)]^{\frac{1}{3}} P_d^{\frac{1}{3}}.
\]

(3.2.3)

Now \( (v-1)P_d = (v-2)\sum_{i=1}^{v} c_{dii}^2 + 2(v-1) \sum_{1 \leq i < j \leq v} c_{dij}^2 - 2 \sum_{1 \leq i < j \leq v} c_{dii} c_{djj} \).

For \( \sum_{i=1}^{v} c_{dii} \) fixed, Lemma 1.4.3 says \( (v-1)P_d \) is minimized, or (3.2.3) maximized, with the \( c_{dii} \) and the \( c_{dij} \) as equal as possible respectively.

Now \( (v-1)P_{d*} = v-2 \), which follows from simple substitution and some algebra. Hence \( \text{tr } C_{d*} - [(v-1)/(v-2)]^{\frac{1}{3}} P_{d*}^{\frac{1}{3}} = (v-2)(e-1)-1 \).

Clearly then for \( d \in \mathcal{D}(v,b-1,v) \) with \( \text{tr } C_d \leq (v-2)(e-1)-1 \) then (iii) holds for those \( d \) and \( d* \) because by Lemma 1.4.3 \( P_d \geq 0 \).

Therefore all we need to do is consider designs \( d \) with \( \text{tr } C_d = \text{tr } C_{d*} - \alpha/(v-1), \alpha = 0,1,2,\ldots,v-2 \).

For \( \alpha = 0 \), or \( d \) with the same trace as \( d* \), we see \( d* \) maximizes (3.2.3) since the \( c_{d*ii} \) are as equal as possible and the \( c_{d*ij} \) are as equal as possible. If \( \text{tr } C_d = \text{tr } C_{d*} \), \( d \) must also be binary, so \( c_{dii} = r_{dii}(k-1)/k = r_{dii}(v-2)/(v-1) \). So the \( c_{dii} \) are as equal as possible if and only if the \( r_{dii} \) are as equal as possible, which is true for \( d* \). If \( (v-2) \sum_{1 \leq i < j \leq v} c_{dij} = \text{tr } C_{d*} = (v-2)(e-1) = \)

\[
\frac{ev(v-1)(v-2)-(v-1)(v-2)}{v-1} = (v-1)(v-2)(e(v-2)-1)
\]

\[+ 2(v-1) \cdot \frac{e(v-2)}{v-1}\]
and \((-v-1)c_{dij}\) is always an integer, then the \(c_{d*ij}\) are as equal as possible.

The difficulty then is when \(\alpha = 1, 2, \ldots, v-2\). Here reducing \(\text{tr } C_d\) a little may minimize \(P_d\) enough to make condition (iii) fail. We will now fix \(\alpha \in \{1, \ldots, v-2\}\) and let \(d\) be arbitrary in the class \(\mathcal{A}(\alpha) \subset \mathcal{A}(v, b-1, v-1)\) where \(\mathcal{A}(\alpha) = \{d: \text{tr } C_d = \text{tr } C_{d*} - \alpha/(v-1)\}\). (3.2.3) will still be maximized (for the fixed trace) if the diagonal and off-diagonal elements of \(C_d\) are as equal as possible for all \(d \in \mathcal{A}(\alpha)\). Actually we will work with a \(C_{d_\alpha}\) which may not correspond to a design \(d \in \mathcal{A}(\alpha)\), but satisfies this smoothness requirement. The hypothetical design, which we will designate \(d_\alpha\), will have the \(C\)-matrix that maximizes (3.2.3) over \(\mathcal{A}(\alpha)\), and (3.2.3) evaluated for \(d_\alpha\) will be less than (3.2.3) evaluated with \(d^*\).

\(d_\alpha\) will have \((v-1)(v-2) + \alpha\ c_{dij}\)'s equal to \((e(v-2)-1)/(v-1)\) and \(2(v-1) - \alpha\ c_{dij}\)'s equal to \(e(v-2)/(v-1)\). Note that for any real design \(\alpha\) would have to be even, but for the purposes of the proof it does not matter. The question of how equal the diagonal elements can be in \(\mathcal{A}(\alpha)\) remains to be answered.

First note that \(\text{tr } C_d\) for \(d \in \mathcal{A}(\alpha)\) is \((v-1)^{-1}\) less than \(\text{tr } C_{d*}\) because the design is no longer binary. \(\text{tr } C_d = \sum_{i=1}^{v} r_{di} - \frac{1}{(v-1)} \sum_{i=1}^{v} \sum_{j=1}^{b} n_{dij}^2\) and we cannot reduce \(\sum_{i=1}^{v} r_{di}\), so the \(n_{dij}\)'s no longer satisfy \(|n_{dij} - (v-1)/v| < 1\) for all \((i, j)\).

Secondly we note that for \(d \in \mathcal{A}(\alpha)\), \(r_{di}(v-1) - \sum_{j=1}^{b} n_{dij}^2\)

\[\leq r_{di}(v-1) - r_{di} = r_{di}(v-2) \leq (r_{di}+1)(v-1) - (r_{di}+1) - \alpha\]
Therefore given \( r_{di} < r_{dj} \), then \((v-1)c_{di} \leq (v-1)c_{djj}\) even if we maximize \( c_{di} \) and minimize \( c_{djj} \) since \( 1 \leq \alpha \leq v-2 \).

We claim that the \( c_{di} \) are most equal in \( \mathcal{A}(\alpha) \) if \( v-1 \) of the
\[
((v-1)c_{di}) = (r-1)(v-1)-(r-1) = (r-1)(v-2)
\]
and the remaining
\[
((v-1)c_{d1}) = r(v-1)-r-\alpha = r(v-2)-\alpha,
\]
where \( r = r_{d1}^o = \ldots = r_{dv}^o e(v-1) \). Let an arbitrary \( d \in \mathcal{A}(\alpha) \) have \( r_{d1} \leq r_{d2} \leq \ldots \leq r_{dv} \), \( r_{d*} = \ldots = r_{d*}, v-1 = r-1 \), \( r_{d*v-1} = r \) is the most equal configuration of the \( r_{di} \), and is the one used in the claim. Also let \( \alpha_i \geq 0 \) be such that \( \sum_{i=1}^{v} \alpha_i = \alpha \). Then \( d \) has \((v-1)c_{di}) = r_{di}(v-1)-\sum_{j=1}^{v} n_{ij} = r_{di}(v-1)-r_{di} \)
\[-\alpha_i = r_{di}(v-2)-\alpha_i, \ i = 1, \ldots, v. \]

Now
\[
(r_{dv}(v-2)-\alpha, \ldots, r_{d1}(v-2)-\alpha)
\]
\[
> (r_{dv}(v-2)-\alpha, (r-1)(v-2), \ldots, (r-1)(v-2)) = (r_{d*}, (v-2), \ldots, r_{d*v-1}(v-2)).
\]
Therefore \( d_{\alpha} \) makes the \( c_{di} \) as equal as possible and maximizes (3.2.3) for all \( d \in \mathcal{A}(\alpha) \). So all that remains is to compare (3.2.3) for \( d^* \) and \( d_{\alpha} \).

\[
(v-1)p^d_{\alpha} = (v-2) \left[ (v-1) \frac{(r-1)^2(v-2)^2}{(v-1)^2} + \frac{(r(v-2)-\alpha)^2}{(v-1)^2} \right] \]
\[
- 2 \left[ \frac{(v-1)(v-2)}{2} \frac{(r-1)^2(v-2)^2}{(v-1)^2} + (v-1) \frac{(r-1)(v-2)(r(v-2)-\alpha)}{(v-1)^2} \right] \]
\[
+ (v-1) \left[ ((v-1)(v-2)+\alpha) \frac{(e(v-2)-1)^2}{(v-1)^2} + (2(v-1)-\alpha) \frac{e^2(v-2)^2}{(v-1)^2} \right] \]
\[
= (v-2) + \frac{\alpha^2(v-2)}{(v-1)^2} - \frac{2\alpha}{(v-1)} + \frac{5\alpha}{(v-1)} \quad (3.2.4)
\]
with algebra and substitution of \( e(v-1) \) for \( r \).

\[
\text{tr } C_{d^*} - \left[ \frac{(v-1)/(v-2)}{2} \right] \frac{1}{\gamma} p_{d^*} \frac{1}{\gamma} = (v-1)(ev-1)-1 \\
\text{tr } C_{d_{\alpha}} - \left[ \frac{(v-1)/(v-2)}{2} \right] \frac{1}{\gamma} p_{d_{\alpha}} \frac{1}{\gamma} = (v-1)(ev-1)- \frac{\alpha}{v-1} - \left[ \frac{(v-1)/(v-2)}{2} \right] \frac{1}{\gamma} p_{d_{\alpha}} \frac{1}{\gamma}
\]

if and only if \( (v-1)p_{d_{\alpha}} \geq (v-2)(1-\alpha/(v-1))^2 \). That is if and only if

\[
(v-1)p_{d_{\alpha}} \geq (v-2) - \frac{2\alpha(v-2)}{(v-1)} + \frac{\alpha^2}{(v-1)^2}
\]

\[
= (v-2) + \frac{\alpha^2}{(v-1)^2} - \frac{2\alpha}{v-1} + \frac{4\alpha}{v-1}
\]

which is obvious from (3.2.4) and the fact that \( \alpha > 0 \).

Since \( \alpha \) was arbitrary in \( \{1,2,\ldots,v-2\} \) we have shown \( C_{d^*} \) maximizes (3.2.3) for all \( d \in \mathcal{D}(v,b-1,v-1) \). Therefore condition (iii) of Theorem 1.4.1 is satisfied and \( d^* \) is \( g_{\mathcal{A}_1} \)-optimal in \( \mathcal{D}(v,b-1,v-1) \).

**Example 3.2.1.** In \( \mathcal{D}(5,4,4) \) \( d^* \) given below is \( g_{\mathcal{A}_1} \)-optimal.

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
2 & 2 & 2 & 3 \\
3 & 3 & 4 & 4 \\
4 & 5 & 5 & 5 \\
\end{array}
\]

**3.3 The Eigenvalue Structure** \( 0 < a < b = \ldots = b \) and \( g_{\mathcal{A}_4} \)-optimality

We will begin with the case of the BIBD plus one block, \( k = v-1 \).

**Theorem 3.3.1:** Let \( d^0 \) be a BIBD in \( \mathcal{D}(v,b,v-1) \), \( v \geq 3 \). Add to \( d^0 \) one binary block of size \( v-1 \) and call the resulting design \( d^* \). Then \( d^* \) is \( g_{\mathcal{A}_4} \)-optimal in \( \mathcal{D}(v,b+1,v-1) \).
Proof: Here we will apply Theorem 2.1.1. From Lemma 3.1.1 we know the eigenvalues of $C_d$ are $0 < v\lambda/k < (v\lambda/k)+1 = \ldots = (v\lambda/k)+1$ where $k = v-1$ and $\lambda_{d^0_{ij}} = \lambda$ for $1 \leq i < j \leq v$. $d^*$ maximizes $\text{tr} \ C_d$ for all $d \in \mathcal{A}(v,b+1,v-1)$ because the blocks in $d^0$ and the block added to $d^0$ are binary. Finally, as mentioned in Section 1.4, Constantine (1981) in his Theorem 3.1 proved the E-optimality of the BIBD plus one binary block. Hence we have satisfied conditions (i) - (iii) of Theorem 2.1.1 and the proof is complete.

Example 3.3.1: In $\mathcal{A}(5,4,6)$ $d^*$ given below is $\mathcal{A}_4$-optimal.

$$
\begin{array}{cccccc}
1 & 1 & 1 & 1 & 2 & 1 \\
2 & 2 & 2 & 3 & 3 & 2 \\
3 & 3 & 4 & 4 & 4 & 3 \\
4 & 5 & 5 & 5 & 5 & 4 \\
\end{array}
$$

d*:

Another case for the eigenvalue structure $0 < a < b = \ldots = b$ is the BIBD plus 2 disjoint binary blocks with $v = 2k$, $(v \geq 4)$. Constantine (1981) referred to an example of Cheng (1979). There the BIBD(8,14,4,7,3) plus two disjoint binary blocks is A-, D- and E-bettered by an RGD that was given by John and Mitchell (1977), which is the union of the two cyclic designs generated by the blocks $(1,2,3,5)$ and $(1,2,3,6)$. However the BIBD(4,6e,2,3e,e), $e \geq 1$, plus two disjoint binary blocks is $\mathcal{A}_3$-optimal in $\mathcal{A}(4,6e+2,2)$. This was proved by Cheng (1979).

The final case for the eigenvalue structure of this thesis coming from Section 3.1 is that of the BIBD minus 1 block, $k = 2$. A BIBD
\( d^0 \in \mathcal{B}(v,b,2) \) must be the union of \( e (e \geq 1) \) copies of the BIBD in \( \mathcal{B}(v,v(v-1)/2,2) \), which contains all pairings of the treatments \( 1 \) through \( v \), so \( b = ev(v-1)/2 \).

Cheng (1981a) and Constantine (1983) proved the \( \mathcal{J}_2^- \) and \( \mathcal{J}_4^- \) optimality, respectively, of \( d^0 \) minus one block if \( e = 1 \). Both proofs were graph theoretic, and do not hold for \( e > 1 \).

Theorem 2.1.1 can be applied, and will be in some cases, but it is the proof of the \( E \)-optimality of \( d^0 \) minus one block that is difficult.

We begin a discussion and a series of lemmas that will become the proof that \( d^* \in \mathcal{B}(v,b-1,2) \), where \( d^* \) is \( d^0 \) minus one block, is \( E \)-optimal for \( v = 3,4,5, \) and \( 6 \). Then Theorem 2.1.1 will be applied in the form of Theorem 3.3.2.

Constantine (1981) proved the \( E \)-optimality of the BIBD minus \( m \) disjoint (binary) blocks for \( vk^{-2} \leq m \leq vk^{-1} \). If \( k = 2 \), \( v/4 \leq 1 \) for \( v = 3 \) and \( 4 \) only. Hence we will get \( \mathcal{J}_4^- \)-optimality for \( v = 3,4, e \geq 1 \).

\[ C_{d^0} = (e/2)(vI_v - \mathcal{J}_v, v) \], and so \( C_{d^*} \), without loss of generality, can be written as

\[
\frac{1}{2}
\begin{bmatrix}
(r-1) & -(\lambda-1) & -\lambda & -\lambda & \cdots & -\lambda \\
-(\lambda-1) & (r-1) & -\lambda & -\lambda & \cdots & -\lambda \\
\lambda & \lambda & \cdots & \lambda & \cdots & \lambda \\
\end{bmatrix}
\]

where \( r = e(v-1) \) and \( \lambda = e \), the replication and pairing parameters of \( d^0 \), respectively.
As in Constantine (1981), let $\mathcal{A}(v,b-1,2) = S_1 \cup S_2 \cup S_3$ where 

$S_1 = \{d \in \mathcal{A}: \text{min}(r_{d_1}) \leq r-2\}$, 
$S_2 = \{d \in \mathcal{A}: \text{min}(r_{d_1}) \geq r-1\}$, and 
$S_3 = \{d \in \mathcal{A}: \text{min}(r_{d_1}) \geq r-1$ 
and $\lambda_{ij} \geq \lambda$ for all $i,j$ such that $r_i = r_j = r-1$.

**Lemma 3.3.1:** If $d^* \in \mathcal{A}(v,b-1,2)$, $v \geq 3$, then $\phi_E(C_{d^*}) \leq \phi_E(C_d)$ for all $d \in S_1 \cup S_2$.

**Proof:** The proof of Constantine's (1981) Theorem 3.3 holds for $1 \leq m \leq v/k$ on $S_1 \cup S_2$. It is for $S_3$ that he needs $m \geq v/k^2$.

**Lemma 3.3.2:** If we let $m$ be the number of treatments with $r_{d_1} = r-1$ for $d \in S_3$, then $\phi_E(C_{d^*}) \leq \phi_E(C_d)$ for any $d$ with $m \geq v/2$. Assume $v \geq 3$.

**Proof:** Here we will apply Proposition 2.1c of Jacroux (1982), which says that if for all $i \neq j$ with $r_{d_i} = r_{d_j} = r-1$ we have 

$\lambda_{d_{ij}} \geq \lambda$, then

$$
\mu_{d_1} \leq \frac{v((r-1)(k-1)-(m-1)\lambda)}{(v-m)k} \quad (3.3.1)
$$

For $k = 2$ the right hand side of (3.3.1) is less than or equal to

$$
\mu_{d^*1} = (v\lambda/k) - 1 = (ev/2) - 1 \quad \text{when}
$$

$$
\frac{v[(e(v-1) - 1) - (m-1)e]}{2(v-m)} = \frac{v}{2(v-m)} (e(v-m) - 1)
$$

$$
\leq \frac{ev}{2} - 1.
$$

This is true when $ve(v-m)-v \leq ev(v-m)-2(v-m)$, if and only if $v \leq 2m$. This ends the proof.
Letting $S_3 = S_{3.1} \cup S_{3.2}$ where $S_{3.1} = \{d \in S_3: \ m < v/2\}$ and $S_{3.2} = \{d \in S_3: \ m \geq v/2\}$, then $d^*$ is E-optimal in $S_1 \cup S_2 \cup S_{3.2}$. So only the $d \in S_{3.1}$ remain.

**Lemma 3.3.3:** For $d \in \omega(v,b-1,2)$, $v \geq 5$, if $r_{di} = r-1$ and $r_{dj} = r+b$ then $\mu_{d1} \leq (ev/2)-1$ for $\lambda_{dij} \leq \lambda-a$, $a \geq (b+3)/2$ ($a,b$ are non-negative integers).

**Proof:** Without loss of generality we let $i = 1, j = 2$ and we average $C_d$ over the permutations of $\{1,2\}$, and $\{3,...,v\}$ separately. Then $\bar{C}_d = \frac{1}{2} \begin{bmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{bmatrix}$

(3.3.2)

where $C_{11} = (r+((b-1)/2)+\lambda-a)I_2 - (\lambda-a)J_{2,2}$. By Lemma 1.4.1 one eigenvalue of $\bar{C}_d$ is $(r+((b-1)/2)+\lambda-a)/2 = (ev/2)+((b-1)/4) - a/2$. But $\mu_{d1} \leq \mu_{d1} \leq \frac{ev}{2} + \frac{b-1}{4} - \frac{a}{2} \leq \frac{ev}{2} - 1$

if and only if $a \geq (b+3)/2$.

**Lemma 3.3.4:** For $d \in \omega(v,b-1,2)$, $v \geq 5$, if $r_{di} = r+x$ and $r_{dj} = r+y$ then $\mu_{d1} \leq (ev/2)-1$ for $\lambda_{dij} \leq \lambda-a$, $a \geq ((x+y)/2)+2$ ($x, y, a$ are nonnegative integers).

**Proof:** Without loss of generality let $i = 1, j = 2$ and average $C_d$ over the permutations of $\{1,2\}$ and $\{3,...,v\}$ separately. Again
$\tilde{C}_d$ is of the form of (3.3.2). From $C_{11}$ and Lemma 1.4.1 $\mu_{d1} \leq \mu_{d1} \leq (e\lambda/2) + ((x+y)/4) - a/2 \leq (e\lambda/2)-1$ if and only if $a \geq ((x+y)/2)+2$.

**Lemma 3.3.5:** For $d \in A(v,b-1,2)$, $v \geq 5$, if $r_{di} = r_{dj} = r-1$ then
$\mu_{d1} \leq (ev/2)-1$ for $\lambda_{dij} \geq \lambda+1$.

**Proof:** Without loss of generality let $i = 1$, $j = 2$ and $\lambda_{d12} = \lambda+a$, $a \geq 1$. Averaging over $C_d$ as in the two previous lemma's we get $\tilde{C}_d$ of the form (3.3.2) with $C_{11} = (r-1+a)I_{2}-(\lambda+a)J_{2,2}$, $C_{12} = -(\lambda-(a+1)/(v-2))J_{2,v-2}$ and $C_{22} = (r\lambda+(2+2a)/((v-2)(v-3)))I_{v-2} - (2+2a)/((v-2)(v-3))J_{v-2,v-2}$. One eigenvalue of $\tilde{C}_d$ via Lemma 1.4.1, is from $C_{12}$ and equals

$$\frac{v}{2} \left( \lambda - \frac{a+1}{v-2} \right) = \frac{ev}{2} - \frac{v(a+1)}{2(v-2)}.$$ 

This is no bigger than $(ev/2)-1$ (and bounds $\mu_{d1} \geq \mu_{d1}$) if and only if $(v(1+a))/(2v-4) \geq 1$ if and only if $a \geq (v-4)/4$ or $a \geq 1$ (since $a$ is an integer and $v \geq 5$).

**Lemma 3.3.6:** For $d \in A(v,b-1,2)$, $v \geq 5$, if $r_{di} = r+x$ and $r_{dj} = r+y$ then $\mu_{d1} \leq (ev/2)-1$ for $\lambda_{dij} \geq \lambda+a$ for $a \geq ((x+y)/2)+2-4/v$. ($x,y$, $a$ are nonnegative integers).

**Proof:** Follows as in Lemma 3.3.5 with $C_{12}$ yielding $\mu_{d1} \leq \mu_{d1} \leq \mu_{d1} \leq \frac{v}{2} \left( e - \frac{2a-(x+y)}{2(v-2)} \right) \leq \frac{ev}{2} - 1$

if and only if

$$a \geq \frac{x+y}{2} + 2 - \frac{4}{v}.$$
Lemma 3.3.7: For $d \in \mathcal{A}(v,b-1,2)$, $v \geq 5$, if $r_{di} = r-1$ and $r_{dj} = r+b$ then $\mu_{di} \leq (ev/2) - 1$ for $\lambda_{dij} \geq \lambda + a$ where $2a-b \geq 3-8/v$ ($a, b$ are nonnegative integers).

Proof: Follows as in Lemma 3.3.5 with $C_{12}$ yielding $\mu_{di} \leq \mu_{d1} \leq \frac{v}{2} (e - \frac{2a-b+1}{2(v-2)}) \leq \frac{ev}{2} - 1$ if and only if $2a-b \geq 3-8/v$.

Lemma 3.3.8: Let $d^* \in \mathcal{A}(5,10e-1,2)$ be a BIBD from $\mathcal{A}(5,10e,2)$ with one block removed. Then $d^*$ is E-optimal in $\mathcal{A}(5,10e-1,2)$.

Proof: Lemma 3.3.1 says we need only look at $d \in S_{3,1}$, and in fact we can eliminate consideration of all but a few of the $d \in S_{3,1}$ with Lemmas 3.3.2 through 3.3.7.

Lemma 3.3.2 with $v = 5$ implies $2 \leq m < 5/2$, or $m = 2$ of the $r_{di}$ will equal $r-1$. The rest will be equal to $r$.

Lemmas 3.3.1 and 3.3.5 say that if $r_{di} = r_{d2} = r-1$ then $\lambda_{d12} = \lambda$.

Lemmas 3.3.3 and 3.3.7 with $b = 0$ eliminate $\lambda_{dij} \leq \lambda - 3/2$ and $\lambda_{dij} \geq \lambda + .7$ for $i = 1, 2$ and $j = 3, 4, 5$. That is $\lambda_{dij}$ for those $i$ and $j$ will be $\lambda - 1$ or $\lambda$.

Lemmas 3.3.4 and 3.3.6 with $x = y = 0$ eliminate $\lambda_{dij} \leq \lambda - 2$ and $\lambda_{dij} \geq \lambda + 1.2$ for $i \neq j \in \{3, 4, 5\}$. That is if $r_{di} = r_{dj} = r$ then $\lambda_{dij} = \lambda - 1, \lambda$ or $\lambda + 1$.

With a little work it is easy to see only two possible cases remain, and we will call them $d_1$ and $d_2$. Without loss of generality we have
\[ 2C_{d_1} = \begin{bmatrix}
(r-1) & -\lambda & -\lambda & -\lambda & -(\lambda-1) \\
(r-1) & -\lambda & -\lambda & -\lambda & -(\lambda-1) \\
r & -(\lambda-1) & -(\lambda+1) & \\
r & -(\lambda+1) & s\text{ym} & \\
r & & & r
\end{bmatrix} \]

and
\[ 2C_{d_2} = \begin{bmatrix}
(r-1) & -\lambda & -\lambda & -\lambda & -(\lambda-1) \\
(r-1) & -\lambda & -(\lambda-1) & -\lambda & \ \\
r & -\lambda & -\lambda & \ \\
r & -(\lambda+1) & \ \\
r & & & r
\end{bmatrix} \]

Now if we think of \( C_{d_1} \) as a blocked matrix with \( C_{11} \) and \( C_{22} \) each being 2 by 2, then we can apply Lemma 2.4.1 to get the eigenvalues of \( C_{d_1} \). They are 0, \((5e-1)/2\) twice, and \((A \pm (B_1^2 + B_2^2)\sqrt{3})/4\) where 
\[ A = 4\lambda + 3(\lambda-1) + 3(\lambda+1) = 10\lambda, \quad B_1 = 0-3(\lambda-1)+3(\lambda+1) = 6 \]
\[ B_2 = 4\cdot2\cdot1\cdot(-1) = -16. \]
The last two eigenvalues are then \((5e \pm (5\sqrt{3})/2, \) and \((5e-(5\sqrt{3})/2 < (5e/2)-1 = \mu_{d+1}. \)

For \( C_{d_2} \) we average over the parameter sets \( \{1,2\}, \{3\} \) and \( \{4,5\} \) to get the blocked matrix
\[ \frac{1}{2} \bar{C}_{d_2} = \begin{bmatrix}
(r-1) & -\lambda & -(\lambda-\frac{1}{2}) & -(\lambda-\frac{1}{2}) \\
(r-1) & -\lambda & -(\lambda-\frac{1}{2}) & -(\lambda-\frac{1}{2}) \\
r & -\lambda & -\lambda & \frac{1}{2} \text{ym} & \\
r & -\lambda & -\lambda & \frac{1}{2} \text{ym} & \\
r & & & & r
\end{bmatrix} \]

Applying Lemma 2.4.1 we get the eigenvalues of \( \bar{C}_{d_2} \) to be 0,
\[(5e \pm 1)/2, \text{ and } (5e-1 \pm 1)/2 \text{ or } (5e-2)/2 \text{ and } 5e/2. \] 
\[(5e-2)/2 = (5e/2)-1 = \mu_{d^{*}l}.\]

Therefore \(d^{*}\) has \(\mu_{d^{*}l} \geq \mu_{dl}\) for all \(d \in S_{3,1}\) and hence for all \(d \in \phi(5,10e-1,2)\). This completes the proof.

For \(v = 6\) we need one more lemma.

**Lemma 3.3.9:** For \(d \in \phi(v,b-1,2), v \geq 5, \text{ if } r_{d1} = r_{d2} = r-1, \lambda_{d2} = \lambda, \ r_{d3} = r \text{ and } \lambda_{d13} = \lambda-1+a, \lambda_{d23} = \lambda-1+b \text{ then } \mu_{dl} \leq (ev/2)-1\) for \(a = b = 0 \text{ and } a+b > 4 (a,b \text{ are nonnegative integers}).\)

**Proof:** We average \(C_d\) over the treatment sets \(\{1,2\}, \{3\}\) and \(\{4,...,v\}\). 
\[
\bar{C}_d = \frac{1}{2} \begin{bmatrix}
(r-1) - \lambda & -(\lambda-1+(a+b)/2) & -\left(\lambda - \frac{a+b}{2(v-3)}\right) J_{1,v-3} \\
-\lambda & (r-1) & -(\lambda-1+(a+b)/2) & -\left(\lambda - \frac{a+b}{2(v-3)}\right) J_{1,v-3} \\
\text{sym} & \text{sym} & r & -\left(\lambda - \frac{a+b-2}{v-3}\right) J_{1,v-3} \\
& & & C_{33}
\end{bmatrix}
\]

The eigenvalues of \(\bar{C}_d\) are \(0, (ev-1)/2, (ev/2) + (a+b-1)/((v-3)(v-4))\) with multiplicity \(v-4\) and then two from Lemma 2.4.1 of the form 
\[(A \pm (B_1^2+B_2)B_3)/4.\]

\[
A = 3(\lambda-1 + \frac{a+b}{2}) + (2+v-3)(\lambda - \frac{a+b}{2(v-3)}) + (1+v-3)(\lambda - \frac{a+b-2}{v-3})
\]

\[
= 2v\lambda - 3 + \frac{2(v-2)}{v-3} - \frac{2(a+b)}{v-3}
\]
\[ B_1 = (\lambda - 1 + \frac{a+b}{2}) - (2 + v - 3)\left(\lambda - \frac{a+b}{2(v-3)}\right) + (1 + v - 3)(\lambda - \frac{a+b-2}{v-3}) \]
\[ = (v-1)/(v-3) \]
\[ B_2 = 4 \cdot 2 \cdot 1(\lambda - 1 + \frac{a+b}{2} - \lambda + \frac{a+b}{2(v-3)}) (\lambda - 1 + \frac{a+b}{2} - \lambda + \frac{a+b-2}{v-3}) \]
\[ = 8(\frac{(v-2)(a+b)}{2(v-3)} - 1)(\frac{(v-1)(a+b)}{2(v-3)} - \frac{v-1}{v-3}). \]

Now \((A-(B_1^2 + B_2^2))\leq (v\lambda/2)\leq 1\ if\ and\ only\ if\ (A-2v\lambda+4)^2 \leq B_1^2 + B_2^2.

With algebra we see this is true if and only if

\[(v-3)[(a+b)^2v-8(a+b)(v-2)+8] \geq 0.\]

But \(F(w) = 2w^2v-8w(v-2)+8 \geq 0\ for\ all\ v\ when\ w = 0 or w \geq 4.\)

Remark 3.3.1: \(\nu_{d_1} \leq (ev/2)-1\ in\ Lemma 3.3.9\ for\ a+b = 2, v = 5\ and\ a+b = 3, v = 5, 6, 7, 8, 9.\)

Lemma 3.3.10: Let \(d^* \in \mathcal{A}(6, 15e-1, 2)\ be\ a\ BIBD\ from\ \mathcal{A}(6, 15e, 2)\) minus one block. Then \(d^*\ is\ E\-optimal\ in\ \mathcal{A}(6, 15e-1, 2).\)

Proof: Lemma 3.3.1 says we need only look at \(S_3\). As in the proof of Lemma 3.3.8 we eliminate all but a few \(d \in S_3.\)

Lemma 3.3.2 with \(v = 6\ says\ that\ exactly\ two\ \(r_{d_1} = r-1.\ Without\ loss\ of\ generality\ let\ \(r_{d_1} = r_{d_2} = r-1.\)

Lemmas 3.3.1 and 3.3.5 force \(\lambda_{d_12} = \lambda.\)

Lemmas 3.3.3 and 3.3.7 with \(b = 0\ force\ \lambda_{d_{ij}} = \lambda-1\ or\ \lambda, i = 1, 2\) and \(j = 3, 4, 5, 6.\)

Lemmas 3.3.4 and 3.3.6 with \(x = y = 0\ force\ \lambda_{d_{ij}} = \lambda-1, \lambda\ or\ \lambda+1\ for\ i \neq j \in \{3, 4, 5, 6\}.\)
There are only two cases to consider at this point for the first two rows of \( C_d \). One case has the first two rows of \( 2C_d = \)
\[
\begin{bmatrix}
(r-1) & -\lambda & -(-\lambda-1) & -\lambda & -\lambda & -\lambda \\
-\lambda & (r-1) & -(-\lambda-1) & -\lambda & -\lambda & -\lambda
\end{bmatrix}
\]
This is eliminated by Lemma 3.3.9. Hence all that remains is the case where the first two rows of \( 2C_d \) look like
\[
\begin{bmatrix}
(r-1) & -\lambda & -(-\lambda-1) & -\lambda & -\lambda & -\lambda \\
-\lambda & (r-1) & -(-\lambda-1) & -\lambda & -\lambda & -\lambda
\end{bmatrix}
\]
Now there are only three choices for \( \lambda_{d34} : \lambda-1, \lambda, \lambda+1 \). \( \lambda-1 \)
will not yield a C-matrix as it forces \( \lambda_{d56} = \lambda-2 \). But \( \lambda_{d34} = \lambda, \lambda+1 \)
give possible designs. The only two cases will be called \( d_1 \) and \( d_2 \).

\[
C_{d_1} = \frac{1}{2} \begin{bmatrix}
(r-1) & -\lambda & -(-\lambda-1) & -\lambda & -\lambda & -\lambda \\
(r-1) & -\lambda & -(-\lambda-1) & -\lambda & -\lambda & -\lambda \\
r & -\lambda & -(-\lambda+1) & -\lambda & -\lambda & -\lambda \\
r & -\lambda & -(-\lambda+1) & -\lambda & -\lambda & -\lambda \\
sym & r & r & r & r & r
\end{bmatrix}
\]
No averaging technique works here using Lemma 1.4.1 or Lemma 2.4.1. However it turns out that \( z' = (1,-1,1,-1,1,-1) \) is an eigenvector of \( C_{d_1} \) and yields the eigenvalue \((r+\lambda-2)/2 = (e(v-1)+e-2)/2 = (6e/2)-1 = u_{d+1}\).
If we try averaging over \( \{1,2\} \), \( \{3,4\} \) and \( \{5,6\} \) we get a block matrix with \( C_{12} = -\left(\lambda - \frac{1}{2}\right)J_{2,2} \), \( C_{13} = -\lambda J_{2,2} = C_{23} \), \( C_{11}, C_{22} \) and \( C_{33} \) yield the eigenvalues \((6e-1)/2\), \((6e+1)/2\) and \(6e/2\) respectively, and of course have 0 also. Lemma 2.4.1 yields \( A = 12\lambda - 2, B_1 = 0, B_2 = 4 \) so the last two eigenvalues of \( \tilde{C}_{d_2} \) are \((12\lambda - 2 \pm 2)/4\) or \((6e/2) - 1\) and \(6e/2\). \((6e/2) - 1 \leq u_{d^*1}\) so we are done.

**Theorem 3.3.2:** Let \( d^0 \in \mathcal{B}(v, ev(v-1)/2, 2) \) be a BIBD. Let \( d^* \) be the design constructed from \( d^0 \) by removing one block from \( d^0 \). Then \( d^* \) is \( J_4 \)-optimal in \( \mathcal{B}(v, (ev(v-1)/2) - 1, 2) \) for \( v = 3, 4, 5 \) and 6.

**Proof:** \( d^* \) maximizes the trace in \( \mathcal{B} \) trivially since \( k = 2 \). The eigenvalues are of the form \( 0 < (ev/2) - 1 < (ev/2) = \ldots = (ev/2) \) by Lemma 3.1.2. E-optimality for \( v = 3 \) and 4 comes from Constantine (1981) as mentioned above. E-optimality for \( v = 5 \) and 6 comes from Lemmas 3.3.8 and 3.3.10, respectively. Therefore by Theorem 2.1.1 we are finished.

**Remark 3.3.2:** Actually the author has proved the E-optimality of these \( d^* \)'s for \( v = 7 \). In this case \( m \) of Lemma 3.3.2 can be 2 or 3, and so a few more lemmas were used, each being proved in a manner
similar to Lemma 3.3.9. About a dozen cases remained to be checked with most not yielding to Lemmas 1.4.1 or 2.4.1. The eigenvectors of these cases were found for $e = 1$ on MINITAB, and they generalized to $e \geq 1$. Each case had one eigenvalue bounded above by $(7e/2)-1$. Because of its tedious nature, this method was not extended to $v \geq 8$.

For the cases $v \geq 7$ and $e \geq 2$ the $A$-, $D$- and $E$-efficiencies were calculated. They may be seen in Chapter 8.
CHAPTER 4

THE BBD PLUS A BLOCK OF SIZE V-1

4.1 Preliminaries and the Cases k = v-1, v

Throughout Chapter 4 let \( d^0 \) and \( d^* \) be as follows. Let \( d^0 \) be a BBD in \( \mathcal{D}(v, b, k) \). Then \( C_{d^0} = (r - \frac{\lambda_0}{k} + \frac{\lambda_1}{k})I_v - \frac{\lambda_1}{k} J_{v, v} \) with \( r, \lambda_0 \) and \( \lambda_1 \) being the unique values of \( r \), \( \lambda_{d^0i} \), \( \lambda_{d^0ii} \) and \( \lambda_{d^0ij} \) (\( 1 \leq i < j \leq v \)) respectively. Recall that this design has maximum trace in \( \mathcal{D}(v, b, k) \) since all of its blocks are binary.

Let \( d^* \) be the design in \( \mathcal{D}(v, b+1, k^*) \) that has \( d^0 \) for its first \( b \) blocks and the treatments 1 through \( v-1 \) in the \((b+1)\)st block, where \((k^*)' = (k_1^b, v-1)\). Then \( C_{d^*} = C_{d^0} + C_0 \) where \( C_0 \) is the C-matrix for the added block. We have

\[
C_0 = \begin{bmatrix}
I_{v-1} - (v-1)^{-1} J_{v-1, v-1}, & \mathcal{O}_{v-1, 1} \\
\mathcal{O}_{1, v-1}, & 0
\end{bmatrix}
\]

and so \( c_{d^*ii} = r-(\lambda_0/k) + (v-2)/(v-1) \) for \( i = 1, \ldots, v-1 \), \( c_{d^*iv, v} = r-(\lambda_0/k) \), \( c_{d^*ij} = -((\lambda_1/k)-(v-1)^{-1}) \) for \( 1 \leq i < j \leq v-1 \) and \( c_{d^*i v} = -((\lambda_1/k) \) for \( i = 1, \ldots, v-1 \). By Lemma 1.4.1 the eigenvalues of \( C_{d^*} \) are 0, \( v\lambda_1/k \) and \( r-(\lambda_0/k)+(v-2)/(v-1)+((\lambda_1/k)+(v-1)^{-1} \) with multiplicities 1, 1 and \( v-2 \) respectively. The last eigenvalue
simplifies to \((n_1/k)+1\) since \(r-(\lambda_0/k) = (v-1)\lambda_1/k\). Note that \(d^*\) has binary blocks and is of maximum trace in \(\mathcal{D}(v,b+1,k^*)\).

Throughout this chapter, without loss of generality we will assume
\[
\min_{1 \leq i \leq v} r_{d_i} = \min_{1 \leq i \leq v} (r_{d^*_i}) \text{ for any } d \in \mathcal{D}(v,b+1,k^*).
\]

**Lemma 4.1.1:** Let \(d, d^* \in \mathcal{D}(v,b+1,k^*)\), \(v \geq 3\), where \(d\) is arbitrary and \(d^*\) is as defined above. Then if
\[
 r_{d^*} - \frac{1}{k} \sum_{j=1}^{b} n_{dj}^2 - \frac{n_{dv}^2}{v-1} \leq \frac{r - \lambda_0}{k}.
\]
we have \(d^*\) E-better than \(d\).

**Proof:** By Theorem 2.2.1 we have
\[
\mu_{d^*} \leq \frac{v}{v-1} (r_{d^*} - \frac{1}{k} \sum_{j=1}^{b} n_{dj}^2 - \frac{n_{dv}^2}{v-1}).
\]
Since \(\mu_{d^*} = (v\lambda_1/k) = (v/(v-1))(r-\lambda_0/k)\), simple algebra gives us the rest.

We will now attempt to get the \(\mathcal{D}_4\)-optimality of \(d^*\) by applying Theorem 2.1.1 and Lemma 4.1.1. The cases \(k = v-1\) and \(k = v\) will be discussed in this section. In Section 4.2 \(k < v-1\) will be covered, and in Section 4.3 we shall see that \(d^*\) is not \(\mathcal{D}_4\)-optimal for many cases of \(k \geq v+1\).

The case of \(k = v-1\)-has been covered in Theorem 3.3.1. Here is the case \(k = v\).

**Theorem 4.1.1:** Let \(d^* \in \mathcal{D}(v,b+1,k^*)\) be a BBD plus one binary block of size \(v-1\) where \((k^*)' = (v-1)' \odot b\). Then \(d^*\) is \(\mathcal{D}_4\)-optimal in \(\mathcal{D}(v,b+1,k^*)\).
Proof: Since \( k = v \), \( d^0 \) is a CBD and for any \( d \in \mathcal{D}(v,b+1,k^*) \),

\[
\min_{1 \leq i \leq v} (r_{di}) \leq r = b < b+1. \text{ If this minimum } r_{di} \text{ is } r_{dv} \text{ then there exists at least one } j \in \{1, \ldots, b, b+1\} \text{ for which } n_{dvj} = 0.
\]

If \( j = b+1 \) then

\[
r_{dv} - \frac{1}{v} \sum_{j=1}^{b} n_{dvj} - \frac{0}{v-1} \leq r_{dv} - \frac{r_{dv}}{v} = \frac{r_{dv}(v-1)}{v} \leq \]

\[
\frac{r(v-1)}{v} = r - \frac{\lambda_0}{v} \text{ since } \lambda_0 = r \text{ for } d^0.
\]

If \( j < b+1 \) then without loss of generality let \( j = 1 \). We have

\[
r_{dv} - \frac{1}{v} \sum_{j=2}^{b} n_{dvj} - \frac{n^2_{dv,b+1}}{v-1} \leq r_{dv} - \frac{1}{v} \sum_{j=1}^{b} n_{dvj} - \frac{n^2_{dv,b+1}}{v} \leq r_{dv} - \frac{r_{dv}}{v} \leq \]

\[
r = \frac{\lambda_0}{k}.
\]

By Lemma 4.1.1 \( d^* \) E-betters all \( d \) in \( \mathcal{D}(v,b+1,k^*) \). Since \( d^* \) is of maximum trace, has binary blocks and the right eigenvalue structure, Theorem 2.1.1 gives us the \( J_4 \)-optimality.

4.2 The Cases \( k < v-1 \)

In these cases we shall prove that \( d^* \) is \( J_4 \)-better than some of the \( d \in \mathcal{D}(v,b+1,k^*) \).

Lemma 4.2.1: \( d^* \) is \( J_4 \)-better than any \( d \in \mathcal{D}(v,b+1,k^*) \) with \( n_{dv,b+1} = 0 \) \((v \geq 3)\).

Proof: We use Lemma 4.1.1. \( r-(\lambda_0/k) = r-(r/k) = r(k-1)/k \), and by Lemma 1.4.3

\[
r_{dv} - \frac{1}{k} \sum_{j=1}^{b} n_{dvj} - \frac{0}{v-1} = r_{dv} - \frac{1}{k} \sum_{j=1}^{b} n_{dvj} \leq r_{dv} - \frac{r_{dv}}{k} = \frac{r_{dv}(k-1)}{k} \leq \frac{r(k-1)}{k}.
\]
Therefore $d^*$ is $E$-better than $d$, and by Section 4.1 and Theorem 2.1.1, $d^*$ is $\w_4$-better.

Lemma 4.2.2: $d^*$ is $E$-better than any $d \in \mathcal{O}(v, b+1, k^*)$, $v \geq 3$, for which $n_{dv, b+1} = \varepsilon \geq 1$ and any one of the following holds:

a) $r - r_{dv} \geq \frac{v-1}{4k(k-1)}$,

b) $k \geq \sqrt[v+1]{\frac{v}{2}}$ and $r_{dv} < r$,

c) For $r_{dv}$ fixed, $0 \leq r - r_{dv} < \frac{v-1}{4k(k-1)}$ with $1 \leq \varepsilon \leq (v-1 - a(r_{dv})^2)/2k$ or $\varepsilon \geq (v-1 + a(r_{dv})^2)/2k$ where $a(r_{dv}) = (v-1)^2 - 4k(k-1)(v-1)(r-r_{dv})$,

d) For fixed $\varepsilon$, $1 \leq r_{dv} \leq r - \frac{\varepsilon}{k-1}(1-\frac{\varepsilon k}{v-1})$,

e) $\varepsilon = 1$ and $1 \leq r_{dv} < r$.

Proof:

Suppose $n_{dv, b+1} = \varepsilon \geq 1$. We have

$$r_{dv} - \frac{1}{k} \sum_{j=1}^{b} n_{dvj}^2 - \frac{\varepsilon^2}{v} \leq r_{dv} - \frac{r_{dv} - \varepsilon}{k} - \frac{\varepsilon^2}{v-1}$$

$$= \frac{r_{dv}(k-1)}{k} + \frac{\varepsilon}{k} - \frac{\varepsilon^2}{v-1}$$

with $1 \leq \varepsilon \leq \min(r_{dv}, v-2)$ and $1 \leq r_{dv} \leq r$. For $E$-optimality of $d^*$ we need

$$\frac{r_{dv}(k-1)}{k} + \frac{\varepsilon}{k} - \frac{\varepsilon^2}{v-1} \leq \frac{r(k-1)}{k}$$

(4.2.1)
by Lemma 1.4.1, which is true if and only if \( r_{dv}(k-1) + \frac{\varepsilon}{k} - \frac{\varepsilon^2}{k/(v-1)} \leq r(k-1) \).

If we fix \( r_{dv} \), (4.2.1) is true if and only if \( A(\varepsilon, r_{dv}) = \frac{\varepsilon^2}{k} - (v-1)\varepsilon + (r-r_{dv})(k-1)(v-1) \geq 0 \). Solving for \( \varepsilon \), \( r-r_{dv} \geq (v-1)/(4k(k-1)) \) implies \( A(\varepsilon, r_{dv}) \geq 0 \) for all \( \varepsilon \). If \( r-r_{dv} < (v-1)/(4k(k-1)) \), then \( A(\varepsilon, r_{dv}) \geq 0 \) if \( 0 \leq \varepsilon \leq (v-1-a(r_{dv})^{3/2})/2k \) or \( \varepsilon \geq (v-1+a(r_{dv})^{3/2})/2k \) where \( a(r_{dv}) = (v-1)^2 - 4k(k-1)(r-r_{dv}) \). This follows from the quadratic formula, and proves a) and c).

If we fix \( \varepsilon \), (4.2.1) is true if and only if \( 1 \leq r_{dv} \leq r - \frac{\varepsilon}{k-1} (1 - \frac{2k}{v-1}) \). In the special case of \( \varepsilon = 1 \), we get \( 1 \leq r_{dv} \leq r - (k-1)^{-1} (1-k/(v-1)) \) or since \( r_{dv} \) is an integer, \( 1 \leq r_{dv} \leq r-1 \). This proves d) and e).

Going back to \( A(\varepsilon, r_{dv}) \), if \( (v-1)/(4k(k-1)) \leq 1 \) then \( d^* \) E-betters all designs except those for which \( r_{dv} = r \), but \( (v-1)/(4k(k-1)) \leq 1 \) if and only if \( k \geq (\sqrt{v} + 1)/2 \). Thus b) holds and the proof is finished.

Looking at the left-hand-side of (4.2.1) we might ask when is

\[
\frac{r_{dv}(k-1)}{k} + \frac{\varepsilon}{k} - \frac{\varepsilon^2}{v-1} \leq \frac{r_{dv}(k-1)}{k} + \frac{\varepsilon-1}{k} - \frac{(\varepsilon-1)^2}{v-1}?
\]

This is true if and only if \( \varepsilon \geq (v-1+k)/2k \). So if \( \varepsilon \geq (v-1+k)/2k \) we can get a larger upper bound for \( n_{dv} \) using Theorem 2.2.1 by letting \( n_{dv,b+1} = \varepsilon-1 \) and not \( \varepsilon \). The maximum for the left-hand-side of (4.2.1) is

\[
\frac{v}{v-1} \left( \frac{r_{dv}(k-1)}{k} + \frac{\varepsilon_0}{k} - \frac{\varepsilon_0^2}{v-1} \right)
\]
where \( \ell_0 = \frac{v-1+k}{2k} - 1 \) if \( \frac{v-1+k}{2k} = \text{int}[\frac{v-1+k}{2k}] \) and \( \ell_0 = \text{int}[\frac{v-1+k}{2k}] \) if \( \frac{v-1+k}{2k} > \text{int}[\frac{v-1+k}{2k}] \).

We can reduce \( n_{dv,b+1} \) to \( \ell = 1 \) if \( (v-1+k)/(2k) \leq 2 \) or \( k \geq (v-1)/3 \).

From this we get the following lemma.

**Lemma 4.2.3:** For \( r_{dv} = \min_{1 \leq i \leq v} (r_{di}) \), \( \sum_{i=1}^{v} r_{di} = vr+v-1 \),

a) \( r_{dv}(k-1) \frac{v}{k} + \frac{2}{k} \frac{v}{v-1} \leq \frac{2}{k} \frac{v}{v-1} \frac{v-1+k}{v-1} - (k-1)^2 \) for \( \ell \geq \frac{v-1+k}{2k} \) and

b) \( r_{dv}(k-1) \frac{v}{k} + \frac{2}{k} \frac{v}{v-1} \) is maximized by \( \ell_0 = \frac{v-1+k}{2k} - 1 \) if \( \ell_0 + 1 = \text{int}[\frac{v-1+k}{2k}] \) and \( \ell_0 \neq \text{int}[\frac{v-1+h}{2k}] \) if \( \frac{v-1+k}{2k} > \text{int}[\frac{v-1+h}{2k}] \).

To summarize what has been said about the optimality of \( d^* \) we have the following theorem.

**Theorem 4.2.1:** Suppose \( d^* \in \mathcal{A}(v,b+1,k^*) \), \( v \geq 3 \), is a BBD plus one binary block of size \( v-1 \). Then \( d^* \) is \( 4 \)-better than all \( d \in \mathcal{A}(v,b+1, k^*) \) except possibly \( d \) for which \( n_{dv,b+1} = \ell \) and

a) \( v \geq 7, k \geq \frac{v-1}{3} > \frac{\sqrt{v+1}}{2}, r_{dv} = r, 0 < \ell < \frac{v-1}{k}, \) (and \( \ell_0 = 1 \)),

b) \( v \geq 7, \frac{v-1}{3} > k \geq \frac{\sqrt{v+1}}{2}, r_{dv} = r, 0 < \ell < \frac{v-1}{k} \) (and \( \ell_0 > 1 \)),

c) \( v \geq 7, \frac{v-1}{3} > \frac{\sqrt{v+1}}{2} > k, 0 \leq r-r_{dv} < \frac{v-1}{4k(k-1)}, (v-1-a(r_{dv})^2)/2k < \ell < (v-1+a(r_{dv})^2)/2k \) (and \( \ell_0 > 1 \)),

d) \( 3 \leq v \leq 6, \frac{v-1}{3} < \frac{\sqrt{v+1}}{2} < k, r_{dv} = r, 0 < \ell < \frac{v-1}{k} \) (and \( \ell_0 = 1 \)).
where $a(r_{dV}) = (v-1)^2 - 4k(v-1)(v-1)(r-r_{d1})$ and $v_0$ is defined in Lemma 4.2.3.

**Proof:** $\frac{v-1}{3} > \frac{\sqrt{v+1}}{2}$ for $v \geq 7$ and $\frac{v-1}{3} < \frac{\sqrt{v+1}}{2}$ for $v \leq 6$. Then a) through d) follow from Lemma 4.2.2, parts a), b) and c), Lemma 4.2.3, Section 4.1 and Theorem 2.1.1.

For the cases of a), c) and d) of Theorem 4.2.1 A-, D- and E-efficiencies will be calculated in Chapter 8.

### 4.3 The Cases $k > v+1$

We begin with two general lemmas for the case of $k > v+1$ and another for this situation analogous to Lemma 4.2.1.

**Lemma 4.3.1:** Let $B(cb+d; b,k) = cb+d-k^{-1}(d(c+1)^2+(b-d)c^2)$ where $k \geq 2$, $c \geq 0$, $b \geq 0$, $0 \leq d \leq b$ and $b$, $c$, $d$ and $k$ are integers. Then $B(c_1b+d_1; b,k) \leq B(c_2b+d_2; b,k)$ if either of the following holds

a) $c_1 = c_2$ and $d_1 = d_2$ or $d_1 < d_2$ with $(k-1)/2 \geq c_2 \geq 0$.

b) $c_1 < c_2$ and $(k-1)/2 \geq c_2 > c_1 \geq 0$.

**Proof:** If $c_1 = c_2$ then $B(c_1b+d_1; b,k) = B(c_1b+d_2; b,k) = \frac{d_2-d_1}{k} (k-(2c_2+1)) \geq 0$ if and only if $d_2 = d_1$ or $d_2 > d_1$ and $k \geq 2c_2+1$. If $c_1 < c_2$, $B(c_1b+d_1; b,k) \leq B(c_1b+b; b,k) = B((c_1+1)b+0; b,k)$. If $c_2 = c_1+1$ we finish with $\leq B((c_1+1)b+d_2; b,k)$. If $c_2 > c_1+1$ we finish with $\leq B((c_1+1)b+b; b,k) \leq \cdots \leq B(c_2b+0; b,k) \leq B(c_2b+d_2; b,k)$. In each case we need $(k-1)/2 \geq c_2 > c_1 \geq 0$ and the inequalities follow as in the case for $c_1 = c_2$. 


**Lemma 4.3.2:** Suppose we are investigating \( d \in \mathcal{A}(v,b;k) \) with \( k \geq v \geq 3 \) and \( \min_{1 \leq i \leq v} (r_{di}) \leq r = \text{int} [bk/v] \). Then \( B(\min_{1 \leq i \leq v} (r_{di}); b,k) \leq B(r; b,k) \).

**Proof:** Let \( r = c_2 b + d_2 \) and \( \min_{1 \leq i \leq v} (r_{di}) = c_1 b + d_1 \). We apply Lemma 4.3.1. If \( c_1 = c_2 \) and \( d_1 = d_2 \) we are done. Otherwise we must show \( (k-1)/2 \geq c_2 \). But \( vr = v(c_2 b + d_2) = v \cdot \text{int}[bk/v] \leq bk \) so \( c_2 \leq (bk-vd_2)/vb \leq (bk)/vb = k/v \leq (k-1)/2 \) with the last inequality following from \( k \geq v \geq 3 \).

**Lemma 4.3.3:** \( d^* \in \mathcal{A}(v,b+1,k^*) \), a BBD plus one binary block of size \( v-1 \), is E-better than any \( d \in \mathcal{A}(v,b+1,k^*) \) with \( n_{dv,j} = 0 \), \( j = 1,2,\ldots,b,b+1 \) (\( v \geq 3 \)).

**Proof:** If \( j = b+1 \), \( r_{dv} - \frac{1}{k} \sum_{j=1}^{b} n_{dvj}^2 - \frac{0}{v-1} \leq B(r_{dv}; b,k) \leq B(r; b,k) = r - \lambda_0/k \) with \( r = \text{int}[bk/v] \). If \( j = 1 \), say, \( r_{dv} - \sum_{j=1}^{b} \frac{n_{dvj}^2}{k} = r_{dv} - \sum_{j=2}^{b} \frac{n_{dvj}^2}{k} - \frac{n_{dv,b+1}^2}{v-1} \leq B(r_{dv}; b,k) \leq B(r; b,k) = r - \lambda_0/k \). Therefore by Lemma 4.1.1 we are done.

This last lemma is most useful when \( j = b+1 \). It says a design that is E-better than \( d^* \) must have its minimum replicated treatments (at least treatment \( v \)) appearing in the block of size \( v-1 \) at least once.

We will now look at the cases \( q = 0, 1, v-1 \) and \( 2 \leq q \leq v-2 \) separately, where \( k = pv+q \) (\( p \geq 1 \)). In each case a design \( \hat{d} \) will be
described as a candidate to be $J_4$-better than $d^*$. Then a theorem will be given where $d^*$ is proven $J_4$-better than a subclass of the $d \in \mathcal{S}(v,b+1,k^*)$, and possibly $d$ is proven $J_4$-better than $d^*$. Proving $d^*$ is $J_4$-better than the subclass will consist of four steps. Lemma 4.3.3 will be applied to show an $E$-better (than $d^*$) $d$ needs $n_{dvj} \geq 1$, $1 \leq j \leq b+1$. Then Lemma 4.1.1 will be applied to show an $E$-better $d$ needs $r_{dv} = r$ for $n_{dv,b+1} \geq 1$. Thirdly Lemma 4.1.1 will be applied to show that an $E$-better $d$ needs $n_{dv,b+1} = 1$ or possibly 2 with some conditions. Finally Theorem 2.1.1 will be applied to $d^*$ (recall it is of maximum trace and has eigenvalues $0 < (v\lambda_1/k) < (v\lambda_1/k) + 1 = ... = (v\lambda_1/k+1)$ and the designs it $E$-better.

For $k = pv$ ($p \geq 2$) $d^0$ has $b$ blocks, each containing $p$ applications of the $v$ treatments. $d^*$ has these blocks as its first $b$ blocks with the $(b+1)$-st block of size $v-1$ and containing treatments 1 through $v-1$. Let $\hat{d}$ have $d^*$'s first $b-1$ blocks, the $b$-th block containing $p$ applications of treatments 1 through $v-2$, $p+1$ applications of $v-1$ and $p-1$ applications of $v$, and its last block containing treatments 1 through $v-2$ and $v$. To create $\hat{d}$ we have traded an application of $v$ from the $b$-th block with the application of $v-1$ in the $(b+1)$-st. Note that $\lambda_0 = \lambda_1 = bp^2$.

**Theorem 4.3.1.** For $k = pv$, $p \geq 2$, $d^*$ is $J_4$-optimal over all $d \in \mathcal{S}(v,b+1,k^*)$ with $v \geq 3$ and $(k^*)' = (pv,k_{p-1}', v-1)$ except possibly those for which $r_{dv} = r$, $n_{dv,b+1} = 1$, $n_{dvj} \geq 1$ ($1 \leq j \leq b$) and $v \geq 3$ with $p \geq 3$ or $v \geq 4$. There exists a $\hat{d}$ with $r_{dv} = r$ and $n_{dv,b+1} = 1$
which E-betters $d^*$ for $p \geq v/(v-2)$ but is E-worse for $p < v/(v-2)$
and is always A- and D-worse.

**Proof:** We apply Lemma 4.3.3.

If $d_{dv} \leq r-1$ and $b \geq 2$ then

$$r_{dv} = \frac{b}{\sum_{j=1}^{b} \frac{n_{dvj}^2}{k} - \frac{n_{dv,b+1}^2}{v-1}} \leq$$

$$B(r_{dv}-1; b,k)+1-(v-1)^{-1} \leq B(r-2; b,k)+1-(v-1)^{-1} = r-2 -$$

$$\frac{\lambda_0 - 2p^2 + 2(p-1)^2}{k} + 1 - \frac{1}{v-1} = r - \lambda_0/k + \left(4p^2 - 1 - \frac{1}{v-1}\right).$$

The first inequality follows from the fact that

$$-\frac{\lambda^2}{k} - \frac{n^2}{v-1} \leq -\frac{(\lambda+1)^2}{k} - \frac{(n-1)^2}{v-1} \quad (4.3.1)$$

for $n \geq 2$, $\lambda \geq 0$. If $b = 1$ then $B(r-2; b,k)+1-(v-1)^{-1} = r - (\lambda_0/k) +$

$$\left(\frac{4p^2 - 1 - \frac{1}{v-1}}{k}\right)$$

since $p \geq 2$. But

$$\frac{4p^2 - 1 - \frac{1}{v-1}}{k} \leq \frac{4p^2 - 1 - \frac{1}{v-1}}{v-1} \leq 0$$

for $p \geq 2$, $v \geq 3$. We apply Lemma 4.1.1, so an E-better $d$ needs

$$d_{dv} = r.$$

If $d_{dv} = r$, $n_{dv,b+1} \geq 1$ and $b \geq 2$ then

$$c_{dvv} = r - \frac{\sum_{j=1}^{b} \frac{n_{dvj}^2}{k} - \frac{n_{dv,b+1}^2}{v-1}} \leq B(r-x; b,k) + x - \frac{x^2}{v-1}$$

$$= r - (\lambda_0/k) + \frac{x(2p-1)}{k} - \frac{x^2}{v-1} \quad (4.3.2)$$

for $x = 2$ or 1 with (4.3.2) larger for $x = 1$. The first inequality
follows from (4.3.1). $\frac{x(2p-1)}{k} - \frac{x^2}{v-1} \leq 0$ for $v = 3$, $p = 2$, $x = 1$ and 2,
but for $x = 2$ only when $v = 3$, $p \geq 3$ or $v \geq 4$, $p \geq 2$. If $b = 1$ and
$n_{dv,b+1} = 1$ we can use (4.3.2) above and the same result holds. If
$b = 1$ and $n_{dv,b+1} \geq 2$ we must again use $p \geq 2$:
\[ r - \frac{b}{k} \sum_{j=1}^{n_{dvj}} \frac{n_{dvj}^2}{k} - \frac{n_{dv,b+1}^2}{v-1} \leq r - (\lambda_0/k) + \frac{4p-4}{k} - \frac{4}{v-1} \]

which is less than \( r - (\lambda_0/k) \) for \( p \geq 2, v \geq 3 \). We apply Lemma 4.1.1, so an E-better \( d \) needs \( r_{dv} = r, n_{dv,b+1} = 1 \) (unless \( v = 3, p = 2 \)).

Now we apply Theorem 2.1.1 to \( d^* \) and all \( d \) except those with \( r_{dv} = r, n_{dv,b+1} = 1, n_{dvj} \geq 1 \) (1 \( \leq j \leq b \)) and \( v = 3, p \geq 3 \) or \( v \geq 4 \).

For the second part of the theorem

\[
C_d = \begin{bmatrix}
C_{11} & C_{12} & C_{13} \\
C_{21} & C_{22} & C_{23} \\
\text{sym} & C_{33}
\end{bmatrix} 
\]

(4.3.3)

where \( C_{11} = (r+1)I_{v-2} - \left( \frac{bp^2}{k} + \frac{1}{v-1} \right)I_{v-2,v-2} \), \( C_{12} = -k^{-1}(bp^2+p)I_{v-2,v-2} \),

\( C_{13} = -(\frac{bp^2-p}{k} + \frac{1}{v-1})I_{v-2} \), \( C_{22} = r+1 - \frac{bp^2+2p+1}{k} \), \( C_{23} = -k^{-1}(bp^2-1) \) and \( C_{33} = r - \frac{bp^2-2p+1}{k} - \frac{1}{v-1} \). From Lemma 2.4.1 the eigenvalues of \( C_d \) are \((v\lambda_1/k)+1 = r+1\) with multiplicity \( v-3, 0 \), and

\[
\frac{v\lambda_1}{k} + \frac{1}{2} - \frac{1}{k} - \frac{1}{2} \left( \frac{v^2p^2-4vp^2+4}{p^2v^2} + \frac{4(p+1)}{pv(v-1)} \right) \]

\( \hat{d} \) is E-worse than \( d^* \) if and only if

\[
\frac{1}{2} - \frac{1}{k} - \frac{1}{2} \left( \frac{v^2p^2-4vp^2+4}{p^2v^2} + \frac{4(p+1)}{pv(v-1)} \right)^{\frac{1}{2}} < 0
\]

which is true if and only if \( p < v/(v-2) \). Straightforward but lengthy calculations comparing \( \phi_A(C_d) \) with \( \phi_A(C_{d^*}) \) and \( \phi_D(C_d) \) with \( \phi_D(C_{d^*}) \) show \( d^* \) is strictly A- and D-better than \( \hat{d} \). The proof is now complete.
For \( k = pv+1 \) \((p \geq 1)\), \( d^0 \) has \( b = ev \) blocks \((e \geq 1)\) and comprises \( e \) copies of the BBD in \( \mathcal{B}(v,v, pv+1) \) with each treatment appearing one more time than all others in exactly one block. Assume the \( b \)-th block of \( d^* \) (or \( d^0 \)) has treatment \( v \) extra-replicated. Then let \( \hat{d} \) be the design with blocks 1 through \( b-1 \) exactly those of design \( d^* \), the \( b \)-th block with \( n_{d,v-1,b} = p+1 \) and \( n_{dib} = p \) for \( i = 1, \ldots, v-2, v \), and the \((b+1)\)-st block with treatments 1 through \( v-2 \) and \( v \). We have traded an application of \( v \) in a block where treatment \( v-1 \) is not extra-replicated (here block \( b \)) with the application of \( v-1 \) in block \( b+1 \).

**Theorem 4.3.2:** For \( k = pv+1 \), \( p \geq 1 \), \( d^* \) is \( J_4 \)-optimal over all \( d \in \mathcal{B}(v, ev+1, k^*) \), (with \( v \geq 3 \) and \((k^*)^\prime = ((pv+1)b, v-1)) \), except possibly those \( d \) for which \( r_{dv} = r, n_{dvj} \geq 1 \) \((1 \leq j \leq b)\), \( n_{dv,b+1} = 1 \) with \( e \geq 1 \) or \( n_{dv,b+1} = 2 \) with \( e \geq 2, p < \frac{v-3}{2} \). There exists a \( \hat{d} \) with \( r_{dv} = r, n_{dv,b+1} = 1 \) that is \( J_4 \)-better and strictly \( A- \), \( D- \) and \( E- \) better than \( d^* \).

**Proof:** We apply Lemma 4.3.3.

If \( r_{dv} \leq r-1 \), \( e = 1 \), \( r_{dv} - \sum_{j=1}^{b} \frac{n_{dvj}^2}{k} - \frac{n_{dv,b+1}^2}{v-1} \leq B(r_{dv}-1; b, k) + \frac{1}{1-(v-1)^{-1}} \leq B(r-2; b, k) + 1 - (v-1)^{-1} = r-(\lambda_0/k) + ((4p/k) - 1 - (v-1)^{-1}) \) by the same argument used preceding (4.3.1). If \( e \geq 2 \), \( c_{dvv} \leq r-(\lambda_0/k)((\frac{4p+2}{k} - 1) - \frac{1}{v-1}) \) by a similar argument, with \((\frac{4p}{k} - 1 - \frac{1}{v-1}) \leq (\frac{4p+2}{k} - 1) - \frac{1}{v-1} \leq 0 \) for \( p \geq 1 \), \( v \geq 3 \). We apply Lemma 4.1.1, so an \( E \)-better \( d \) must have \( r_{dv} = r \) for \( n_{dv,b+1} \geq 1 \).
Assume \( r_{dv} = r \). If \( n_{dv,b+1} = 1, e \geq 1, \) then
\[
\sum_{j=1}^{b} \frac{n_{dvi}^2}{k} - \frac{1}{v^{-1}} \leq r - \frac{\lambda_0^2(p+1)^2 + p^2}{k} - \frac{1}{v^{-1}} > r - \lambda_0/k. \]
If \( n_{dv,b+1} = 2, e = 1 \) (and \( v \geq 4 \)),
\[
r - \sum_{j=1}^{b} \frac{n_{dvi}^2}{k} - \frac{4}{v^{-1}} \leq r - \frac{\lambda_0^2(p+1)^2 + p^2 - p^2 + (p-1)^2}{k} - \frac{4}{v^{-1}} = r - (\lambda_0/k) + \frac{4p}{k} - \frac{4}{v^{-1}} \leq r - (\lambda_0/k) \text{ for } p \geq 1, v \geq 4. \]
If \( n_{dv,b+1} = 2, e \geq 1 \) (and \( v \geq 4 \) then
\[
c_{dvv} \leq r - \frac{\lambda_0^2(p+1)^2 + 2p^2}{k} - \frac{4}{v^{-1}} = r - (\lambda_0/k) + \frac{4p+4}{k} - \frac{4}{v^{-1}} \leq r - (\lambda_0/k) \text{ if and only if } p \geq (v-3)/2. \]
If \( 3 \leq n_{dv,b+1} \leq v-2 \) (and \( v \geq 5 \) then the three cases \( e = 1, e = 2 \) and \( e \geq 3 \) all yield \( d^* \) as
E-better with similar arguments. In each case above we used the fact
that \( c_{dvv} = r - \sum_{j=1}^{b} \frac{n_{dvi}^2}{k} - \frac{n_{dv,b+1}^2}{v^{-1}} \leq B(r-x; b, k) + x - \frac{x^2}{v^{-1}} \) for \( n_{dv,b+1} \geq x = 3, 2 \) or 1 with the last statement increasing as \( x \) decreases. The argument follows as at (4.3.2). We apply Lemma 4.1.1, so an E-better
\( d \) must have \( r_{dv} = r, n_{dv,b+1} = 1 \), or \( n_{dv,b+1} = 2 \) with \( p < (v-3)/2 \)
and \( e \geq 2 \).

Now we apply Theorem 2.1.1 to \( d^* \) and all \( d \) that \( d^* \) E-bettered to
prove the first part of the theorem.

For the second part of the theorem \( C_d \) is of the form given at
(4.3.3). The eigenvalues of \( C_d \) are \( (v\lambda_1/k) + 1 \) with multiplicity
\( v-3 \), and using Lemma 2.4.1,
\[
\frac{v\lambda_1}{k} + \frac{1}{2} + \frac{1}{2} [\left( \frac{2p(v-2)}{k} - 1 \right)^2 + \frac{4p}{k} \left( \frac{2p(v-2)}{k} - \frac{v-2}{v-1} \right) \frac{1}{k^2}].
\]
\( d \) is of full trace, as is \( d^* \). Furthermore for \( v \geq 3, p \geq 1, \)
\[ 0 \leq \frac{1}{2} \left[ \left( \frac{2p(v-2)}{k} - 1 \right)^2 + \frac{4p}{k} \left( \frac{2p(v-2)}{v-1} - \frac{v-2}{v-1} \right)^3 \right] \leq \frac{1}{2}. \]

Therefore the eigenvalues of \( \hat{d} \) majorize those of \( d^* \), and are not a permutation of them, making \( \hat{d} \) \( J_4 \)-better and strictly \( A- \), \( D- \) and \( E- \) better than \( d^* \).

**Remark 4.3.1:** \( \hat{d} \) is \( J_4 \)-better than all \( d \) for which \( d^* \) is \( J_4 \)-better. But \( \hat{d} \) has not been proven \( J_4 \)-optimal in \( \varnothing(v,ev+1,k^*) \) and its eigenvalue structure is not that of Theorem 2.1.1.

For \( k = pv+v-1 \), \( p \geq 1 \), \( d^0 \) comprises \( e \) copies (\( e \geq 1 \)) of the unique BBD in \( \varnothing(v,v,pv+v-1) \). \( d^* \) is \( d^0 \) union the block of size \( v-1 \) with treatments \( 1 \) through \( v-1 \). Without loss of generality assume the \( b \)-th block of \( d^* \) contains \( p+1 \) copies of treatments \( 1 \) through \( v-2 \) and \( v \), with \( p \) copies of \( v-1 \). To create \( \hat{d} \) trade an application of \( v \) from the \( b \)-th block with the application of \( v-1 \) from the \((b+1)\)-st block. Blocks \( 1 \) through \( b-1 \) of \( \hat{d} \) remain identical to blocks \( 1 \) through \( b-1 \) of \( d^* \). Here \( b = ev \) and \( \hat{d} \) is of maximum trace in \( \varnothing(v,ev+1,k^*) \).

**Theorem 4.3.3:** For \( k = pv+v-1 \), \( p \geq 1 \), \( d^* \) is \( J_4 \)-optimal over all \( d \in \varnothing(v,ev+1,k^*) \), (with \( v \geq 3 \) and \((k^*)' = ((pv+v-1)b,v-1))\) except possibly those for which \( r_{dv} = r \), \( n_{dv,b+1} = 1 \), and \( n_{dvj} > 1 \), \( 1 \leq j \leq b \). There exists a \( \hat{d} \) with \( r_{dv} = r \) and \( n_{dv,b+1} = 1 \) that is \( J_4 \)-better and strictly \( A- \), \( D- \) and \( E- \) better than \( d^* \).

**Proof:** We apply Lemma 4.3.3.
If \( r_{dv} \leq r-1, e \geq 1 \), \( r_{dv} - \frac{b}{k} \sum_{j=1}^{j} \frac{n_{dv,j}^2}{v-1} - \frac{n_{dv,b+1}^2}{v-1} \leq B(r_{dv}-1; b, k) + 1 - (v-1)^{-1} \leq B(r-2; b, k) + 1 - (v-1)^{-1} = r-(\lambda_0/k) + \left(\frac{4p+2}{k} - 1 - \frac{1}{v-1}\right).

\[
\frac{4p+2}{k} - 1 - \frac{1}{v-1} < 0 \text{ for } k = pv+q, \ p \geq 1, \ v \geq 3 \text{ and } 2 \leq q \leq v-1, \text{ so }
\]

\( r_{dv} \) must equal \( r \).

If \( r_{dv} = r, n_{dv,b+1} \geq 1 \) then \( r_{dv} - \frac{b}{k} \sum_{j=1}^{j} \frac{n_{dv,j}^2}{v-1} - \frac{n_{dv,b+1}^2}{v-1} \leq
\]

\( r-(\lambda_0/k) + \frac{x(2p+1)}{k} - \frac{x^2}{v-1} \) for \( x = 1 \) or 2, the larger of which occurs for \( x = 1, \ x(2p+1) - \frac{x^2}{v-1} < 0 \) for \( x \geq 2, \ p \geq 1, \ v \geq 3 \), but is in fact greater than zero for \( x = 1 \). For \( v = 3 \) \( x \) can not be greater than 1.

We apply Lemma 4.1.1 to the preceding paragraphs, so an E-better \( d \) must have \( r_{dv} = r \) and \( n_{dv,b+1} = 1 \).

We now apply Theorem 2.1.1 to \( d^* \) and the \( d \) for which \( d^* \) is E-better to finish the first part of the proof.

\( C_d \) is of the form given at (4.3.3), and has eigenvalues 0, \((v\lambda_1/k)+1\) with multiplicity \( v-3 \), and from Lemma 2.4.1

\[
\frac{v\lambda_1}{k} + \frac{1}{2} + \frac{1}{2} \left(1 - \frac{4v(v-2)p(p+1)}{k^2(v-1)}\right)^{\frac{1}{2}}.
\]

Since \( 0 \leq \frac{1}{2} \left(1 - \frac{4v(v-2)p(p+1)}{k^2(v-1)}\right)^{\frac{1}{2}} < \frac{1}{2} \), these eigenvalues majorize and are not a permutation of those of \( d^* \), making \( \hat{d} \) \( \mathcal{J}_{4} \)-better and strictly A-, D- and E-better than \( d^* \). So ends the proof.

For \( k = pv+q, \ 2 \leq q \leq v-2 \), we can still construct a \( \hat{d} \) which is potentially \( \mathcal{J}_{4} \)-better than \( d^* \). However for these \( q \) \( C_d \) is of the form
\[
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & C_{14} \\
C_{22} & C_{23} & C_{24} \\
C_{33} & C_{34} \\
C_{44}
\end{bmatrix}
\]

where

- \( C_{11} = (r - \frac{\lambda_0}{k} + \frac{\lambda_1}{k} + 1)I_{q-1} - \left(\frac{\lambda_1}{k} + \frac{1}{v-1}\right)J_{q-1, q-1} \)
- \( C_{12} = -\left(\frac{\lambda_1}{k} + \frac{1}{v-1}\right)J_{q-1, (v-2)-(q-1)} \)
- \( C_{13} = -\left(\frac{\lambda_1}{k} + \frac{p+1}{k}\right)I_{q-1} \)
- \( C_{14} = -\left(\frac{\lambda_1}{k} + \frac{p+1}{k} + \frac{1}{v-1}\right)I_{q-1} \)
- \( C_{22} = -\left(\frac{\lambda_1}{k} + \frac{p}{k}\right)I_{v-q-1} \)
- \( C_{24} = -\left(\frac{\lambda_1}{k} + \frac{p}{k} + \frac{1}{v-1}\right)I_{v-q-1} \)
- \( C_{33} = r - \frac{\lambda_0}{k} + 1 - \frac{2p+1}{k} \)
- \( C_{34} = -\frac{\lambda_1}{k} \) and \( C_{44} = r - \frac{\lambda_0}{k} + \frac{2p+1}{k} - \frac{1}{v-1} \).

To get \( \hat{d} \), without loss of generality assume the \( b \)-th block of \( d^* \) has treatments 1 through \( q-1 \) and \( v \) as the extra-replicated treatments, with the \( (b+1) \)-st block as usual. Then trade the application of \( v \) in the \( b \)-th block with that of \( v-1 \) in the \( (b+1) \)-st. The first \( q-1 \) rows of \( C_{\hat{d}} \) correspond to the treatments in the \( b \)-th block that were extra-replicated with treatment \( v \).

In this case we cannot apply Lemma 2.4.1 to \( C_{\hat{d}} \), and as was mentioned in Section 2.4, the 4-by-4 blocked matrix equivalent to Lemma 2.4.1 is unwieldy.

We shall be content here to give a theorem that shows \( d^* \) is \( S_4 \)-optimal over many of the \( d \in \mathcal{D}(v, b+1, k^*) \), and then give some examples indicating that \( \hat{d} \) with \( r_{d^*} = r \), \( n_{d^*, b+1} = 1 \) is at least \( A_- \), \( D_- \) and \( E \)-better than \( d^* \).
Theorem 4.3.4: For $k = pv + q$, $2 \leq q \leq v - 2$, $p \geq 1$, $v \geq 3$, $d^*$ is $\lambda_4$-optimal over all $d \in \mathcal{O}(v, b + 1, \lambda^*)$ with $(\lambda^*)' = ((pv + q)_1^b, v - 1)$ except possibly those for which $r_{dv} = r$, $n_{vdj} \geq 1$ $(1 \leq j \leq b)$ and $n_{vd, b + 1} = 1$ or $n_{vd, b + 1} = 2$ with $p < (v - 1 - 2q)/2$.

Proof: We apply Lemma 4.3.3.

The second paragraph in the proof of Theorem 4.3.3 holds for $2 \leq q \leq v - 1$, so $d^*$ E-betters $d$ for which $n_{dv, b + 1} \geq 1$ and $r_{dv} < r$.

If $r_{dv} = r$ and $n_{dv, b + 1} \geq 1$ then $r - \sum_{j=1}^{b} \frac{n_{dvj}^2}{k} - \frac{n_{dv, b + 1}^2}{v - 1} \leq B(r; x, b, k) + \frac{x^2}{v - 1}$ for $n_{dv, b + 1} \geq x = 3, 2, 1$ with the last term increasing as $x$ decreases. For $x = 3$, $e = 1$ (two cases) and $x = 3$, $e \geq 2$ we get $c_{dvv} \leq r - (\lambda_0/k) + \frac{6p + 1}{k} - \frac{9}{v - 1}$ or $r - (\lambda_0/k) + \frac{6p + 3}{k} - \frac{9}{v - 1}$, and the larger of which is less than $r - (\lambda_0/k)$ for $p \geq 1$, $v \geq 3$ (recall $n_{dv, b + 1} \geq 3$ only for $v \geq 5$). For $x = 2$ we have $c_{dvv} \leq r - (\lambda_0/k) + \frac{4p + 2}{k} - \frac{4}{v - 1}$. $4p + 2/k - 4/v - 1 \leq 0$ for $p \geq (v - 1 - 2q)/2$. For $x = 1$ the bound on $c_{dvv}$ is never less than or equal to $r - (\lambda_0/k)$.

We apply Lemma 4.1.1, so an E-better $d$ must have $n_{dv, b + 1} = 1$ or $n_{dv, b + 1} = 2$ and $p < (v - 1 - 2q)/2$.

Now apply Theorem 2.1.1 to $d^*$ and the designs that it E-betters.

Remark 4.3.2: For $q = 2$, $d^0$ is $e$ copies $(e \geq 1)$ of the unique BBD in $\mathcal{O}(v, v(v - 1)/2, pv + 2)$, so we can get the general form for $\lambda_0$, $\lambda_1$, $C_d^*$ and $C_d$. For $3 \leq q \leq v - 2$ this can not be accomplished in general.
Example 4.3.1  $q = 2$ or $q = v-2$. $\hat{d}$, $d^* \in A(4,6+1,k^*)$ with $k = 6$, and $\hat{d}$ is A-, D- and E-better.

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
d^*: & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 \\
2 & 3 & 3 & 3 & 3 & 3 & 4 & 4 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
\hat{d}: & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
1 & 1 & 1 & 2 & 2 & 2 & 2 & 3 \\
2 & 3 & 3 & 3 & 3 & 4 & 4 & 4 \\
\end{array}
\]

Example 4.3.2  $q = 3$ or $q = v-2$. $\hat{d}$, $d^* \in A(5,10+1,k)$ with $k = 8$, and $\hat{d}$ is A-, D- and E-better than $d^*$, which is not given here.

\[
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
\hat{d}: & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 5 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
1 & 1 & 1 & 1 & 1 & 2 & 2 & 2 \\
2 & 2 & 2 & 3 & 3 & 4 & 3 & 4 \\
3 & 4 & 4 & 4 & 5 & 4 & 5 & 5 \\
\end{array}
\]
In Chapter 8 the following efficiencies for cases from this section will be presented: for $k = pv$ the $A$- and $D$- efficiencies of $d^*$ and the $E$-efficiencies of $d^*$ and $\hat{d}$, and for $k = pv+q > v$ the $A$, $D$- and $E$-efficiencies for $d^* (1 < q < v-1)$ and $\hat{d} (q = 1$ and $q = v-1)$. 
CHAPTER 5
THE BBD MINUS ONE OBSERVATION

5.1 Preliminaries

Throughout Chapter 5 define \( d^0 \) and \( d^* \) as follows. Let \( d^0 \) be a BBD in \( \mathcal{B}(v, b, k) \). Then \( C_{d^0} = (r - \frac{\lambda_0}{k} + \frac{\lambda_1}{k})I_v - \frac{\lambda_1}{k}J_{v,v} \) with \( r^0 = r, \lambda^0_{i} = \lambda_0 (1 \leq i \leq v) \) and \( \lambda^0_{ij} = \lambda_1 (1 \leq i < j \leq v) \). \( d^0 \) has all its blocks being binary and is of maximum trace in \( \mathcal{B}(v, b, k) \).

Let \( d^* \) be the design in \( \mathcal{B}(v, b, k^*), (k^*)' = (k-1, k_2^{b-1}) \), whose last \( b-1 \) blocks are exactly those of \( d^0 \) but whose first block is that of \( d^0 \) but with an application of treatment 1 removed. Without loss of generality we will assume the first block of \( d^0 \) has treatments 1 through \( k \) if \( k < v \) or has treatments 1 through \( q \) extra-replicated if \( k = pv+q, p \geq 1 \).

\( d^* \) is of maximum trace in \( \mathcal{B}(v, b, k^*) \) since \( n_{d^0 \mid 1} = \max_{1 \leq i \leq v} n_{d^0 \mid 1} \) and the blocks of \( d^* \) are binary. Note that if \( k \leq v \), treatment 1 could only be removed from a block in which it appeared, but if \( k > v+1 \) we could remove an observation of 1 where it was not equi-replicated. This case gives a design \( \tilde{d} \) that is E-worse (and \( \mathcal{B}_4 \)-worse) than \( d^* \), and will be considered later.
For $d^*$ the $r_{d^*i}$ are as equal as possible with $r_{d^*1} = r-1$,
$r_{d^*i} = r$, $2 \leq i \leq v$. For any $d \in \mathcal{D}(v,b,k^*) \min_{1 \leq i \leq v} (r_{d1}) \leq r-1$, so without loss of generality assume $r_{d1}$ equals this minimum for each $d$, though it may not be unique.

Note that we must have $k \geq 3$ by what was said in Section 1.5. For $k = 2$ loss of one observation in a block makes the other observation useless.

The cases $k < v$ and $k \geq v$ will be discussed separately in Sections 5.2 and 5.3 respectively.

5.2 The Cases $3 < k \leq v-1$

If the first block of $d^0$ contained treatments 1 through $k$, then

$$C_{d^*} = \begin{bmatrix}
C_{11} & C_{12} & C_{13} \\
C_{22} & C_{23} \\
sym & C_{33}
\end{bmatrix}$$

where $C_{11} = r - 1 - \frac{1}{k} \sum_{j=2}^{b} r_{d^*1j} = (r-1)(k-1)/k$, $C_{12} = \frac{1}{k} \sum_{j=2}^{b} r_{d^*1j}$,

$$C_{13} = -\frac{\lambda_1}{k} J_{v-k,1}^t$$

$C_{22} = (\frac{(r-1)(k-1)}{k} + 1 + \frac{\lambda_1}{k}) I_{k-1} - \frac{\lambda_1}{k} I_{k-1} + \frac{\lambda_1}{k} J_{k-1,k-1}$, $C_{23} = -\frac{\lambda_1}{k} J_{k-1,v-k}$ and $C_{33} = \frac{r(k-1)}{k} + \frac{\lambda_1}{k} I_{v-k}$.

From Lemma 2.4.1 we can get the eigenvalues of $C_{d^*}$. They are $0, \frac{r(k-1)}{k} + \frac{\lambda_1}{k}$ with multiplicity $v-k-1, \frac{(r-1)(k-1)}{k} + 1 - \frac{1}{k} + \frac{\lambda_1}{k} = \frac{v\lambda_1}{k}$ with multiplicity $k-2$, and $(A + (B_1^2 + B_2^2)^{\frac{1}{2}})/2k$.
where \( A = 2v\lambda_1-k \), \( B_1 = k-2 \), and \( B_2 = 4(k-1) \). \( B_1^2 + B_2 = k^2 \) so the last two eigenvalues become \((v\lambda_1/k) - 1\) and \((v\lambda_1/k)\). Thus \( \lambda_{d^*} \) has eigenvalues \( 0 < (v\lambda_1/k) - 1 < (v\lambda_1/k) = \ldots = (v\lambda_1/k) \), in the form of Theorem 2.1.1.

**Theorem 5.2.1:** \( d^* \) is \( J_4 \)-better than all \( d \in \mathcal{D}(v,b,k^*) \), \((k^*)' = (k-1,k_1b-1)\), except possibly a design with \( r_{di} \geq r-1 \) \((1 \leq i \leq v)\), all blocks binary, and if \( r_{di} = r_{dj} = r-1 \), \( \lambda_{dij} \geq \lambda_1 \) \((v > k \geq 3)\).

**Proof:** Suppose \( d \) has \( r_{d1} \leq r-2 \). By Theorem 2.2.1

\[
\mu_{d1} \leq \frac{v}{v-1} \left( r_{d1} - \frac{n_{d11}}{k-1} - \sum_{j=2}^{b} \frac{n_{d1j}}{k} \right).
\]

(5.2.1)

There is some \( n_{d1j} = 0 \), since \( k_j < v \) \((1 \leq j \leq b)\) and \( r_{d1} \leq r-2 \). If \( n_{d1b} = 0 \), say, then we continue with (5.2.1)

\[
= \frac{v}{v-1} \left( r_{d1} - \frac{n_{d11}}{k-1} - \frac{b-1}{k} \sum_{j=2}^{b} \frac{n_{d1j}}{k} \right) \leq \frac{v}{v-1} \left( r_{d1} - \sum_{j=1}^{b-1} \frac{n_{d1j}}{k} \right)
\]

\[
\leq \frac{v}{v-1} \left( r_{d1} - \frac{r_{d1}}{k} \right) \leq \frac{v}{v-1} \left( r-2 \right) \left( k-1 \right)
\]

\[
= \frac{v\lambda_1}{k} - \frac{2v(k-1)}{k(v-1)} < \frac{v\lambda_1}{k} - 1
\]

for \( k \geq 3, v \geq 3 \). If \( n_{d11} = 0 \) we continue with (5.2.1)

\[
= \frac{v}{v-1} \left( r_{d1} - \sum_{j=2}^{b} \frac{n_{d1j}}{k} \right) \leq \frac{v}{v-1} \left( r_{d1} - \frac{r_{d1}}{k} \right)
\]

and the rest follows identically.
Suppose $n_{d11} \geq 2$, $r_{d1} = r-1$. Again

$$
\mu_{d1} \leq \frac{v}{v-1} \left( r-1 - \frac{n_{d11}^2}{k-1} - \frac{\sum_{j=2} b j n_{d1j}^2}{k^2} \right) \leq \frac{v}{v-1} \left( r-1 - \frac{r-1-n_{d11}}{k} - \frac{n_{d11}^2}{k-1} \right)
$$

$$
= \frac{v}{v-1} \left( \frac{(r-1)(k-1)}{k} + \frac{n_{d11} n_{d11}}{k} - \frac{n_{d11}^2}{k-1} \right)
$$

$$
= \frac{v \lambda_1}{k} - \frac{v}{v-1} \left( \frac{n_{d11}^2}{k-1} - \frac{n_{d11}}{k} + \frac{1}{k-1} \right). \quad (5.2.2)
$$

Now $(a^2/(k-1))a/k$ is increasing in $a \geq 1$, $k \geq 3$. If $n_{d11} = 2$, then (5.2.2) is less than $(v \lambda_1/k)-1$, and so must be for $n_{d11} \geq 2$.

Suppose $n_{d1j} \geq 2$ for some $j \in \{2, \ldots, v\}$ with $r_{d1j} = r-1$. Without loss of generality let $j = b$.

$$
\mu_{d1} \leq \frac{v}{v-1} \left( r-1 - \frac{n_{d11}^2}{k-1} - \frac{n_{d1b}^2}{k} - \frac{b}{k} \sum_{j=2} n_{d1j}^2 \right)
$$

$$
\leq \frac{v}{v-1} \left( r-1 - \frac{r-1}{k} + \frac{n_{d11} n_{d11}}{k} - \frac{n_{d11}^2}{k-1} + \frac{n_{d1b} n_{d1b}}{k} - \frac{n_{d1b}^2}{k} \right)
$$

$$
= \frac{v \lambda_1}{k} - \frac{v}{v-1} \left( \frac{n_{d11}^2}{k-1} - \frac{n_{d11}}{k} + \frac{n_{d1b}}{k} - \frac{n_{d1b}}{k} \right). \quad (5.2.3)
$$

$a(a-1)/k$ is increasing for $a \geq 1$. The proof that (5.2.3) is less than $\mu_{d1}$ is analogous to that for (5.2.2), except that $n_{d11} = 0, 1$ are considered separately.

Suppose $r_{d1} = r_{d2} = r-1$, say, and $\lambda_{d12} \leq \lambda_{11}$. We also know $n_{d1j}, n_{d2j}$ are 0 or 1 for all $j$. Theorem 2.2.1 says

$$
\mu_{d1} \leq \frac{1}{2} \left( r-1 - \frac{n_{d11}^2}{k-1} - \frac{b}{k} \sum_{j=2} n_{d1j} + r-1 - \frac{n_{d21}^2}{k-1} - \frac{b}{k} \sum_{j=2} n_{d2j}^2 + \frac{n_{d11} n_{d12}}{k-1} + \frac{b}{k} \sum_{j=2} n_{d1j} n_{d2j} \right). \quad (5.3.4)
$$
If at least one of \( n_{dl1}, n_{d21} \) is zero, (5.3.4) is

\[
\leq r-1 - \frac{b}{2} \sum_{j=1}^{\frac{n_d}{k}} \frac{2}{k} + \sum_{j=1}^{\frac{n_d}{k}} \frac{n_{d2j}}{k} + \sum_{j=2}^{\frac{n_d}{k}} \frac{n_{dlj}n_{d2j}}{k}
\]

\[
\leq r-1 - \frac{r-1}{k} + \frac{\lambda_{1-1}}{k} = \frac{r(k-1)}{k} + \frac{\lambda_{1-1}}{k} - \frac{k-1}{k} - \frac{1}{k} = \frac{\nu_{\lambda_{1-1}}}{k} - 1.
\]

If \( n_{dl1} = n_{d21} = 1 \), then (5.3.4) is

\[
\leq r-1 - \frac{r-2}{k} - \frac{1}{k-1} + \frac{\lambda_{1-2}}{k} + \frac{1}{k-1} = \frac{r(k-1)}{k} - 1 + \frac{\lambda_{1}}{k} = \frac{\nu_{\lambda_{1}}}{k} - 1.
\]

Therefore we have shown \( d^* \) is E-better than all \( d \in D(v,b,k) \) except possibly those with the characteristics given in the hypothesis. Application of Theorem 2.1.1 to \( d^* \) and the designs it E-betters completes the proof.

Remark 5.2.1: The proof that \( d^* \) E-betters any design with \( n_{dl1} \geq 2 \) also shows \( d^* \) is E-better than a design with \( n_{dl1} = 1 \) if \( v \leq k(k-1)/(k-2) \). The right hand side equals 6 for \( k = 3 \) and 4, and is increasing after that. So \( d^* \) E-betters designs with \( r_{dl} = r-1, n_{dl1} = 1 \) for \( v = 5 \) and \( v = 6 \), \( v = 7 \) with \( k = 6 \), but not necessarily for \( v = 7 \) with \( k = 3, 4 \) or 5.

A-, D- and E-efficiencies for \( d^* \) will be presented in Chapter 8.

5.3 The Cases \( k > v \)

An application of treatment 1 is removed from block 1, where it is extra-replicated. Since the forms of \( C_{d^0} \) and \( C_{d^*} \) are different for each \( k = pv+q \ (0 \leq q \leq v-1, \ p \geq 1) \) we must consider the cases separately.
For $k = v$, by Theorem 4.1.1, $d^*$ is $\mathcal{A}_4$-optimal in $\mathcal{A}(v,b,k^*)$.

For $k = pv$, $p \geq 2$, $C_{d^*}$ is of the form
\[
\begin{bmatrix}
    r - \frac{\lambda_0}{k} - 1 + \frac{p^2}{k} - \frac{(p-1)^2}{k-1} & -\frac{\lambda_1}{k} - \frac{p^2}{k} + \frac{p(p-1)}{k-1} \\
    \text{sym} & (r - \frac{\lambda_0}{k} + \frac{\lambda_1}{k}) I_{v-1} - (\frac{\lambda_1}{k} - \frac{p^2}{k} + \frac{p(p-1)}{k-1}) I_{v-1, v-1}
\end{bmatrix}
\]

From Lemma 1.4.1 the eigenvalues of $C_{d^*}$ are 0, $r - \frac{\lambda_0}{k} + \frac{\lambda_1}{k} = \frac{v\lambda_1}{k}$ with multiplicity $v-2$, and
\[
v(\frac{\lambda_1}{k} - \frac{p^2}{k} + \frac{p(p-1)}{k-1}) = \frac{v\lambda_1}{k} - v(\frac{p^2}{k} - \frac{p(p-1)}{k-1}) = \frac{v\lambda_1}{k} - \frac{v}{v-1} (1 - (\frac{p^2}{k} - \frac{(p-1)^2}{k-1})).
\]

These are of the form of Theorem 2.1.1.

**Theorem 5.3.1:** $d^*$ is $\mathcal{A}_4$-optimal over all $d \in \mathcal{A}(v,b,k^*)$ where $(k^*)' = (pv-1, pv_{b-1})$, $p \geq 1$ and $v \geq 3$.

**Proof:** We assume $p \geq 2$, as the $p = 1$ case was proved in Theorem 4.1.1. Recall that $d^*$ has the most balanced $r_{d1}$: $r_{1}, r_{2}, \ldots, r_{v}$ with $r = bp^2$. Let $d \in \mathcal{A}(v,b,k^*)$ be arbitrary and recall $r_{d1} = \min_{1 \leq i \leq v} (r_{d1})$.

Theorem 2.2.1 says that
\[
\mu_{d1} \leq \frac{v}{v-1} \left( r_{d1} - \frac{n_{d11}^2}{k-1} - \sum_{j=2}^{v} \frac{n_{d1j}^2}{k-1} \right) = \frac{v}{v-1} \left( r_{d1} - \frac{n_{d11}^2}{k-1} \right) + \sum_{j=2}^{v} \frac{n_{d1j}^2}{k-1} + \frac{n_{d11}^2}{k-1}
\]
\[
\begin{align*}
\leq & \frac{\nu}{v-1} \left( B(r_{d11} - n_{d11} \mid b-1, k) + n_{d11} - \frac{n_{d11}^2}{k-1} \right) \\
\leq & \frac{\nu}{v-1} \left( B(r-1 - n_{d11} \mid b-1, k) + n_{d11} - \frac{n_{d11}^2}{k-1} \right) \\
= & \frac{\nu}{v-1} \left( r-1 - \sum_{j=2}^{b} \frac{\tilde{n}_{1j}^2}{k} - \frac{n_{d11}^2}{k-1} \right) \\
\quad (5.3.1)
\end{align*}
\]

where the \( \tilde{n}_{1j} \geq 0 \) (2 \( \leq j \leq b \)), are as equal as possible, and

\[
\sum_{j=2}^{b} \tilde{n}_{1j} + n_{d11} = r-1.
\]

If the \( \tilde{n}_{1j} \) (2 \( \leq j \leq b \)) and \( n_{d11} \) are as equal as possible, then they take on the values \( p-1, p, \ldots, p \). In this case

\[
\sum_{j=2}^{b} \frac{\tilde{n}_{1j}^2}{k} + \frac{n_{d11}^2}{k-1}
\quad (5.3.2)
\]

is minimized if the \( p-1 \) is assigned to \( n_{d11} \). Now we must show (5.3.1) is maximized when the \( \tilde{n}_{1j} \) and \( n_{d11} \) are as equal as possible with \( n_{d11} = p-1 \).

There are two other cases for the values of the \( \tilde{n}_{1j} \) and \( n_{d11} \). The first is for \( n_{d11} < p-1 \) and \( \tilde{n}_{1j} \geq p \) (2 \( \leq j \leq b \)) with at least one \( \tilde{n}_{1j} > p \). The second is that \( n_{d11} > p-1 \) and \( \tilde{n}_{1j} \leq p \) (2 \( \leq j \leq b \)) with at least one \( \tilde{n}_{1j} < p \).

If the first case is true, let \( \tilde{n}_{1j} = p+\varepsilon \), \( \varepsilon \geq 1 \), and \( n_{d11} = p-1-m \), \( m \geq 1 \). Then

\[
\frac{\tilde{n}_{1j}^2}{k} + \frac{n_{d11}^2}{k-1} = \frac{(p+\varepsilon)^2}{k} + \frac{(p-1-m)^2}{k} > \frac{(p+\varepsilon-1)^2}{k} + \frac{(p-m)^2}{k-1} =
\]

\[
\frac{(\tilde{n}_{1j}-1)^2}{k} + \frac{(n_{d11}+1)^2}{k-1}
\]
for \( v \geq 3, p \geq 2 \). This means we can minimize (5.3.2) by pushing \( n_{d11} \) up to \( p-1 \) while making the remaining \( n_{d1j} \) equal to \( p \).

If the second case is true, let \( \tilde{n}_{1j} = p-\ell, \ell \geq 1 \) with \( n_{d11} = p-1+m, m \geq 1 \). Then

\[
\frac{(p-\ell)^2}{k} + \frac{(p-1+m)^2}{k-1} > \frac{(p-\ell+1)^2}{k} + \frac{(p+m-2)^2}{k-1}
\]

for \( v \geq 3, p \geq 2 \). Again we maximize (5.3.2) by making the \( n_{d1j} \) (1 \( \leq j \leq b \)) equal to \( p-1, p, \ldots, p \), respectively.

Therefore (5.3.2) is

\[
\leq \frac{v}{v-1} \left( r-1 - \frac{(b-1)p^2}{k} - \frac{(p-1)^2}{k-1} \right) = \mu_{d^*1},
\]

the most balanced case, that of \( d^* \). \( d^* \) is E-best in \( D(v,b,k^*) \) so by Theorem 2.1.1, we are finished with the proof.

For \( k = pv+1, p \geq 1 \), \( C_{d^*} \) is of the form

\[
\begin{bmatrix}
\frac{\lambda_0}{k} - \frac{(p+1)^2}{k} - \frac{p^2}{k-1} & \left[ \frac{\lambda_1}{k} - \frac{p(p+1)}{k} + \frac{p^2}{k-1} \right] \overline{v-1} \\
\text{sym} & (r - \frac{\lambda_0}{k} + \frac{\lambda_1}{k}) I_{v-1} - \left( -\frac{\lambda_0}{k} - \frac{p^2}{k} + \frac{p^2}{k-1} \right) \overline{v-1, v-1}
\end{bmatrix}
\]

when we assume treatment 1 was extrareplicated in the first block.

Recall that \( r = e(p+1) + e(v-1)p, \) where \( b = ev, e \geq 1 \).

The eigenvalues of \( C_{d^*} \) are derived via Lemma 1.4.1, and are

0, \((v\lambda_1/k)\) with multiplicity \( v-2 \), and

\[
\frac{v\lambda_1}{k} - v\left( \frac{p(p+1)}{k} - \frac{p^2}{k-1} \right) = \frac{v\lambda_1}{k} - \frac{v}{v-1} \left( 1 - \frac{(p+1)^2}{k} - \frac{p^2}{k-1} \right)
\]

They are of the form of Theorem 2.1.1.
Theorem 5.3.2: \( d^* \) is \( J_4 \)-optimal over all \( d \in \mathcal{A}(v,b, k^*) \) where
\( (k^*)' = (pv,(pv+1))_{b-1} \), \( p \geq 1 \) and \( v \geq 3 \).

Proof: \( d^* \) has the most balanced configuration of \( r_{dl} \). Let \( d \in \mathcal{A}(v,b, k^*) \) be arbitrary.

\[
\mu_{dl} \leq \frac{v}{v-1} \left( r_{dl} - \frac{n_{dl1}^2}{k-1} - \sum_{j=2}^{b} \frac{n_{dlj}^2}{k} \right)
\leq \frac{v}{v-1} \left( r_{dl} - \frac{1}{k-1} \sum_{j=2}^{b} \frac{n_{lj}}{k} - \frac{n_{dl1}^2}{k} \right)
\tag{5.3.3}
\]

where the argument and definition of the \( \tilde{n}_{lj} \) follow that of (5.3.1). The only difference is that the most equal the \( \tilde{n}_{lj} \) (2 \( \leq j \leq b \)) and \( n_{dl1} \) can be is for e-1 of them to equal \( p+1 \) and \( e(v-1)+1 \) of them to equal \( p \). (In \( d^0 \), for each treatment we had \( e \) blocks with \( n_{dlj} = p+1 \) and \( e(v-1) \) blocks with \( n_{dlj} = p \).). In this most balanced case, (5.3.2) is minimized if a \( p \) is assigned to \( n_{dl1} \). Now we show (5.3.3) is maximized by this case.

One other possibility is that \( n_{dl1} < p \), e-1 of the \( \tilde{n}_{lj} \geq p+1 \), e\((v-1)\) of the \( \tilde{n}_{lj} \geq p \), and at least one inequality on the \( \tilde{n}_{lj} \) is strict. Another possibility is that \( n_{dl1} > p \), e-1 of the \( \tilde{n}_{lj} \leq p+1 \), e\((v-1)\) of the \( \tilde{n}_{lj} \leq p \), and at least one inequality on the \( \tilde{n}_{lj} \) is strict.

For the first case,
\[
\frac{\tilde{n}_{lj}^2}{k} + \frac{n_{dl1}^2}{k-1} = \frac{(p+1)^2}{k} + \frac{(p-m)^2}{k} + \frac{(p+e-1)^2}{k} + \frac{(p-m+1)^2}{k-1}
\]

for \( e \geq 1 \), \( m \geq 1 \), \( v \geq 3 \), \( p \geq 1 \). Therefore we maximize (5.3.3) by pushing \( n_{dl1} \) up to \( p \), keeping the other \( n_{dlj} \) as equal as possible.
For the second case,
\[
\frac{n_{11}^2}{k} + \frac{n_{d11}^2}{k-1} = \frac{(p+1-\varepsilon)^2}{k} + \frac{(p+m)^2}{k-1} > \frac{(p+2-\varepsilon)^2}{k} + \frac{(p+m-1)^2}{k-1}
\]
for \( \varepsilon \geq 1, m \geq 1, v \geq 3, p \geq 1 \). We maximize (5.3.3) by pushing \( n_{d11} \) down to \( p \), keeping the other \( n_{dlj} \) as equal as possible.

Finally then, (5.3.3) is
\[
\leq \frac{v}{v-1} \left( (r-1) \left( \frac{(p+1)^2}{k} + e(v-1)p^2 \right) - \frac{p^2}{k-1} \right) = \mu_{d\ast 1},
\]
and by Theorem 2.1.1 the proof is complete.

If we think of running an experiment with a BBD, \( k = pv+1 \) and \( p \geq 1 \), then we have just shown that if an observation must be thrown out, and it occurred in a block where that treatment was extra-replicated, the resulting design is \( \omega^2 \)-optimal. If the observation was from a block where the treatment was not extra-replicated, how much worse is the resulting design? If we call this design \( \tilde{d} \) and assume treatment 1 was lost in block 1, then \( \tilde{C}_d \) is of the form
\[
\begin{bmatrix}
C_{11} & C_{12} & C_{13} \\
C_{21} & C_{22} & C_{23} \\
\text{sym} & & C_{33}
\end{bmatrix}
\]
(5.3.4)

where
\[
C_{11} = r - \frac{\lambda_0}{k} - 1 + \frac{p^2}{k} - \frac{(p-1)^2}{k-1}, \quad C_{12} = -\left( \frac{1}{k} - \frac{p(p+1)}{k} + \frac{(p+1)(p-1)}{k-1} \right),
\]
\[
C_{13} = -\left( \frac{1}{k} - \frac{p^2}{k} + \frac{p(p-1)}{k-1} \right), \quad C_{22} = r - \frac{\lambda_0}{k} + \frac{(p+1)^2}{k} - \frac{(p+1)^2}{k-1},
\]
\[
C_{23} = -\left( \frac{1}{k} - \frac{p^2}{k} + \frac{p(p-1)}{k-1} \right), \quad C_{33} = \begin{bmatrix} \end{bmatrix}
\]
\[ C_{23} = -\left(\frac{\lambda_1}{k} - \frac{p(p+1)}{k} + \frac{p(p+1)}{k-1}\right)_{v-2}, \quad C_{33} = (r - \frac{\lambda_0}{k} + \frac{\lambda_1}{k})_{v-2} - \left(\frac{\lambda_1}{k} - \frac{p^2}{k-1}\right)_{v-2} \text{, and without loss of generality treatment } 2 \]

was extra-replicated in block 1. Using Lemma 2.4.1 we get as eigenvalues of \( C_d \): 0, \((v\lambda_1/k)\) with multiplicity \( v-2 \), and

\[ \mu_{d1} = \frac{v\lambda_1}{k} - \frac{vp^2(v-1)}{k(k-1)} \quad \mu_{d^*} = \frac{v\lambda_1}{k} - \frac{vp^2(v-1)}{k(k-1)} = \mu_{d^*}. \]

The eigenvalues of \( C_d \) differ from those of \( C_{d^*} \) only in that \( \mu_{d1} < \mu_{d^*} \) by exactly the amount that \( \text{tr}(C_d) < \text{tr}(C_{d^*}) \).

For \( k = pv+q, \ 2 \leq q \leq v-1 \), \( C_{d^*} \) is of the form of (5.3.4) except that

\[ C_{11} = (r - \frac{\lambda_0}{k} - 1 + (p+1)^2 - \frac{p^2}{k-1}, \quad C_{12} = -\left(\frac{\lambda_1}{k} - \frac{(p+1)^2}{k} + \frac{p(p+1)}{k-1}\right)_{v-2}, \]

\[ C_{13} = -\left(\frac{\lambda_1}{k} - \frac{p(p+1)}{k} + \frac{p^2}{k-1}\right)_{v-2}, \]

\[ C_{22} = (r - \frac{\lambda_0}{k} + \frac{\lambda_1}{k})_{v-2} - \left(\frac{\lambda_1}{k} - \frac{(p+1)^2}{k} + \frac{p(p+1)}{k-1}\right)_{v-2}, \]

\[ C_{23} = -\left(\frac{\lambda_1}{k} - \frac{p(p+1)}{k} + \frac{p(p+1)}{k-1}\right)_{v-2}, \]

\[ C_{33} = (r - \frac{\lambda_0}{k} + \frac{\lambda_1}{k})_{v-2} - \left(\frac{\lambda_1}{k} - \frac{p^2}{k} + \frac{p^2}{k-1}\right)_{v-2}, \]

and treatments 1 through \( q \) were assumed to be extra-replicated in block 1. Using Lemma 2.4.1 we get the eigenvalues 0, \((v\lambda_1/k)\) of multiplicity \( q-2 + v-q-1 = v-3 \), and two from \((A + (B_1^2 + B_2^2)^{1/3})/2\). The last two eigenvalues are

\[ \frac{v\lambda_1}{k} \quad \text{and} \quad \frac{\lambda_1}{k} - 1 + \frac{vp(p+1)}{k(k-1)}. \]
so the eigenvalues of \( C_{d^*} \) are still of the form \( 0 < a < b' = \ldots = b \).

**Theorem 5.3.3:** \( d^* \in \mathcal{D}(v,b,k^*) \) where \((k^*)' = (pv+q-1, (pv+q)_{b-1})\), 
\( 2 \leq q \leq v-1 \), is \( \mathcal{D}_4 \)-optimal over all \( d \) except possibly those with 
\[ r_{d^{*}1} = \min_{1 \leq i \leq v} (r_{di}) = r-1 \] and \( n_{d11} = p \) or \( p+1 \). \( (v \geq 3) \).

**Proof:** \( d^* \) has the most balanced configuration of \( r_{di} \). As in the previous two theorems we will show an \( E \)-better design must have 
\( r_{d^{*}1} = r-1 \), but we can no longer carry through the same arguments. 
This is because \( C_{d^*} \) is not a 2-by-2 blocked matrix.

Let \( d \in \mathcal{D}(v,b,k^*) \) be arbitrary. Theorem 2.2.1 says

\[
\mu_{d1} \leq \frac{v}{v-1} \left( r_{d1} - \frac{b}{v} \sum_{j=2}^{n_{d11}} \frac{n_{d1j}}{k} - \frac{n_{d11}}{k-1} \right)
\]

\[
\leq \frac{v}{v-1} \left( B(r_{d1} - n_{d11}; b-1, k) + n_{d11} - \frac{n_{d11}}{k-1} \right)
\]

\[
\leq \frac{v}{v-1} \left( B(r_{d1} - n_{d11}; b-1, k) + n_{d11} - \frac{n_{d11}}{k-1} \right)
\]

\[
= \frac{v}{v-1} \left( r_{d1} - \sum_{j=2}^{n_{d11}} \frac{n_{d1j}}{k} - \frac{n_{d11}}{k-1} \right)
\]

\[
= \frac{v}{v-1} \left( r_{d1} - \sum_{j=2}^{n_{d11}} \frac{n_{d1j}}{k} - \frac{n_{d11}}{k-1} \right)
\]

\[
\left(5.3.5\right)
\]

where the arguments and definition of the \( \tilde{n}_{1j} \) follow as at (5.3.1).

If for \( d^0 \) we let \( n_{d^01j} = p+1 \) for \( b_0 \) blocks, and \( n_{d^01j} = p \) for \( b-b_0 \) blocks, then for \( d^* \) \( n_{d^*1j} = p+1 \) for \( b_0-1 \) blocks and \( p \) for \( b-b_0 \) blocks.

(5.3.2) is again minimized if a \( p \) is assigned to the \( n_{d11} \) and the \( \tilde{n}_{1j} \) \( (2 \leq j \leq b) \) equal \( p \) or \( p+1 \).
In the Theorems 5.3.1 and 5.3.2 we showed that any other configuration made the equivalent of 5.3.5 less than that for the most balanced case, which was bounded by \((v\lambda_1/k)-1\). Of course less balanced configurations of the \(n_{d_{1j}}\) still give bounds lower than that for the balanced case, but the balanced case bound is too large.

If \(n_{d_{11}} = p\) (5.3.5) is

\[
\leq \frac{v}{v-1} \left( r-1 - \frac{(b_0-1)(p+1)^2 + (b-b_0)p^2}{k} - \frac{p^2}{k-1} \right)
\]

but this is greater than \((v\lambda_1/k)-1\) for all \(v, p\) and \(q\).

If \(n_{d_{11}} \leq p-1\) (5.3.5) is

\[
\leq \frac{v}{v-1} \left( r-1 - \frac{b_0(p+1)^2 + (b-b_0-1)p^2}{k} - \frac{(p-1)^2}{k-1} \right) \leq \mu_{d_1^*}.
\]

If \(n_{d_{11}} \geq p+2\) (5.3.5) is

\[
\leq \frac{v}{v-1} \left( r-1 - \frac{(b_0-3)(p+1)^2 + (b-b_0+2)p^2}{k} - \frac{(p+2)^2}{k-1} \right) \leq \mu_{d_1^*}
\]

If \(n_{d_{11}} = p+1\) (5.3.5) is

\[
\leq \frac{v}{v-1} \left( r-1 - \frac{(b_0-2)(p+1)^2 + (b-b_0+1)p^2}{k} - \frac{(p+1)^2}{k-1} \right) \leq \mu_{d_1^*}
\]

if and only if \(q^2 - q(v+1) + 2v(p+1) \geq 0\). This holds for \(q = 2\) and \(q = v-1\) (any \(v\) and \(p\)) but not for all \(v, p\) and \(q\).

Therefore \(d^*\) is E-better than all \(d\) except possibly some for which \(r_{d_{11}} = r-1\), and \(n_{d_{11}} = p\) or \(p+1\). Theorem 2.1.1 is now applied to the E-bettered (by \(d^*\)) designs.
Remark 5.3.1: For the cases of \( q = 2 \) and \( q = v-1 \) an E-better (than \( d^* \)) \( d \) must have for any minimum replicated treatment \( r_{di} = r-1 \) and \( n_{di1} = p \). These are two cases where \( d^* \) can be specified in general but no E-better \( \tilde{d} \) has been found, nor has it been proven not to exist.

If we remove treatment 1 from a block in \( d^0 \) where 1 was not extra-replicated, and call the design \( \tilde{d} \), then \( C_{\tilde{d}} \) is of the form (5.3.4). In this case

\[
C_{11} = r - \frac{\lambda_0}{k} - 1 + \frac{p^2}{k} - \frac{(p-1)^2}{k-1}, \quad C_{12} = -\left( \frac{\lambda_1}{k} - \frac{p(p+1)}{k} + \frac{p^2-1}{k-1} \right), \quad C_{13} = -\left( \frac{\lambda_1}{k} - \frac{p^2}{k} + \frac{p(p-1)}{k-1} \right) v-q-1,
\]

\[
C_{22} = (r - \frac{\lambda_0}{k} + \frac{\lambda_1}{k}) I_q \left( -\frac{\lambda_1}{k} + \frac{(p+1)^2}{k} - \frac{(p+1)^2}{k-1} \right), \quad C_{23} = -\left( \frac{\lambda_1}{k} - \frac{p(p+1)}{k} + \frac{p(p+1)}{k-1} \right) v-q-1
\]

where treatments 2 through \( q+1 \) were extra-replicated in block 1.

Lemma 2.4.1 yields the eigenvalues 0, \( \mu_{\tilde{d}_i} = \mu_{d^*i} (2 \leq i \leq v-1) \), and

\[
\mu_{\tilde{d}1} = \frac{\nu_1}{k} - 1 + \frac{vp(p+1)}{k(k-1)} - \frac{2}{k-1} < \mu_{d^*1}.
\]

In Chapter 8 A-, D- and E- efficiencies will be calculated for \( \tilde{d} \) when \( k = pv+1 \), and \( d^* \) and \( \tilde{d} \) for \( k = pv+q, 2 \leq q \leq v-1 \).
CHAPTER 6
THE BBD PLUS M BLOCKS

6.1 Preliminaries

Throughout Chapter 6 define $d^0$ and $d^*$ as follows. Let $d^0 \in \mathcal{B}(v,b,k)$ be a BBD with $k = pv+q$ ($1 \leq q \leq v-1$). Add to $d^0$ $m$ binary blocks which are disjoint with respect to their extra-replicated treatments, with $1 \leq m \leq v/q$. If we call the new design $d^*$, then $d^* \in \mathcal{B}(v,b+m,k)$ and $C_{d^*} = C_{d^0} + C_0$ where $C_0$ is the C-matrix for the blocks added.

Lemma 6.1.1: $C_0$ is of the form

$\begin{bmatrix}
A & B & \ldots & B & B & D \\
A & B & \ldots & B & D \\
A & B & \ldots & B & D \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
A & B & \ldots & B & D \\
A & D \\
\end{bmatrix}
$

where $A = (mp+1)I_q - \left(\frac{(m-1)p^2 + (p+1)^2}{k}\right)J_{q,q}$, $B = \left(\frac{(m-2)p^2 + 2p(p+1)}{k}\right)J_{q,q}$

$D = \left(\frac{(m-1)p^2 + p(p+1)}{k}\right)J_{v-mq,v-mq}$ and $E = mpI_{v-mq} - \frac{mp^2}{k}J_{v-mq,v-mq}$. 
Proof: The mq treatments extra-replicated in the added blocks appear mp+1 times in those blocks; the remaining treatments appear only mp times. The extra-replicated treatments are paired \((p+1)^2+(m-1)p^2\) times within added blocks, \(2p(p+1)+(m-2)p^2\) times between added blocks, and \((m-1)p^2+p(p+1)\) times with the remaining treatments. The non-extra-replicated treatments are paired mp^2 times in the added blocks.

Lemma 6.1.2: \(C_{d^*}\) has the eigenvalues 0,

\[
\frac{v\lambda_1}{k} + mp \text{ with multiplicity } v-mq-1 \text{ (unless } mq \geq v-1),
\]

\[
\frac{v\lambda_1}{k} + \frac{vp(mp+1)}{k} \text{ with multiplicity } 1 \text{ (unless } mq = v),
\]

\[
\frac{v\lambda_1}{k} + mp + 1 - \frac{q}{k} \text{ with multiplicity } m-1,
\]

\[
\frac{v\lambda_1}{k} + mp + 1 \text{ with multiplicity } m(q-1),
\]

where \(\lambda_i = \lambda_{d^*_{ij}}\) \((i \neq j)\) and for each case of m and q all realized eigenvalues are distinct. The eigenvalue structure is 0 < a < b = ... = b for m = 2 with q = v/2, m = 1 with q = v-1 or m = v-1 with q = 1. The eigenvalue structure is 0 < a = ... = a < b if m = 1 with q = 1 and is 0 < a = ... = a if m = v and q = 1.

Proof: \(C_{d^*} = (r - \frac{\lambda_0}{k} + \frac{\lambda_1}{k})I_v - \frac{\lambda_1}{k} J_{v,v}\) with \(\lambda_0 = \lambda_{d^*_{ij}}\) and \(r = r_{d^*_{ij}}\) \((1 \leq i \leq v)\). To \(C_{d^*} = C_{d^*} + C_0\) we apply Lemma 6.1.1 and Lemma 1.4.2 to get the eigenvalues.

It is easy to show that
\[
\frac{v\lambda_1}{k} + mp \leq \frac{v\lambda_1}{k} + \frac{vp(mp+1)}{k} < \frac{v\lambda_1}{k} + mp + 1 - \frac{q}{k}
\]

with equality between the first pair of eigenvalues only when mq = v, but in that case they are not realized.

To get the parameter values for which the various eigenvalue structures occur consider mq = v, mq = v-1 and 1 \leq mq \leq v-2 as separate cases. If mq = v, at most the two largest nonzero eigenvalues occur with m = 2 and m = v yielding results. If mq = v-1 at most the three largest eigenvalues occur with m = 1 and q = 1 giving results. If mq \leq v-2 then m = q = 1 gives the eigenvalue structure of Cheng. This ends the proof.

Note that for q = 0 all values of b \geq 2 allow a BBD in \mathcal{B}(v,b,pv) which is \mathcal{A}_U-optimal. Therefore we only consider 1 \leq q \leq v-1.

In Sections 6.2, 6.3 and 6.4 we will consider the cases mq = v, mq = v-1 and 1 \leq mq \leq v-2, respectively. Throughout \lambda_0, \lambda_1 and r are defined as above.

6.2 The Case mq = v

For mq = v the eigenvalues of C_{d^*} are 0, \((v\lambda_1/k) + mp + 1 - (q/k)\) and \((v\lambda_1/k) + mp + 1\) with multiplicities 1, m-1 and m(q-1) respectively.

All eigenvalues are equal if m = v and q = 1, in which case \(d^0\) comprises e copies (e \geq 1) of the unique BBD in \mathcal{B}(v,v,pv+1), and \(d^*\) will then be e+1 copies of the same. Therefore \(d^*\) is a BBD and we can apply Corollary 1.4.4, making \(d^*\) \mathcal{A}_U-optimal in \mathcal{B}(v,ev+v,pv+1).
Note that if \( m = 1 \) and \( q = v \) we get \( J_4 \)-optimality of \( d^* \), but this is really just the case of \( q = 0 \).

The eigenvalue structure \( 0 < a < b = \ldots = b \) is achieved in one case, that of \( m = 2 \) and \( q = v/2 \). The nonzero eigenvalues of \( d^* \) become

\[
\frac{v\lambda_1}{k} + mp + 1 - \frac{v}{2k} \quad \text{and} \quad \frac{v\lambda_1}{k} + mp + 1
\]

with multiplicities \( 1 \) and \( v-2 \) respectively. Two examples will show, however, that each \( m \) and \( q \) pair must be dealt with separately.

**Lemma 6.2.1:** If \( k = pv+q \), \( p \geq 1 \), \( 1 \leq q \leq v-1 \), \( d^0 \) is a BBD in \( \Omega(v,b,pv+q) \) and \( d^* \) is \( d^0 \) plus \( m \) disjoint binary blocks with \( mq = v \), then \( d^* \) is E-better than any \( d \in \Omega(v,b+m,pv+q) \) with \( \min_{1 \leq i \leq v} (r_{di}) \leq r + mp \), and any \( d \) for which \( \lambda_{dij} \leq \lambda_1 + mp^2 + 2p-(q-1) \) whenever \( r_i = r, j = r + mp + 1, \quad v \geq 3 \).

**Proof:** Using Theorem 1.4.3, and assuming without loss of generality that \( r_{d1} = \min_{1 \leq i \leq v} (r_{di}) \) for an arbitrary \( d \),

\[
\begin{align*}
\mu_{d1} & \leq \frac{v}{v-1} \left( r_{d1} - \sum_{j=1}^{b+m} \frac{n_{dij}}{k} \right) \leq \frac{v}{v-1} B(r_{d1}; b+m, k) \\
& \leq \frac{v}{v-1} B(r+mp; b+m, k) = \frac{v}{v-1} \left( r+mp - \frac{\lambda_0 + mp^2}{k} \right) \\
& = \frac{v\lambda_1}{k} + \frac{vmmp^2}{k} + \frac{v^2p}{k(v-1)} \leq \frac{v\lambda_1}{k} + mp + 1 - \frac{q}{k}.
\end{align*}
\]

If for \( d \), \( r_{d1} = r_{d2} = r + mp + 1 \) and \( \lambda_{d12} \leq \lambda_1 + mp^2 + 2p - (q-1) \),
\[ \mu_d 1 \leq r^{mp+1} - \frac{1}{2} \left( \sum_{j=1}^{b+m} \frac{n_{d1j}}{k} + \sum_{j=1}^{b+m} \frac{n_{d2j}}{k} \right) + \frac{\lambda_{d12}}{k} \]

\[ \leq r^{mp+1} - \frac{\lambda_0 + mp^2 + 2p + 1}{k} + \frac{\lambda_1 + mp^2 + 2p - (q-1)}{k} \]

\[ = \frac{v\lambda_1}{k} + mp + 1 - \frac{q}{k} \leq \frac{v\lambda_1}{k} + mp + 1. \]

If \( m = 1 \), \( \frac{v\lambda_1}{k} + mp + 1 \) is the smallest nonzero eigenvalue of \( d^* \).

For \( m \geq 2 \), \( \frac{v\lambda_1}{k} + mp + 1 - \frac{q}{k} \) is the smallest nonzero eigenvalue of \( d^* \). In either case \( \mu_d 1 \leq \mu_{d^* 1} \).

**Lemma 6.2.2**: If \( k = pv + 2, \ p \geq 1, \ m = v/2, \ d^0 \) and \( d^* \) are as in Lemma 6.2.1, then \( d^* \) is E-optimal in \( (v, b+m, pv+2) \). \( v \geq 3 \).

**Proof**: The nonzero eigenvalues of \( C_{d^*} \) are \((v\lambda_1/k) + mp + 1 - (2/k)\) and \((v\lambda_1/k) + mp + 1\) with multiplicities \((v/2)-1\) and \((v/2)\), respectively.

We also have

\[
C_{d^*} = \frac{1}{k} \begin{bmatrix}
    a & -(b+1) & -b & -b & \ldots & -b & -b \\
    a & -b & -b & \ldots & -b & -b \\
    a & -(b+1) & \ldots & -b & -b \\
    a & \ldots & -b & -b \\
    \vdots & \vdots & \vdots & \vdots \\
    a & -(b+1) & \ldots & \ldots & \ldots \\
    \end{bmatrix}
\]

(6.2.1)

with \( a = (v-1)\lambda_1 + k(mp+1) - (mp^2 + 2p + 1) \) and \( b = \lambda_1 + mp^2 + 2p \).
Lemma 6.2.1 says $d^*$ E-bests all $d$ with $\min_{1 \leq i \leq v} (r_{d, i}) \leq r + mp$. This means all $r_{d, i}$ must equal $r + mp + 1 = r_{d^*, i}$ $(1 \leq i \leq v)$ for an E-better (than $d^*$) $d$. Also Lemma 6.2.1 says $d^*$ E-betters any $d$ with $r_{d, i} = r_{d^*, i} = r + mp + 1$ and $\lambda_{d, ij} \leq \lambda_{d^*, ij} = \lambda_1 + mp^2 + 2p - 1$. Therefore all that is left are $d$ with all replications equal to $r + mp + 1$ and $\lambda_{d, ij} \geq \lambda_1 + mp^2 + 2p$.

However at most one of these can be equal to $\lambda_1 + mp^2 + 2p + 1$ in each row. Therefore an E-better $d$ must have $C_d$ of the form (6.2.1). So $d^*$ must be E-optimal in $\mathcal{A}(v, b+m, pv+2)$.

**Remark 6.2.1:** The $d^*$ in Lemma 6.2.2 is unique up to a permutation of the treatment labels.

**Theorem 6.2.1:** Let $v = 4$, $k = 4p+2$ $(p \geq 1)$, $d^0$ be a BBD in $\mathcal{A}(4, b, 4p+2)$ and $d^*$ be $d^0$ plus $m = 2$ disjoint binary blocks. Then $d^*$ is $\mathcal{J}_4$-optimal in $\mathcal{A}(4, b+2, 4p+2)$.

**Proof:** $d^*$ has maximum trace, eigenvalues of the form $0 < a < b=b$ by the proof of Lemma 6.2.2 with $(v/2) = 2$, and $d^*$ is E-optimal by Lemma 6.2.2. Therefore Theorem 2.1.1 gives us $\mathcal{J}_4$-optimality.

**Example 6.2.1:** $d^* \in \mathcal{A}(4, 8, 6)$ given below is $\mathcal{J}_4$-optimal.

$$
\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\
\hline
\text{d*}: & 4 & 4 & 4 & 4 & 4 & 4 & 4 \\
1 & 1 & 1 & 2 & 2 & 3 & 1 & 3 \\
2 & 3 & 4 & 3 & 4 & 4 & 2 & 4
\end{array}
$$
Example 6.2.1 is the $k > v$ extension of the $A_4$-optimal BIBD$(4, 6e, 2, 3e, e)$ (here $e = 1$) from Cheng (1979) that was mentioned in Section 3.3. In that same section the example of the BIBD$(8, 14, 4, 7, 3)$ plus two disjoint binary blocks being $A$-, $D$- and $E$-bettered by an RGD can also be extended to $k > v$.

Example 6.2.2: Let $d^0 \in O(8, 14, 8p+4)$ be the BBD constructed by adding $p$ copies of each treatment to each block of the BIBD$(8, 14, 4, 7, 3)$. Then $d^* \in O(8, 16, 8p+4)$ is $d^0$ plus two disjoint binary blocks. The eigenvalues of $C_{d^*}$ are $0, (8\lambda_1/k) + (16p(p+1)/k) = (8A+24)/k$, and $(8\lambda_1/k) + 2p + 1 = (8A+28)/k$ where $\lambda_1 = 14p^2+14p+3$ and $A = 16p(p+1)$.

Now let $\hat{d}$ be the RGD in $O(8, 16, 4)$ mentioned above with $p$ copies of each treatment added to each block. $\hat{d} \in O(8, 16, 8p+4)$ and the eigenvalues of $C_{\hat{d}}$ are $0, (8A+26)/k, (8A+28-\sqrt{2})/k, (8A+28)/k$, and $(8A+28+\sqrt{2})/k$ with multiplicities 1, 2, 2, 1 and 2 respectively. $\hat{d}$ is easily seen to be $E$-better than $d^*$ and with algebra is seen to be $A$- and $D$-better.

The eigenvalue structure $0 < a = \ldots = a < b$ is not realized for $mq = v$.

$A$-, $D$- and $E$-efficiencies for the cases of $m = 2$ and $q = v/2$ will be presented in Chapter 8, along with the $A$- and $D$-efficiencies for $q = 2$.

6.3 The Case $mq = v-1$

We begin with a lemma.
Lemma 6.3.1: For $k = pv + q$, $1 \leq q \leq v-1$, $p \geq 1$, $v \geq 3$, $mq \leq v-1$ and $d^*$ equal to $d^0$ plus $m$ disjoint binary blocks ($d^0$ a BBD in $\mathcal{B}(v,b, pv+q)$), then for any $d \in \mathcal{B}(v,b+m, pv+q)$

$$\mu_{d1} \leq \frac{v\lambda_1}{k} + \frac{v}{v-1} \left( mp + \frac{mp^2}{k} \right)$$

$$= \frac{v\lambda_1}{k} + \frac{vp}{k} \left( mp + \frac{mq}{v-1} \right).$$

Proof: Since $mq \leq v-1$, $\min \{ r_{d1} \} \leq r+mp$ for any $d \in \mathcal{B}(v,b+m, pv+q)$. Using Theorem 1.4.3, and assuming $l$ is the minimum replicated treatment,

$$\mu_{d1} \leq \frac{v}{v-1} \left( r_{d1} - \sum_{j=1}^{b+m} \frac{n_{d1j}}{k} \right) \leq \frac{v}{v-1} B(r_{d1}; b+m, k)$$

$$\leq \frac{v}{v-1} B(r+mp; b+m, k) = \frac{v}{v-1} (r+mp - \frac{\lambda_0 + mp^2}{k}).$$

Algebra completes the proof.

When $mq = v-1$ the eigenvalues of $C_{d^*}$ are $0, \frac{v\lambda_1}{k} + \frac{vp(mp+1)}{k}, \frac{v\lambda_1}{k} + mp + 1 - \frac{q}{k}$, and $\frac{v\lambda_1}{k} + mp + 1$ with multiplicities $l, l, m-1$ and $m(q-1)$, respectively.

Theorem 6.3.1: For $k = pv + q$, $1 \leq q \leq v-1$, $p \geq 1$, $v \geq 3$, $mq = v-1$, $d^*$ is E-optimal in $\mathcal{B}(v,b+m, pv+q)$.

Proof: By Lemma 6.3.1 for any $d \in \mathcal{B}(v,b+m, pv+q)$

$$\mu_{d1} \leq \frac{v\lambda_1}{k} + \frac{vp}{k} \left( mp + \frac{mq}{v-1} \right),$$

the smallest nonzero eigenvalue of $d^*$. This completes the proof.

Now from Lemma 6.1.2 we have $m = 1$ with $q = v-1$ and $m = v-1$ with $q = 1$ as the only two cases giving $d^*$ with one of the three eigenvalue
structures from Theorems 1.4.1, 1.4.4 and 2.1.1. The structure of the eigenvalues of $C_{d^*}$ is $0 < a < b = \ldots = b$ in both cases, with

$$a = \frac{\lambda_1}{k} + \frac{vp}{k} (mp+1).$$

**Theorem 6.3.2:** For $k = pv+q$, $p \geq 1$, $v \geq 3$, $m = T$ with $q = v-1$ or $m = v-1$ with $q = 1$, $d^*$ is $\mathcal{A}_4$-optimal in $\mathcal{A}(v,b+m,pv+q)$.

**Proof:** Theorems 6.3.1 and 2.1.1 with the fact that $C_{d^*}$ is of maximum trace gives the result.

A- and D-efficiencies for these E-optimal designs when $mq = v-1$, $m \neq 1$, $q \neq 1$ will be presented in Chapter 8.

### 6.4 The Cases $1 < mq < v-2$

From Lemma 6.1.2 only one of the cases $1 \leq mq \leq v-2$ yields two distinct eigenvalues, and that is $m = q = 1$. The eigenvalue structure for $d^*$ with $mq = 1$ is that of Theorem 1.4.1 with $C_{d^*}$ having eigenvalues $0$, $\frac{\lambda_1}{k} + p$ with multiplicity $v-2$, and $\frac{\lambda_1}{k} + \frac{vp(p+1)}{k}$.

For this case $d^0$ is $e$ copies ($e \geq 1$) of the unique BBD in $\mathcal{A}(v,v,pv+1)$, and $d^*$ is $d^0$ plus a binary block with, say, treatment 1 extra-replicated. There is a $\mathcal{A}$ in $\mathcal{A}(v, ev+1, pv+1)$ with $d^*$ that is $E$-better than $d^*$ when $p \geq v-1$, but can only be constructed when $p \geq v-2$. We construct $\mathcal{A}$ by taking $e-1$ copies of that unique BBD in $\mathcal{A}(v,v,pv+1)$, plus $v$ binary blocks with treatment 1 extra-replicated, and one block with treatments 2 through $v$ appearing $p+1$ times and treatment 1 appearing $p-(v-2)$ times.
\[ C_d = C_{d0} + C_0 \] where

\[ C_0 = \begin{bmatrix}
  \left( p+1 \right)^2 + v^2 - 3v + 2 - \frac{p(p+1)-(v-2)}{k} & \frac{(p+1)(v-2)}{k} \\
  \text{sym} & \left( p + \frac{1}{k} \right) I_{v-1} - \frac{p^2+1}{k} J_{v-1,v-1}
\end{bmatrix} \]

and so \( \hat{d} \) has the eigenvalues 0, \( \frac{v\lambda_1}{k} + p + \frac{1}{k} \) with multiplicity \( v-2 \),
and \( \frac{v\lambda_1}{k} + \frac{wp(p+1)}{k} - \frac{v(v-2)}{k} \).

We need \( p \geq v-1 \) for

\[ \frac{v\lambda_1}{k} + \frac{wp(p+1)}{k} - \frac{v(v-2)}{k} \geq \frac{v\lambda_1}{k} + p + \frac{1}{k}, \]

with strict inequality if \( p > v-1 \) and equality for \( p = v-1 \). Note that for \( p \geq v-1 \) \( \hat{d} \) is strictly \( E \)-better than \( d^* \) but that
\( \text{tr} \ C_{\hat{d}} < \text{tr} \ C_{d^*} \) by \( k^{-1}(v-1)(v-2) \). We summarize what is known about \( d^* \) and \( \hat{d} \) in this case of \( mq = 1 \) in the following theorem.

**Theorem 6.4.1:** \( d^* \) is \( E \)-better than any \( d \in D(v, ev+1, pv+1) \) for which 1) \( \min_{1 \leq i \leq v} (r_{di}) \leq r+p-1 \) or 2) \( \lambda_{dij} \leq \lambda_1 + p^2 \) when \( r_{di} = r_{dj} = r+p \). For \( p \geq v-1 \) there exists a \( \hat{d} \) (with \( r_{d1} = r+p+1 \), \( r_{dij} = r+p \) for \( 2 \leq i \leq v \) and \( \lambda_{dij} = \lambda_1 + p^2+1 \) for \( 2 \leq i < j \leq v \)) which is strictly \( E \)-better than \( d^* \). Therefore \( d^* \) is not \( g_{\lambda} \)-optimal for \( p \geq v-1 \).

**Proof:** That \( d^* \) is \( E \)-better than the \( \hat{d} \) satisfying 1) or 2) follows from Lemma 6.4.1 at the end of the chapter. The results for \( \hat{d} \) follow from the previous discussion.
Remark 6.4.1: From a few examples for this case of \( m_q = 1 \) \( \hat{d} \) appears to be \( A- \) and \( D- \) worse than \( d^* \). One such example follows.

**Example 6.4.1:** \( d^*, \hat{d} \in \mathcal{A}(3, 4, 7) \). \( \hat{d} \) is \( E- \) better but \( A- \) and \( D- \) worse than \( d^* \).

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 \\
3 & 3 & 3 & 3 \\
3 & 3 & 1 & 1 \\
3 & 3 & 3 & 3 \\
\end{array}
\]

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 \\
1 & 2 & 3 & 1 \\
\end{array}
\]

Sometimes, as for the following example, when \( m_q = 1 \) with \( p < v-1 \) we can prove \( d^* \) is \( g_{\mathcal{A}_1} \)-optimal using Theorem 1.4.1.

**Example 6.4.2:** \( d^* \in \mathcal{A}(3, 4, 4) \) is \( g_{\mathcal{A}_1} \)-optimal:

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 \\
1 & 2 & 3 & 1 \\
\end{array}
\]

In Chapter 8 the \( A-, D- \) and \( E- \) efficiencies of \( d^* \) and the \( E- \) efficiency of \( \hat{d} \) (\( p \geq v-1 \)) are presented for this case of \( m_q = 1 \).

For \( 2 \leq m_q \leq v-2 \) \( C_{d^*} \) has three (\( m = 1 \) or \( q = 1 \)) or four distinct eigenvalues. It turns out that some of the ideas used to construct \( \hat{d} \) for \( m_q = 1 \), or other techniques, can be used
to construct an E-better (than $d^*$) design. Nothing has been generalized, but here are two such examples.

**Example 6.4.3:** $\hat{d}$ is E-better but A- and D-worse than $d^*$ in $\mathcal{D}(4,6,9)$. $m = 2$, $q = 1$, $mq = 2 = v-2$ and only $\hat{d}$ is given here.

\[
\begin{array}{ccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 \\
2 & 2 & 2 & 2 & 2 & 3 & 3 \\
\hat{d}: & 3 & 3 & 3 & 3 & 3 & 3 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 \\
1 & 1 & 1 & 2 & 2 & 2 & 2 \\
\end{array}
\]

**Example 6.4.4:** $\hat{d}$ is E-better but A- and D-worse than $d^* \in \mathcal{D}(5,11,7)$. Given here are only the last two plots of each (binary) block to display the extra-replicated treatments. $m = 1$, $q = 2$ and $mq = 2 = v-3$.

\[
\begin{array}{ccccccccccc}
1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 & 1 \\
\text{d*:} & 2 & 3 & 4 & 5 & 3 & 4 & 5 & 4 & 5 & 5 & 2 \\
\hat{d}: & 1 & 1 & 1 & 1 & 1 & 3 & 3 & 3 & 3 & 4 & 4 \\
2 & 2 & 2 & 2 & 4 & 4 & 4 & 5 & 5 & 5 & 5 \\
\end{array}
\]

**Remark 6.4.2:** The $\hat{d}$ for $\mathcal{D}(4,6,9)$ is not of full trace. The $d^*$ for $\mathcal{D}(4,6,5)$ is the $p = 1$ case corresponding to Example 6.4.3, and
can be proven E-optimal. The \( \hat{d} \) for \( \Theta(5,11,12) \) is of full trace.
The \( d^* \) for \( \Theta(5,11,7) \) is the \( p = 1 \) case corresponding to Example 6.4.4, and can be proven E-optimal.

We conclude this chapter with the following lemma which was needed for Theorem 6.4.1 and \( mq = 1 \) but is also true for \( 2 \leq mq \leq v-2 \).

**Lemma 6.4.1**: If \( k = pv+q, 1 \leq q \leq v-1, p \geq 1, v \geq 3, \)

\( 1 \leq mq \leq v-2 \), then \( d^* \) is E-better than any \( d \in \Theta(v,b+m,pv+q) \) with

\[
\min_{1 \leq i \leq v} (r_{di}) \leq r+mp-1 \text{ or with } \lambda_{dij} \leq \lambda_1+mp^2 \text{ when } r_{di*} = r_{djj} = r+mp.
\]

**Proof**: Using Theorem 1.4.3 (i),

\[
\nu_{d1} \leq \frac{v}{v-1} B\left( \min_{1 \leq i \leq v} (r_{di}); b+m,k \right)
\]

\[
\leq \frac{v}{v-1} B(r+mp-1; b+m,k)
\]

\[
= \frac{v}{v-1} (r+mp-1- \frac{\lambda_0+(m+1)p^2-(p+1)^2}{k})
\]

\[
= \frac{v}{v-1} \frac{\lambda_1}{k} + \frac{v}{v-1} (mp-1- \frac{mp^2-2p-1}{k}) < \frac{v}{k} \frac{\lambda_1}{k} + mp.
\]

Using Theorem 1.4.3 (ii),

\[
\nu_{d1} \leq r+mp- \frac{\lambda_0+mp^2}{k} + \frac{\lambda_1+mp^2}{k}
\]

\[
= \frac{v}{k} \frac{\lambda_1}{k} + mp.
\]

This completes the proof.
CHAPTER 7
THE BBD MINUS M BLOCKS

7.1 Preliminaries

Throughout Chapter 7 define \( d^0 \) and \( d^* \) as follows. Let \( d^0 \in \mathcal{B}(v,b,k) \) be a BBD with \( k = pv + q \) \((1 \leq q \leq v-1)\). Remove from \( d^0 \) \( m \) blocks disjoint with respect to their extra-replicated treatments, with \( 1 \leq m \leq v/q \). We will call this new design \( d^* \).

\[ d^* \in \mathcal{B}(v,b-m,k) \text{ and } C_{d^*} = C_{d^0} - C_0 \text{ where } C_0 \text{ is given in Lemma 6.1.1.} \]

Lemma 7.1.1: \( C_{d^*} \) has the eigenvalues 0,

\[
\frac{v\lambda_1}{k} = mp - 1 \text{ with multiplicity } m(q-1),
\]

\[
\frac{v\lambda_1^*}{k} = mp - 1 + \frac{q}{k} \text{ with multiplicity } m-1,
\]

\[
\frac{v\lambda_1}{k} = \frac{vp(mp+1)}{k} \text{ with multiplicity } 1 \text{ (unless } mq = v),
\]

\[
\frac{v\lambda_1}{k} = mp \text{ with multiplicity } v-mq-1 \text{ (unless } mq \geq v-1),
\]

where \( \lambda_1 = \lambda_{d^0_{ij}} \) \((i \neq j)\) and for each case of \( m \) and \( q \) all realized eigenvalues are distinct. The eigenvalue structure is \( 0 < a < b = \ldots = b \) if \( m = q = 1 \), \( 0 < a = \ldots = a < b \) if \( m = 2 \) with \( q = v/2 \), \( m = 1 \) with \( q = v-1 \) or \( m = v-1 \) with \( q = 1 \), and the nonzero eigenvalues are all equal if \( m = v \) and \( q = 1 \).
Proof: \( C_{d^0} = \left( r - \frac{\lambda_0}{k} + \frac{\lambda_1}{k}\right) I_v - \frac{\lambda_1}{k} J_{v,v} \) with \( \lambda_0 = \lambda_{d^0_{ii}} \) and \( r = r_{d^0_{ii}} \) \( (1 \leq i \leq v) \), and \( C_{d^*} = C_{d^0} - C_0 \). We apply Lemma 6.1.1 and Lemma 1.4.2 to get the eigenvalues.

\[
\frac{\nu \lambda_1}{k} - mp \geq \frac{\nu \lambda_1}{k} - \frac{vp(mp+1)}{k} > \frac{\nu \lambda_1}{k} - mp - 1 + \frac{q}{k}
\]

with equality between the largest two only when \( mq = v \), in which case those eigenvalues are not realized.

Looking at \( mq = v \), \( mq = v-1 \) and \( 1 \leq mq \leq v-2 \) separately we get the cases mentioned in the hypothesis for the eigenvalue structures of Theorems 2.1.1, 1.4.1 and 1.4.4.

Remark 7.1.1: We must of course have \( m \leq b-2 \) in order for \( d^* \) to have at least two blocks. It may also not be possible to remove two disjoint binary blocks. The BBD in \( \omega(6,10,6p+3) \) does not have two disjoint binary blocks.

Note that for any \( b \geq 2 \) there is a BBD in \( \omega(v,b,pv) \) which is \( \omega \)-optimal, so we only consider \( 1 \leq q \leq v-1 \).

Before going on we present the following lemma.

**Lemma 7.1.2:** \( d^* \) is \( E \)-better than any \( d \in \omega(v,b-m,k) \) for which

1) \( \min_{1 \leq i \leq v} (r_{d_i}) \leq r - mp - 2 \),
2) \( \lambda_{d_{ij}} \leq \lambda_1 - mp^2 - 2p \) when \( r_{d_i} = r_{d_j} = r - mp - 1 \) and \( q = 1 \), or
3) \( \lambda_{d_{ij}} \leq \lambda_1 - mp^2 - 2p - 1 \) when \( r_{d_i} = r_{d_j} = r - mp - 1 \) and \( q \geq 2 \), \( k = pv+q \) with \( p \geq 1 \), \( 1 \leq q \leq v-1 \), \( v \geq 3 \) and \( 1 < mq \leq v-1 \).

Proof: From Theorem 1.4.3 part (i) and assuming without loss of generality that \( r_{d_i} = \min_{1 \leq i \leq v} (r_{d_i}) \),
\[
\nu d_1 \leq \frac{v}{v-1} \left( r d_1 - \frac{b-m}{k} \sum_{j=1}^{n} \frac{n_{d1j}^2}{k} \right) \leq \frac{v}{v-1} B(r d_1; b-m, k)
\]
\[
\leq \frac{v}{v-1} B(r-mp-2; b-m, k).
\]
(7.1.1)

For \( q = 1 \) and \( b = v \) (7.1.1) equals
\[
\frac{v}{v-1} \left( r - mp-2 - \frac{\lambda_0 - (m-1)p^2 - (p+1)^2 - p^2 + (p-1)^2}{k} \right)
\]
\[
= \frac{v \lambda_1}{k} - \frac{v}{v-1} \left( mp+2 - \frac{mp^2 + 4p}{k} \right) \leq \frac{v \lambda_1}{k} - mp-1 + \frac{1}{k},
\]
the smallest nonzero eigenvalue of \( d^* \) (since \( mq > 1 \)). For \( q \geq 2 \) or \( q = 1 \) with \( b \geq 2v \) (7.1.1) equals
\[
\frac{v}{v-1} \left( r - mp-2 - \frac{\lambda_0 - (m-1)p^2 - 2(p+1)^2 + p^2}{k} \right)
\]
\[
= \frac{v \lambda_1}{k} - \frac{v}{v-1} \left( mp+2 - \frac{mp^2 + 4p^2}{k} \right) \leq \frac{v \lambda_1}{k} - mp-1 \leq
\]
the smallest nonzero eigenvalue of \( d^* \).

If \( r d_1 = r d_2 = r - mp - 1 \) and \( q = 1 \), by Theorem 1.4.3 part (ii) we have
\[
\nu d_1 \leq r - mp - 1 - \frac{1}{2} \sum_{j=1}^{b-m} \left( - \frac{n_{d1j}^2 + n_{d2j}^2}{k} \right) + \frac{\lambda d12}{k}
\]
\[
\leq r - mp - 1 - \frac{\lambda_0 - (m-1)p^2 - (p+1)^2}{k} + \frac{\lambda_1 - mp^2 - 2p}{k}
\]
\[
= \frac{v \lambda_1}{k} - mp-1 + \frac{1}{k} = \frac{v \lambda_1}{k} - mp-1 + \frac{q}{k},
\]
the smallest nonzero eigenvalue of \( d^* \). If \( r d_1 = r d_2 = r - mp - 1 \) and \( q \geq 2 \),
\[ \nu_d \leq r \cdot m \cdot p \cdot l - \frac{\lambda_0 - m \cdot p^2 - 2 \cdot p \cdot l}{k} + \frac{\lambda_1 - m \cdot p^2 - 2 \cdot p \cdot l}{k} \]

\[ = \frac{\nu \lambda_1}{k} \cdot m \cdot p \cdot l - \frac{1}{k} \cdot l - \frac{1}{k} = \frac{\nu \lambda_1}{k} \cdot m \cdot p \cdot l, \]

the smallest nonzero eigenvalue of \( d^* \). This completes the proof.

The cases of \( mq = v \), \( mq = v - 1 \) and \( 1 \leq mq \leq v - 2 \) will be covered in Sections 7.2, 7.3 and 7.4 respectively.

7.2 The Case \( mq = v \)

When \( mq = v \) \( C_d^* \) is of the form

\[
\begin{bmatrix}
A & B & \ldots & B \\
A & \ldots & B \\
& & & \\
& & & \\
& & & \\
A & & & \\
\end{bmatrix}
\]

where \( A = (r - \frac{\lambda_0}{k} + \frac{\lambda_1}{k} \cdot m \cdot p \cdot l) I_q - \frac{\lambda_1 - m \cdot p^2 - 2 \cdot p \cdot l}{k} J_{q, q} \) and \( B = \frac{\lambda_1 - m \cdot p^2 - 2 \cdot p}{k} J_{q, q} \). \( d^* \) is just a GD PBBD with \( m \) groups of size \( q \) where in the notation of GD designs, \( \lambda_2 = \lambda_1 + 1 \). Then by Theorem 4.1 of Cheng (1978), \( C_d^* \) is the unique E-optimal matrix in \( C = \{ C_d ; d \in \mathcal{B}(v, b - m, pv + q) \} \).

For \( m = v \) and \( q = 1 \) (and \( b \geq 2v \)) \( d^* \) is a BBD in \( \mathcal{B}(v, ev - v, pv + 1) \) for \( e \geq 2 \). Corollary 1.4.4 says \( d^* \) is \( J_{U} \)-optimal. This is the case of \( 0 < a = \ldots = a \).

For \( m = 2 \) and \( q = v/2 \) \( d^* \) has the eigenvalue structure \( 0 < a = \ldots = a < b \). Then \( d^* \) is a MB GD PBBD and by Corollary 3.1.1 of Cheng (1978) \( d^* \) is \( gJ_{1} \)-optimal.
No other cases of \( mq = v \) considered here give one of the three
eigenvalue structures of Lemma 7.1.1. \( m = 1 \) and \( q = v \) gives
\( 0 < a = \ldots = a \) but \( k = pv + v \) allows a BBD for any \( b \geq 2 \), so we have
required \( q \) to be less than \( v \).

In Chapter 8 the A- and D-efficiencies of these \( E \)-optimal
designs will be presented.

7.3 The Case \( mq = v-1 \)

When \( mq = v-1 \) only two cases give \( d^* \) with an
eigenvalue structure mentioned in Lemma 7.1.1. They are \( m = v-1 \) with \( q = 1 \)
and \( m = 1 \) with \( q = v-1 \). Each \( d^* \) has the eigenvalue structure
\( 0 < a = \ldots = a < b \).

For \( m = v-1 \) with \( q = 1 \) we require \( d^0 \in \mathcal{B}(v, ev, pv+1) \) with
\( e \geq 2 \). \( d^* \) is then in \( \mathcal{B}(v, ev-(v-1), pv+1) = \mathcal{B}(v, (e-1)v+1, pv+1) \)
and equivalently could be constructed by adding one balanced block
to a BBD in \( \mathcal{B}(v, (e-1)v, pv+1) \). So this is just the case of \( mq = 1 \)
of Section 6.4.

For \( m = 1 \) with \( q = v-1 \) \( C_{d^*} \) has eigenvalues \( 0, (v\lambda_1/k)_p-1 \)
and \( (v\lambda_1/k)-(vp(p+1)/k) \) with multiplicities \( 1, v-2 \) and \( 1 \)
respectively.

**Theorem 7.3.1**: \( d^* \) is \( E \)-better than any \( d \in \mathcal{B}(v, ev-1, pv+v-1) \)
for which 1) \( \min_{1 \leq i \leq v} (r_{d^*_{di}}) \leq r-p-2 \) or 2) \( \lambda_{d^*_{dij}} \leq \lambda_1-p^2-2p-1 \) when
\( r_{d^*_{di}} = r_{d^*_{dj}} = r-p-1 \). For \( p \geq v-2 \) there exists a \( \hat{d} \) (with \( r_{d^*_{d1}} = r-p, \)
\( r_{d^*_{di}} = r-p-1 \) for \( 2 \leq i \leq v \) and \( \lambda_{d^*_{dij}} = \lambda_1-p^2-2p \) for \( 2 \leq i < j \leq v \)
which is strictly \( E \)-better than \( d^* \). Therefore \( d^* \) is not \( g \mathcal{B}_{\lambda_1} \)-
optimal for \( p \geq v-2 \).
Proof: That \( d^* \) E-betters the designs satisfying 1) and 2) follows from Lemma 7.1.2.

If \( d^0 \) is \( e \) copies \((e \geq 1)\) of the unique BBD in \( \mathcal{B}(v,v,pv+v-1) \) let \( \hat{d} \) be \( e-1 \) copies of that BBD in \( v \) blocks plus \( v-2 \) blocks with \( p \) applications of treatment 1 and \( p+1 \) applications of treatments 2 through \( v \), and finally a block with \( p+v-1 \) applications of treatment 1 and \( p \) applications of treatments 2 through \( v \).

\[
C_{\hat{d}} = \begin{bmatrix}
  c_{11} & -c_{12}^{1,1} \\
  -c_{12}^{1,v-1} & C_{22}
\end{bmatrix}
\]

where \( c_{11} = \frac{(v-1)e\Delta}{k} - \frac{v-1}{k}(p^2+p+v-2) \), \( c_{12} = \frac{e\Delta}{k} - \frac{p^2+p+v-2}{k} \), and

\[
C_{22} = \left( \frac{ev\Delta}{k} - \frac{vp^2+(2v-1)p+v-2}{k} \right) I_{v-1} - \left( \frac{e\Delta}{k} - \frac{p^2+2p}{k} \right) J_{v-1,v-1}.
\]

Here \( \Delta = vp^2+2(v-1)p+v-1 \) and is the \( \lambda_{ij} \) \((i \neq j)\) for the unique BBD in \( \mathcal{B}(v,v,pv+1) \). Therefore \( e\Delta = \lambda_1 \) of \( d^0 \).

Therefore by Lemma 1.4.1 the eigenvalues of \( C_{\hat{d}} \) are 0,

\[
\frac{ve\Delta}{k} - \frac{vp^2+(2v-1)p+v-2}{k} = \frac{v\lambda_1}{k} - (p+1) + \frac{1}{k}
\]

with multiplicity \( v-2 \) from \( C_{22} \) and

\[
v c_{12} = v \left( \frac{e\Delta}{k} - \frac{p^2+p+v-2}{k} \right)
\]

\[
= \frac{v\lambda_1}{k} - (p+1) + \frac{(v-1)(p+1)-v(v-2)}{k}
\]

To construct this design, we only need \( p \geq 1 \), but the single nonzero eigenvalue is at least as big as the eigenvalue with
multiplicity \(v-2\) if \(p \geq v-2\) with equality for \(p = v-2\) and strict inequality for \(p > v-2\).

Since \(\frac{v\lambda_1}{k} - (p+1) + \frac{1}{k} > \frac{v\lambda_1}{k} - (p+1) = \mu_{d^{*}1}\), \(\hat{d}\) is strictly \(E\)-better than \(d^{*}\) and \(d^{*}\) cannot be \(g_{\frac{1}{4}}\)-optimal.

**Remark 7.3.1:** For \(p < v-2\) Theorem 1.4.1 cannot be applied in general. In trying to prove that the \(d^{*}\) in the next example was \(g_{\frac{1}{4}}\)-optimal, the design \(\hat{d}\) of that example had a strictly larger value of \(\text{tr} \ C_{d} - \left(\frac{v-1}{v-2}\right)^{\frac{1}{2}} [\text{tr} (C_{d}^{2}) - (\text{tr} C_{d})^{2}/(v-1)]^{\frac{1}{2}}\) than did \(d^{*}\). \(d^{*}\) is \(E\)-optimal in \(\mathcal{D}(4,3,7)\).

**Example 7.3.1:** \(m = 1, q = v-1, p < v-2\). \(d^{*}, \hat{d} \in \mathcal{D}(4,3,7)\).

See Remark 7.3.1.

\[
\begin{array}{ccc}
1 & 1 & 1 \\
2 & 2 & 2 \\
3 & 3 & 3 \\
d^{*}: & 4 & 4 & 4 \\
1 & 1 & 1 \\
2 & 2 & 3 \\
3 & 4 & 4 \\
\end{array}
\]

\[
\begin{array}{ccc}
1 & 1 & 1 \\
2 & 2 & 2 \\
3 & 3 & 3 \\
\hat{d}: & 4 & 4 & 4 \\
1 & 1 & 3 \\
2 & 2 & 3 \\
3 & 4 & 4 \\
\end{array}
\]

It is also true that for \(mq = v-1\) and \(m > 1\) that \(d^{*}\) is not necessarily \(E\)-optimal, and hence not necessarily \(g_{\frac{1}{4}}\)- or \(g_{\frac{1}{4}}\)-optimal.

**Example 7.3.2:** \(m = 2, q = 2\). \(d^{*} \in \mathcal{D}(5,8,7)\) is \(E\)-worse than \(\hat{d}\) given below.
The $A_-$, $D_-$ and $E$-efficiencies for $d^*$ and the $E$-efficiency for $d$ ($p \geq v-2$) with $m = 1$ (and $q = v-1$) will be presented in Chapter 8.

### 7.4 The Cases $1 < mq < v-2$

For all these $m$ and $q$ only $m = q = 1$ gives a $d^*$ with one of the eigenvalue structures in Lemma 7.1.1. The eigenvalues are $0$,

\[
\frac{v\lambda_1}{k} - \frac{vp(p+1)}{k} \quad \text{and} \quad \frac{v\lambda_1}{k} - \frac{p}{k},
\]

with multiplicities 1, 1 and $v-2$ respectively.

**Theorem 7.4.1:** $d^* \in \mathcal{D}(v,ev-1,pv+1)$, the BBD in $\mathcal{D}(v,ev,pv+1)$ with one block removed, is $\lambda_4$-optimal in $\mathcal{D}(v,ev-1,pv+1)$. $v > 3$.

**Proof:** $d^*$ is of maximum trace and has eigenvalue structure $0 < a < b = \ldots = b$. For any $d \in \mathcal{D}(v,ev-1,pv+1)$,

\[
\mu_d \leq \frac{v}{v-1} (r-p-1-\frac{\lambda_0-(p+1)^2}{k}) = \frac{v\lambda_1}{k} - \frac{v}{v-1} \left( p+1-\frac{(p+1)^2}{k} \right)
\]

\[
= \frac{v\lambda_1}{k} - \frac{vp(p+1)}{k} = \mu_{d^*}.
\]

Therefore by Theorem 2.1.1 we are done.
The following examples illustrate that $d^*$ for $2 \leq mq \leq v-2$ is not necessarily $E$- and hence $g_4^-$ or $h_4^-$-optimal.

**Example 7.4.1:** $\hat{d}$ given below is strictly $E$-better than $d^*$ in $\mathcal{D}(4,5,6)$. Here $m = 1$ and $q = v-2 = 2$.

\[
\begin{array}{cccc}
1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 \\
\hat{d}: & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 \\
1 & 1 & 2 & 3 \\
2 & 2 & 4 & 3 & 4 \\
\end{array}
\]

**Example 7.4.2:** $\hat{d}$ is strictly $E$-better than $d^*$ in $\mathcal{D}(6,13,8)$, where both designs have only the extra-replicated treatments of their (binary) blocks displayed. Here $m = 2$ and $q = (v-2)/2 = 2$.

\[
\begin{array}{cccccccc}
d^*: & 1 & 1 & 1 & 1 & 2 & 2 & 2 & 3 & 3 & 4 & 4 \\
2 & 3 & 4 & 5 & 6 & 3 & 4 & 5 & 6 & 5 & 6 & 5 & 6 \\
\hat{d}: & 1 & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 3 & 4 & 4 & 5 \\
2 & 2 & 2 & 3 & 4 & 5 & 6 & 4 & 5 & 6 & 5 & 6 & 6 \\
\end{array}
\]

**Example 7.4.3:** $\hat{d}$ is strictly $E$-better than $d^*$ in $\mathcal{D}(7,6,10)$, where both designs have only the extra-replicated treatments of their (binary) blocks displayed. Here $m = 1$ and $q = v-4$.

\[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
d^*: & 5 & 6 & 7 & 1 & 2 & 3 \\
6 & 7 & 1 & 2 & 3 & 4 \\
\end{array}
\]
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CHAPTER 8

EFFICIENCIES FOR PREVIOUS CHAPTERS

8.1 Preliminaries

In this section we present two methods used to calculate most of the A-, D- and E-efficiencies in this chapter, as well as four lemmas used to prove the efficiencies are strictly increasing as a parameter related to the number of blocks increases.

In general we take a possibly nonexistent or nonconstructible design $d_B$ whose eigenvalues $\bar{\mu}_i$ for $1 \leq i \leq v-1$ will be used to construct the lower bound for $\phi(C_d)$. Then the true efficiency of a design $d$ is greater than or equal to

$$e_\phi(d) = \frac{\phi(C_d_B)}{\phi(C_d)}$$

for any of $A$, $D$ and $E$. This $e_\phi(d)$ is really a lower bound since the $\phi$-best $d \in D(v,b,k)$ may not be as good as $d_B$. Sometimes $d_B$ will be a $J_4$-optimal $d^\star$ so $\bar{\mu}_i = \mu_{d^\star i}$ for $1 \leq i \leq v-1$. Sometimes $d_B$ will have $\bar{\mu}_1 = \mu_{d^\star 1}$ for an $E$-optimal $d^\star$ with $\bar{\mu}_2 = \ldots = \bar{\mu}_{v-1} = (v-2)^{-1}(\text{tr } C_{d^\star}-\mu_{d^\star 1})$.

In the numerical examples given the value of $e_\phi(d)$ was calculated to seven or more places but is truncated to four decimal places for presentation. If $e_\phi(d)$ had nonzero terms in the fifth decimal place and beyond we shall write $e_\phi(d) > .9876$ (for example).
The first method for finding the \( \tilde{\mu}_1 \) of \( d_B \) involves Theorem 2.2.1. The bound (i) for \( u_{d1} \) is made as large as possible by taking

\[
\min_{1 \leq i \leq v} (r_{di}) = \min_{1 \leq i \leq \nu} (r_{d^*i}) \text{ and minimizing } \sum_{j=1}^{v-1} (n_{d^*j}/k_j) \text{ if } r_{d^*} = \min_{1 \leq i \leq v} (r_{di}).
\]

If this maximum bound does not force the trace of \( C_d \) to be less than \( \max_{d \in \Theta} (\tr(C_{d^*}) - \tilde{\mu}_1) \), then we let \( \tilde{\mu}_1 \) equal this bound and

\[\tilde{\mu}_2 = \ldots = \tilde{\mu}_{v-1} = (v-2)^{-1}(\tr(C_{d^*} - \tilde{\mu}_1)).\]

Then Theorem 2.1.1 says this \( d_B \) is \( \mathcal{B}_{v-1} \)-optimal and hence \( A-, D- \) and \( E- \)optimal. We call this Method 1.

Method 2 is just letting \( \tilde{\mu}_1 = \ldots = \tilde{\mu}_{v-1} = (v-1)^{-1}(\tr(C_{d^*})). \) This is used if pushing bound (i) of Theorem 2.2.1 up too far decreases the trace of \( C_d \). This can happen if there are blocks of different sizes. See for example the case of Theorem 4.2.1 part c) and its corresponding Theorem 8.3.2.

Now we present four lemmas used to prove that most of the efficiencies are strictly increasing in \( e, \lambda_1, \) or \( r \) while the remaining parameters \( v, p \) and \( q \) remain fixed. \( e \) was used as the number of copies of the unique BIBD in \( \mathcal{B}(v,v(v-1)/2,2) \), so as \( e \) increases we are increasing the number of blocks of \( d^* \) in \( \mathcal{B}(v, ev(v-1)/2-1,2) \) in Section 8.2. As \( \lambda_1 \) and \( r \) increase for fixed \( v, p \) and \( q \) we are increasing the number of pairs in \( d^0 \) used to construct \( d^* \). But \( \lambda_1 = r(k-1)/(v-1) = bk(k-1)/(v(v-1)) \) for \( d^0 \) so both increase as \( b \) increases, for \( b \) the number of blocks in \( d^0 \).

Lemma 8.1.1: For \( A(x) \) a differentiable function of \( x > 0, b_i \) a constant with respect to \( x \) (for \( i = 0,1,2,3 \), \( 1 \leq M, N \leq v-2, \) and \( v \geq 3 \) an integer we have
\[
\frac{d}{dx} \left( \frac{1}{A(x) - b_2} + \frac{v-2}{A(x) - b_1} \right) > 0
\]

if all of the following hold:

(i) \( b_3 \geq b_2 \geq b_1 \geq b_0 \geq 0 \) with at least one of the first three inequalities being strict,

(ii) \( A'(x) > 0, A(x) - b_3 > 0 \), and

(iii) \( M(b_3 - b_2) + (v-2)M(b_3 - b_1) \geq N(b_2 - b_0) + (v-2)N(b_1 - b_0) \).

**Proof:** Suppose \( M(b_3 - b_2) \leq N(b_2 - b_0) \). Then taking the derivative we get

\[
A'(x) \left( \frac{M}{A(x) - b_3} + \frac{N}{A(x) - b_0} \right)^{-2} \left[ \frac{M(b_3 - b_2)}{(A(x) - b_3)^2(A(x) - b_2)^2} + \frac{M(v-2)(b_3 - b_1)}{(A(x) - b_3)^2(A(x) - b_1)^2} - \frac{N(b_2 - b_0)}{(A(x) - b_2)^2(A(x) - b_0)^2} - \frac{N(v-2)(b_1 - b_0)}{(A(x) - b_1)^2(A(x) - b_0)^2} \right]
\]

\[
= A'(x) \left( \frac{M}{A(x) - b_3} + \frac{N}{A(x) - b_0} \right)^{-2} \times \]

\[
(M(b_3 - b_2)[(A(x) - b_3)^{-2}(A(x) - b_2)^{-2} - (A(x) - b_2)^{-2}(A(x) - b_0)^{-2}]
\]

\[
+ (N(b_2 - b_0) - M(b_3 - b_2))[(A(x) - b_3)^{-2}(A(x) - b_1)^{-2} - (A(x) - b_2)^{-2}(A(x) - b_0)^{-2}] \]
\[ + N(v-2)(b_1-b_0)[(A(x)-b_3)^{2}(A(x)-b_1)^{2}- (A(x)-b_1)^{2}(A(x)-b_0)^{2}] \\
\frac{(M(b_3-b_2)+(v-2)(b_3-b_1))-N(b_2-b_0+(v-2)(b_1-b_0))}{(A(x)-b_3)^2(A(x)-b_1)^2} \}, \]

This is strictly greater than zero by conditions (i) and (ii). The proof for \( M(b_3-b_2) > N(b_2-b_0) \) is analogous.

**Remark 8.1.1:** Lemma 8.1.1 can actually be proved with a weak majorization argument that would allow for a more general result with a different condition (i).

**Lemma 8.1.2:** For \( A(x) \) a differentiable function of \( x > 0 \), \( b_1 \) a constant with respect to \( x \) (for \( i = 0,1,2,3 \), \( 1 \leq M, N \leq v-2 \) and \( v \geq 3 \) an integer we have

\[
\frac{d}{dx} \left[ \frac{(A(x)-b_3)^M}{(A(x)-b_2)} \frac{A(x)-b_0}{(A(x)-b_1)^N} \right] > 0
\]

if all of the following hold:

(i) \( b_3 \geq b_2 \geq b_1 \geq b_0 \geq 0 \) with at least one of the first three inequalities being strict,

(ii) \( A'(x) > 0, A(x)-b_3 > 0 \), and

(iii) \( M(b_3-b_2) > n(b_1-b_0) \).

**Proof:** Taking the derivative we get

\[
\left( \frac{A(x)-b_3}{A(x)-b_2} \right)^{M-1} \left( \frac{A(x)-b_0}{A(x)-b_1} \right)^{N-1} \frac{A'(x)}{(A(x)-b_2)^2(A(x)-b_1)^2} \times \\
[M(b_3-b_2)(A(x)-b_0)(A(x)-b_1)-N(b_1-b_0)(A(x)-b_2)(A(x)-b_3)]
\]
which is strictly greater than zero given conditions (i), (ii) and (iii).

**Remark 8.1.2:** A stronger result with the same proof is to replace (i) and (iii) with

\[M(b_3-b_2)(A(x)-b_0)(A(x)-b_1)-N(b_1-b_0)(A(x)-b_2)(A(x)-b_3) > 0\]

and (ii) with \(A'(x) > 0, A(x)-b_i > 0\) for \(i = 0, 1, 2, 3\).

**Lemma 8.1.3:** For \(A(x)\) a differentiable function of \(x > 0\), \(a_i\) a constant with respect to \(x\) (for \(i = 0, 1, 2, 3\)), and \(v > 3\) an integer we have

\[
\frac{d}{dx} \left[ \frac{A(x)+a_3}{A(x)+a_1} \left( \frac{A(x)+a_0}{A(x)+a_2} \right)^{v-2} \right] > 0
\]

if all of the following hold:

(i) \(a_3 \geq a_2 \geq a_1 \geq a_0 > 0\) with at least one of the first and third inequalities being strict,

(ii) \(A'(x) > 0, A(x)+a_0 > 0\), and

(iii) \((v-2)(a_2-a_0) \geq (a_3-a_1)\).

**Remark 8.1.3:** The proof of Lemma 8.1.3 is analogous to the proof of Lemma 8.1.2. Also (i) and (iii) could be replaced by

\[(v-2)(a_2-a_0)(A(x)+a_3)(A(x)+a_1)-(a_3-a_1)(A(x)+a_2)(A(x)+a_0) > 0\]

and (ii) by \(A'(x) > 0, A(x)+a_i > 0\) for \(i = 0, 1, 2, 3\) to give a stronger result.

**Lemma 8.1.4:** For \(A(x)\) a differentiable function of \(x > 0\), \(A'(x) > 0, b_3 > b_2 \geq 0\) we have
\[
\frac{d}{dx} \left( \frac{A(x)-b_3}{A(x)-b_2} \right) > 0.
\]

**Proof:** The proof is straightforward.

In the following sections we present selected efficiencies from Chapters 3 through 7.

### 8.2 Efficiencies for Chapter 3

Recall that \( d^0 \in \mathcal{B}(v, ev(v-1)/2, 2) \) is \( e \) copies \( (e \geq 1) \) of the BIBD in \( \mathcal{B}(v, v(v-1)/2, 2) \), and \( d^* \) was \( d^0 \) minus one (binary) block. \( d^* \) has been proven \( J_4 \)-optimal for \( e = 1 \) and all \( v \) by others and \( e \geq 2 \) for \( v = 3, 4, 5 \) and 6 by the author.

**Theorem 8.2.1:** The A-, D- and E-efficiencies for \( d^* \in \mathcal{B}(v, ev(v-1)/2 - 1, 2) \) calculated with Method 1 are

\[
e_A(d^*) = \frac{1}{ev-2} + \frac{v-2}{ev(v-1)},
\]

\[
e_D(d^*) = \left( \frac{ev-2}{ev-v(v-1)} \right) \left( \frac{ev}{ev-(v-1)} \right)^{v-2},
\]

\[
e_E(d^*) = \frac{ev-2}{ev-v(v-1)},
\]

with each strictly increasing in \( e \geq 1 \) and converging to 1 as \( e \) goes to infinity. \( v \geq 3 \).
Proof: From Section 3.1 \( \mu_{d^*1} = (v \lambda_1/2) - 1 \) and \( \mu_{d^*2} = \ldots = \mu_{d^*,v-1} = (v \lambda_1/2) \). Method 1 gives \( \mu_{d1} \leq \frac{v}{v-1} \left( r-1 - \frac{r-1}{2} \right) = \frac{v \lambda_1}{2} - \frac{v}{2(v-1)} = \lambda_1 \) and \( \mu_2 = \frac{v \lambda_1}{2} - \frac{1}{2(v-1)} \). We also have \( \lambda_1 = e \) and \( r = e(v-1) \).

Lemmas 8.1.1, 8.1.2 and 8.1.4 are used to get the efficiencies increasing in \( e \) with \( b_0 = 0, b_1 = (v-1)^{-1}, b_2 = \frac{v}{v-1}, b_3 = 2 \) and \( A(e) = ev \). The convergence to 1 is trivial.

Remark 8.2.1: Because \( d^* \) is intuitively the best design for any reasonable criteria the \( A-, D- \) and \( E- \) efficiencies were evaluated for \( 7 \leq v \leq 50 \) and \( 1 \leq e \leq 5 \). These were proved to be increasing in \( e \) for fixed \( v \), and appear to be increasing in \( v \) for fixed \( e \). In Table 8.2.1, p. 130, are presented \( e_A(d^*), e_D(d^*), \) and \( e_E(d^*) \) for \( v = 7, 8, 9, 10, 15, 20, 30, 40, 50 \) and \( e = 1, 2, 3, 4, 5 \). They are truncated to four decimal places unless more places prove illustrative. Fewer than four places indicates equality with \( e_E(d^*) \).

8.3 Efficiencies for Chapter 4

For \( k < v-1 \) we have the four cases given in Theorem 4.2.1, some of which will require different bounds than others. Recall that for \( d^* \) a BBD plus a binary block of size \( (v-1) \), the eigenvalues of \( C_{d^*} \) are \( \mu_{d^*1} = \frac{v \lambda_1}{k} \) \( \left( \frac{r(k-1)}{k} \right) \) and \( \mu_{d^*2} = \ldots = \mu_{d^*,v-1} = \frac{v \lambda_1}{k} + 1 = \frac{v}{v-1} \left( \frac{r(k-1)}{k} + \frac{v-1}{v} \right) \).

Theorem 8.3.1: The \( A-, D- \) and \( E- \) efficiencies for \( d^* \in \mathcal{B}(v,b+1,k^*), \) \( (k^*)' = (k^*_b', v-1), v \geq 7 \) with \( v-1 > k \geq \frac{v-1}{3} \) or \( 4 \leq v \leq 6 \) with
Table 8.2.1: BIBD in $\omega(v, ev(v-1)/2, 2)$ minus one block

<table>
<thead>
<tr>
<th>$v$</th>
<th>$e = 1$</th>
<th>$e = 2$</th>
<th>$e = 3$</th>
<th>$e = 4$</th>
<th>$e = 5$</th>
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<tbody>
<tr>
<td>*</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td></td>
<td>D</td>
<td>D</td>
<td>D</td>
<td>D</td>
<td></td>
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<td>E</td>
<td>E</td>
<td>E</td>
<td>E</td>
<td>E</td>
</tr>
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<td>0.9989</td>
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<tr>
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<td>0.9689</td>
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</tr>
<tr>
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<td>0.9996</td>
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<td>0.9749</td>
<td>0.9944</td>
<td>0.9976</td>
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<td>0.9991</td>
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<td>0.9625</td>
<td>0.9722</td>
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</tr>
<tr>
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<td>0.99994</td>
<td>0.99997</td>
<td>0.99998</td>
</tr>
<tr>
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<td>0.9990</td>
<td>0.9997</td>
<td>0.9998</td>
<td>0.9994</td>
<td>0.9996</td>
</tr>
<tr>
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<td>0.9876</td>
<td>0.9918</td>
<td>0.9938</td>
<td>0.9951</td>
</tr>
<tr>
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<td>0.9993</td>
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</tr>
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<td>0.9901</td>
<td>0.9934</td>
<td>0.9950</td>
<td>0.9960</td>
</tr>
</tbody>
</table>
\( v-1 > k \geq 2 \), are calculated by Method 1 and equal

\[
e_{A}(d^*) = \frac{(r+ \frac{v-1-k}{(k-1)(v-1)})^{-1} + (v-2)(r+ \frac{k(v-1)}{v(k-1)})^{-1}}{1 + (v-2)(r+ \frac{k(v-1)}{v(k-1)})^{-1}}
\]

\[
e_{D}(d^*) = \left( \frac{r}{r+ \frac{v-1-k}{(k-1)(v-1)}} \right) \left( \frac{r+ \frac{k(v-1)}{v(k-1)}}{r+ \frac{k(v-1)}{v(k-1)} - \frac{v-1-k}{(k-1)(v-1)(v-2)}} \right)^{-1}
\]

\[
e_{E}(d^*) = \frac{r}{r+ \frac{v-1-k}{(k-1)(v-1)}}
\]

with each strictly increasing in \( r \) and converging to \( \bar{u} \) as \( r \) goes to infinity.

**Proof:** Method 1 gives \( \bar{u}_{d1} \leq \bar{u}_1 = \frac{v}{v-1} \left( \frac{r(k-1)}{k} + \frac{v-1-k}{k(v-1)} \right) \) and \( \bar{u}_2 = \frac{v}{v-1} \left( \frac{r(k-1)}{k} + \frac{v-1-k}{v(k-1)(v-2)} \right) \) since the cases of this theorem are cases a) and d) of Theorem 4.2.1. Therefore the largest one can make \( \bar{u}_1 \) is by putting the treatment with \( \min_i r_{d_i} = r \) in the block of size \( v-1 \) exactly once.

Lemmas 8.1.1, 8.1.2 and 8.1.4 show the efficiencies are increasing in \( r \) with \( b_0 = 0 \), \( b_1 = \frac{v-1-k}{(k-1)(v-1)(v-2)} \), \( b_2 = \frac{k(v-1)}{v(k-1)} - \frac{v-1-k}{(k-1)(v-1)} \), \( b_3 = \frac{k(v-1)}{v(k-1)} \) and \( A(r) = r+ \frac{k(v-1)}{v(k-1)} \).

**Theorem 8.3.2:** The A-, D- and E-efficiencies for \( d^* \in \mathcal{A}(v,b+1,k^*) \) with \( k^* \) as in Theorem 8.3.1, \( v \geq 7 \) with \( 2 \leq k < \frac{\sqrt{v+1}}{2} \) are calculated by Method 2 and equal
\[ e_A(d^*) = \frac{(v-1)(r + \frac{k(v-2)}{v(k-1)})^{-1}}{r + (v-2)(r + \frac{k(v-2)}{v(k-1)})} , \]

\[ e_D(d^*) = \left( \frac{r}{r+ \frac{k(v-2)}{v(k-1)}} \right) \left( \frac{r+ \frac{k(v-1)}{v(k-1)}}{r+ \frac{k(v-2)}{v(k-1)}} \right)^{v-2} , \]

\[ e_E(d^*) = \left( \frac{r}{r+ \frac{k(v-2)}{v(k-1)}} \right) , \]

with each strictly increasing in \( r \) and converging to 1 as \( r \) goes to infinity.

**Proof:** This is the case of Theorem 4.2.1 c) and so Method 2 is used giving
\[ \bar{\mu}_1 = \ldots = \bar{\mu}_{v-1} = \frac{v}{v-1} \left( \frac{r(k-1)}{k} + \frac{v-2}{v} \right) . \]

Lemmas 8.1.1, 8.1.2 and 8.1.4 show the efficiencies are increasing in \( r \) with \( b_0 = 0, b_1 = b_2 = \frac{k}{v(k-1)}, b_3 = \frac{k(v-1)}{v(k-1)} \) and
\[ A(r) = r + \frac{k(v-1)}{v(k-1)} \]

**Remark 8.3.1:** For the case of \( v \geq 7 \) and \( \frac{v-1}{3} > k \geq \frac{\sqrt{v+1}}{2} \), we know a design E-better than \( d^* \) needs \( \min_{1 \leq i \leq v} (r_{di}) = r \) but we can put more than one allocation of a minimum treatment in the block of size \( v-1 \) to get a bigger \( \bar{\mu}_1 \). Therefore,

\[ \bar{\mu}_1 = \frac{v}{v-1} \left( \frac{r(k-1)}{k} + \frac{v}{v-1} \right) \]

\[ + \frac{v^2}{v(v-1)(v-2)} - \frac{v^2}{v(v-1)(v-2)} , \]

\[ \bar{\mu}_i = \frac{v}{v-1} \left( \frac{r(k-1)}{k} + \frac{v-1}{v} + \frac{v^2}{v(v-1)(v-2)} - \frac{v^2(v-k)}{kv(v-2)} \right) , \]
for \(2 \leq i \leq v-1\) (where \(z_0\) is the value of \(n_{di,b+1}\) which maximizes \(\nu_1\) when \(r_{di} = r\)) should be used to calculate at least the E-efficiency. Since \(z_0 > 1\) does decrease the overall trace of \(C_d\), Methods 2 is suggested for the A- and D-efficiencies.

The following examples illustrate how good the efficiencies are even for small values of \(v\) and \(r\).

**Example 8.3.1:** Take \(d^* \in \rho(4,7,k^*)\) with \(k = 2\). Here \(r = 3\) and it is a case from Theorem 8.3.1. \(e_A(d^*) = .9791, e_D(d^*) = (.9)(27/26)^2 > .9705\) and \(e_E(d^*) = .9\). To give some idea of how fast these efficiencies are increasing, for \(d^* \in \rho(4,13,k^*)\) with \(k = 2\) (\(r = 6\)) \(e_A(d^*) > .9937, e_D(d^*) > .9909\) and \(e_E(d^*) = (18/19) > .9473\).

**Example 8.3.2:** Take \(d^* \in \rho(10,11,k^*)\) with \(k = 2\). Here \(r = 9\) and this is actually the smallest value of \(v\) for which Theorem 8.3.2 can be used since for \(7 \leq v \leq 9\) we have \(\sqrt{v+1} \leq 2\). Here \(e_A(d^*) > .9967, e_D(d^*) > .9860\) and \(e_E(d^*) > .8490\).

For \(k = pv\) (\(p \geq 2\)) it was shown in Theorem 4.3.1 that \(d^*\) was sometimes E-worse than but always A- and D-better than \(\hat{d}\). Here we will calculate E-efficiencies for \(d^*\) and \(\hat{d}\) and A- and D-, efficiencies for \(d^*\) only.

**Theorem 8.3.3:** For \(d^*, \hat{d} \in \rho(v,b+1,k^*)\) with \(k = pv\) (\(p \geq 2\), \(v \geq 3\)) the E-efficiencies calculated by Method 1 and the A- and D-efficiencies calculated by Method 2 are

\[
e_E(d^*) = \frac{\lambda_1}{\lambda_1^+ \frac{pv}{v-1} (2p-1) - \frac{1}{v-1}}
\]
\[
e^{\hat{d}}_E = \frac{\lambda_1 + \frac{p}{2} - \frac{1}{v} - \frac{p}{2} \left( \frac{v^2p^2 - 4vp^2 + 4}{vp(v-1)} \right)^{\frac{1}{2}}}{\lambda_1 + \frac{pv}{v-1} \left( \frac{2p-1}{pv} - \frac{1}{v-1} \right)},
\]

\[
e^{\hat{d}}_A(d^*) = \frac{(v-1)(v\lambda_1 + k(v-2))^{-1}}{v\lambda_1 + (v-2)(v\lambda_1 + k)^{-1}},
\]

\[
e^{\hat{d}}_D(d^*) = \left( \frac{v\lambda_1}{v\lambda_1 + k(v-2)} \right)^{v-2},
\]

with each strictly increasing in \(\lambda_1\) and converging to 1 and \(\lambda_1\) increases to infinity.

**Proof:** Method 1 gives \(\bar{\mu}_i = \frac{v\lambda_1}{k} + \frac{v}{v-1} \left( \frac{2p-1}{k} - \frac{1}{v-1} \right)\) with \(k = pv\), and we use this \(\bar{\mu}_i\) for \(e^{\hat{d}}_E(d^*)\) and \(e^{\hat{d}}_E(d^*)\). Lemma 8.1.4 is applied to \(e^{\hat{d}}_E(d^*)\), and to \(e^{\hat{d}}_E(d^*)\) for \(p \geq v/(v-2)\) (which makes \(e^{\hat{d}}_E(d^*) \geq e^{\hat{d}}_E(d^*)\)).

Method 2 gives \(\bar{\mu}_i = \frac{v\lambda_1}{k} + \frac{v-2}{v-1} (1 \leq i \leq v-1)\) and we apply Lemmas 8.1.1 and 8.1.2 to \(e^{\hat{d}}_A(d^*)\) and \(e^{\hat{d}}_D(d^*)\), respectively, with \(b_0 = 0, b_1 = b_2 = k(v-1)^{-1}, b_3 = k\) and \(A(\lambda_1) = v\lambda_1\).

**Remark 8.3.2:** For \(k = pv\), putting an application of a minimum replicated treatment \(i\) with \(r_{di} = r\) in the block of size \(v-1\) pushes up \(\bar{\mu}_i\) with Method 1 but decreases the overall trace of \(C_d\). Therefore Method 2 is used for \(e^{\hat{d}}_A(d^*)\) and \(e^{\hat{d}}_D(d^*)\).

For \(k = pv+q\) \((p \geq 1, 1 \leq q \leq v-1)\) we have the following theorems.
Theorem 8.3.4: The A-, D- and E-efficiencies for $d^* \in \mathcal{D}(v,b+1,k^*)$ with $k = pv+q$ ($p \geq 1$, $1 \leq q \leq v-1$) calculated with Method 1 are

$$e_A(d^*) = \frac{\left(\lambda_1 + \frac{k}{v-1} \left(\frac{2p+1}{k} - \frac{1}{v-1}\right)\right)^{-1} + (v-2)\left(\lambda_1 + \frac{k}{v} - \frac{k}{(v-1)(v-2)} \left(\frac{2p+1}{k} - \frac{1}{v-1}\right)\right)^{-1}}{\lambda_1 + (v-2) \left(\lambda_1 + \frac{k}{v} - \frac{k}{(v-1)(v-2)} \left(\frac{2p+1}{k} - \frac{1}{v-1}\right)\right)^{-1}},$$

$$e_D(d^*) = \left(\frac{\lambda_1 + \frac{k}{v-1} \left(\frac{2p+1}{k} - \frac{1}{v-1}\right)}{\lambda_1 + \frac{k}{v-1} \left(\frac{2p+1}{k} - \frac{1}{v-1}\right)}\right)^{v-2},$$

$$e_E(d^*) = \frac{\lambda_1}{\lambda_1 + \frac{k}{v-1} \left(\frac{2p+1}{k} - \frac{1}{v-1}\right)},$$

with each strictly increasing in $\lambda_1$ and converging to 1 as $\lambda_1$ goes to infinity. $v \geq 3$.

Proof: Analogous to that of Theorem 8.3.1 with $\bar{\nu}_1 = \frac{v\lambda_1}{k} + \frac{v}{v-1} \left(\frac{2p+1}{k} - \frac{1}{v-1}\right)$, $\bar{\nu}_2 = \frac{v\lambda_1}{k} + 1 - \frac{v}{(v-1)(v-2)} \left(\frac{2p+1}{k} - \frac{1}{v-1}\right)$, $b_0 = 0$, $b_1 = \frac{k}{(v-1)(v-2)} \left(\frac{2p+1}{k} - \frac{1}{v-1}\right)$, $b_2 = \frac{k}{v} - \frac{k}{v-1} \left(\frac{2p+1}{k} - \frac{1}{v-1}\right)$, $b_3 = \frac{k}{v}$ and $A(\lambda_1) = \lambda_1 + \frac{k}{v}.$

Theorem 8.3.5: The A-, D- and E-efficiencies for $d \in \mathcal{D}(v,b+1,k^*)$ calculated with Method 1 for $k = pv+1$ and $k = pv+v-1$, respectively, are
\[ e_A(\hat{d}) = \frac{\left( \lambda_1^+ + \frac{k}{v-1} \left( \frac{2p+1}{k} - \frac{1}{v-1} \right) \right)^{-1} + (v-2) \left( \lambda_1^+ + \frac{k}{v} \left( \frac{2p+1}{k} - \frac{1}{v-1} \right) \right)^{-1}}{\left( \lambda_1^+ + \frac{k}{2v} \left( 1-S(1) \right) \right)^{-1} + (\lambda_1^+ + \frac{k}{2v} \left( 1+S(1) \right))^{-1} + (v-3) \left( \lambda_1^+ + \frac{k}{v} \right)^{-1}}. \]

\[ e_B(\hat{d}) = \frac{\left( (\lambda_1^+ + \frac{k}{2v} - \frac{1}{v-1}) (\lambda_1^+ + \frac{k}{v} - \frac{1}{v-1}) \right)^{-2}}{\left( \lambda_1^+ + \frac{k}{v-1} \left( \frac{2p+1}{k} - \frac{1}{v-1} \right) \right)(\lambda_1^+ + \frac{k}{v} \left( \frac{2p+1}{k} - \frac{1}{v-1} \right) \left( \frac{2p+1}{k} - \frac{1}{v-1} \right))} \]

\[ e_E(\hat{d}) = \frac{\lambda_1^+ + \frac{k}{2v} \left( 1-S(1) \right)}{\lambda_1^+ + \frac{k}{v-1} \left( \frac{2p+1}{k} - \frac{1}{v-1} \right)}, \quad \text{where} \]

\[ S(1) = \left( \frac{2p(v-2)}{k} - 1 \right)^2 + \frac{4p}{k} \left( \frac{2p(v-2)}{k} - \frac{v-2}{v-1} \right)^2, \quad \text{and} \]

\[ e_A(\hat{d}) = \]

\[ e_B(\hat{d}) = \]

\[ e_E(\hat{d}) = \frac{\lambda_1^+ + \frac{k}{2v} \left( 1-S(2) \right)}{\lambda_1^+ + \frac{k}{v-1} \left( \frac{2p+1}{k} - \frac{1}{v-1} \right)^2} \]

\[ S(2) = \left( 1 - \frac{4v(v-2)p(p+1)}{k^2(v-1)} \right)^{\frac{1}{3}} \]

with the E-efficiencies known to be strictly increasing in \( \lambda_1 \) and all converging to 1 as \( \lambda_1 \) goes to infinity, \( p \geq 1 \) and \( v \geq 3 \).
Proof: \( \mu_1 \) and \( \mu_2 \) were given in Theorem 8.3.4, the eigenvalues for \( \hat{d} \) are found in the proofs of Theorems 4.3.2 and 4.3.3, and Lemma 8.1.4 is applied to the E-efficiencies.

Remark 8.3.3: Recall that for \( k = pv+q \) and \( q = 2 \) or \( q = v-1 \) \( \hat{d} \) was shown to be strictly \( J_{4} \)-better than \( d* \) in Theorems 4.3.2 and 4.3.3 respectively. Therefore even if we do not know how \( e_A(\hat{d}) \) and \( e_D(\hat{d}) \) behave, we know they are bounded between the efficiencies for \( d* \) and 1. For \( 2 \leq q \leq v-2 \) \( C_\hat{d} \) at (4.3.4) would not yield the eigenvalues for \( \hat{d} \) in general. It is for these reasons that Theorem 8.3.4 is necessary.

The following examples with \( q = 1, 2 \) and \( v-1 \) respectively, show \( d* \) may be very good even for small \( v, p \) and \( q \).

Example 8.3.3: \( d*, \hat{d} \in \mathcal{S}(4,4+1,k*) \) with \( k = v+1 = 5 \). \( e_A(d*) > .9937 \) and \( e_A(\hat{d}) > .9989 \). \( e_D(d*) > .9908 \) and \( e_D(\hat{d}) > .9984 \). \( e_E(d*) = 54/58 > .9310 \) and \( e_E(\hat{d}) > .9908 \).

Example 8.3.4: \( d*, \hat{d} \in \mathcal{S}(4,6+1,k*) \) with \( k = v+2 = 6 \). See Example 4.3.1. \( e_A(d*) > .9985 \) and \( e_A(\hat{d}) > .9996 \). \( e_D(d*) > .9978 \) and \( e_D(\hat{d}) > .9994 \). \( e_E(d*) = 39/40 = .975 \) and \( e_E(\hat{d}) > .9957 \).

Example 8.3.5: \( d*, \hat{d} \in \mathcal{S}(4,4+1,k*) \) with \( k = v+v-1 = 7 \). \( e_A(d*) > .9985 \) and \( e_A(\hat{d}) > .9999 \). \( e_D(d*) > .9978 \) and \( e_D(\hat{d}) > .9998 \). \( e_E(d*) = 54/55 = .9818 \) and \( e_D(\hat{d}) > .9996 \).
8.4 Efficiencies for Chapter 5

For \(3 \leq k \leq v-1\) recall that \(d^*\), a BIBD with one observation removed, had \(C_{d^*}\) with eigenvalues \(0, \frac{\nu \lambda_1}{k} - 1 = \frac{v}{v-1} \left( \frac{r(k-1)}{k} - \frac{v-1}{v} \right)\), and \(\frac{\nu \lambda_1}{k} = \frac{v}{v-1} \left( \frac{r(k-1)}{k} \right)\) with multiplicity \(v-2\).

Theorem 8.4.1: The A-, D- and E-efficiencies for \(d^* \in S(v,b,k^*)\) with \((k^*)' = (k_{b-1}'-1,k-1)\), \(3 \leq k \leq v-1\), as calculated with Method 1 are

\[
e_A(d^*) = \frac{1}{r-1} + (v-2) \left( \frac{r}{v} - \frac{(v-k)}{v(v-2)(k-1)} \right),
\]

\[
e_D(d^*) = \left( \frac{r-1}{v(k-1)} \right) \left( \frac{r}{v-k} \right) \left( \frac{r}{v(v-2)(k-1)} \right),
\]

\[
e_E(d^*) = \frac{r-k(v-1)}{v(k-1)}.
\]

with each strictly increasing in \(r\) and converging to 1 as \(r\) goes to infinity. \(v \geq 3\).

Proof: Method 1 gives \(\bar{\mu}_{d^1} = \frac{v}{v-1} \left( r-1 - \frac{r-1}{k} \right) = \frac{v}{v-1} \left( \frac{r(k-1)}{k} - \frac{k-1}{k} \right) = \bar{\mu}_1\) and \(\bar{\mu}_2 = \frac{v}{v-1} \left( \frac{r(k-1)}{k} \right) - \frac{v-k}{k(v-2)(k-1)}\) since \(\bar{\mu}_1\) is made largest by not putting the minimum replicated treatment in the smaller \((b\text{-th})\) block.

Lemmas 8.1.1, 8.1.2 and 8.1.4 are applied with \(b_0 = 0\), \(b_1 = \frac{v-k}{v(v-2)(k-1)}\), \(b_2 = 1\) and \(b_3 = \frac{k(v-1)}{v(k-1)}\) with \(A(r) = r\).
Example 8.4.1: \( d^* \in \mathcal{D}(5,10, k^*) \) with \( k = v-2 = 3 \): \( e_A(d^*) > .9961, \) 
\( e_D(d^*) > .9927 \) and \( e_E(d^*) = 24/25 = .96. \)

Example 8.4.2: \( d^* \in \mathcal{D}(5,5, k^*) \) with \( k = v-1 = 4 \): \( e_A(d^*) > .9968, \) 
\( e_D(d^*) > .9942 \) and \( e_E(d^*) = 44/45 = .9777. \)

Example 8.4.3: \( d^* \in \mathcal{D}(4,4, k^*) \) with \( k = v-1 = 3 \): \( e_A(d^*) > .9840, \) 
\( e_D(d^*) > .9778 \) and \( e_E(d^*) = 15/16 = .9375. \) \( d^* \in \mathcal{D}(4,8, k^*) \) with \( k = 3 \): 
\( e_A(d^*) > .9969, \) \( e_D(d^*) > .9956 \) and \( e_E(d^*) = 39/40 = .975. \)

For \( k = pv, p \geq 1 \) \( d^* \) was proved \( J_4 \)-optimal in Theorem 5.3.1. For 

\( k = pv+q, p \geq 1, 1 \leq q \leq v-1 \) \( d^* \) was proved \( J_4 \)-optimal only for the case \( q = 1 \). However for all \( 1 \leq q \leq v-1 \) the design \( \hat{d} \) (where the observation was removed from a block where it was not equireplicated) was always shown to be \( J_4 \)-worse than \( d^* \). Now we shall give the A-, D- and E-efficiencies for \( d^* (2 \leq q \leq v-1) \) and \( \hat{d} (1 \leq q \leq v-1) \).

Theorem 8.4.2: The A-, D- and E-efficiencies for \( \tilde{d} \in \mathcal{D}(v, ev, k^*) \) with \( e \geq 1 \) and \( k = pv+1 \) are

\[
e_A(\tilde{d}) = \frac{(v\lambda_1-p(v-1))^{-1} + \frac{v-2}{v\lambda_1}}{(v\lambda_1-p(v-1)-\frac{2k}{k-1})^{-1} + \frac{v-2}{v\lambda_1}},
\]

\[
e_D(\tilde{d}) = e_E(\tilde{d}) = \frac{v\lambda_1-p(v-1)-2k/(k-1)}{v\lambda_1-p(v-1)},
\]

with each strictly increasing in \( \lambda_1 \) and converging to 1 as \( \lambda_1 \) goes to infinity. \( v \geq 3. \)
Proof: Since Theorem 5.3.1 showed $d^*$ to be $J_4$-optimal, we use its eigenvalues for $\tilde{\mu}_i \ (1 \leq i \leq v-1)$. $\mu_{d^*1} = \frac{\lambda_1}{k} - \frac{p(v-1)}{k}$ and $\mu_{d^*2} = \cdots = \mu_{d^*,v-1} = \frac{\lambda_1}{k}$, whereas $\mu_{d1} = \frac{\lambda_1}{k} - \frac{p(v-1)}{k} - \frac{2}{k-1}$ and $\mu_{d^*_i} = \mu_{d^*1}$ for $2 \leq i \leq v-1$.

Lemmas 8.1.1 and 8.1.4 are applied to prove the increasing property.

Example 8.4.4: $\bar{d} \in \mathcal{A}(v,5,k^*)$ with $k = v+1 = 6$: $e_A(\bar{d}) > .9775$ and $e_E(\bar{d}) = 143/155 > .9225$.

Theorem 8.4.3: The $A_-$, $D_-$, and $E$-efficiencies for $d^* \in \mathcal{A}(v,b,k^*)$ with $(k^*)' = (k_{b-1}'b,k-1)$, $k = pv+q$, $2 \leq q \leq v-1$, as calculated by Method 1 are

$$e_A(d^*) = \frac{(v\lambda_1-k+\frac{vp(p+1)}{k-1}) + \frac{(v-q)(q-1)}{(v-2)(v-1)(k-1)^2}}{(v\lambda_1-k+\frac{vp(p+1)}{k-1}) + \frac{v-2}{v\lambda_1}},$$

$$e_D(d^*) = \frac{v\lambda_1-k+\frac{vp(p+1)}{k-1}}{v\lambda_1-k+\frac{vp(p+1)}{k-1} + \frac{(v-q)(q-1)}{(v-2)(v-1)(k-1)}} \left( \frac{v\lambda_1}{v\lambda_1 - \frac{(v-q)(q-1)}{(v-2)(v-1)(k-1)}} \right)^{-v-2},$$

$$e_E(d^*) = \frac{v\lambda_1-k+\frac{vp(p+1)}{k-1}}{v\lambda_1-k+\frac{vp(p+1)}{k-1} + \frac{(v-q)(q-1)}{(v-1)(k-1)}},$$

with each strictly increasing in $\lambda_1$ and converging to 1 as $\lambda_1$ goes to infinity. $v \geq 3$. 
Proof: By Method 1 \( \mu_{d1} \leq \tilde{\mu}_1 = \frac{\lambda_1}{k-1} + \frac{v(p+1)}{k(k-1)} + \frac{(v-q)(q-1)}{(v-1)(v-2)k(k-1)} = \)

\( \mu_{d1} + \frac{(v-q)(q-1)}{(v-1)k(k-1)}. \) \( \tilde{\mu}_2 = \mu_{d*2} - \frac{(v-q)(q-1)}{(v-1)(v-2)k(k-1)}. \) We apply

Lemmas 8.1.1, 8.1.2 and 8.1.4 with \( b_0 = 0, b_1 = \frac{(v-q)(q-1)}{(v-1)(v-2)(k-1)}, b_2 = k - \frac{v(p+1)}{k-1} - \frac{(v-q)(q-1)}{(v-1)(k-1)}, b_3 = k - \frac{v(p+1)}{k-1} \) and \( A(\lambda_1) = \frac{\nu \lambda_1}{\lambda_1}. \)

Theorem 8.4.4: The A-, D- and E-efficiencies of \( d \in \mathcal{D}(v,b,k^*) \) of Theorem 8.4.3 as calculated by Method 1 are

\[
e_A(d) = \frac{(\lambda_1 - k^+ \frac{v(p+1)}{k-1} + \frac{(v-q)(q-1)}{(v-1)(k-1)} - 1 + (v-2)(\lambda_1 - \frac{(v-q)(q-1)}{(v-1)(v-2)(k-1)})^{-1}}{(\nu \lambda_1 - k^+ \frac{v(p+1)}{k-1} - \frac{2k-1}{k-1})^{-1}} + \frac{v-2}{v} \frac{1}{\lambda_1}.
\]

\[
e_D(d) = \left( \frac{\lambda_1 - k^+ \frac{v(p+1)}{k-1} - \frac{2k-1}{k-1}}{\nu \lambda_1 - k^+ \frac{v(p+1)}{k-1} + \frac{(v-q)(q-1)}{(v-1)(k-1)}} \right)^{-1} \left( \frac{v-1}{(v-2)(v-1)(k-1)} \right)^{v-2}.
\]

\[
e_E(d) = \frac{\lambda_1 - k^+ \frac{v(p+1)}{k-1} - \frac{2k-1}{k-1}}{\nu \lambda_1 - k^+ \frac{v(p+1)}{k-1} + \frac{v(q)(q-1)}{(v-1)(k-1)}},
\]

with each strictly increasing in \( \lambda_1 \) and converging to 1 as \( \lambda_1 \) goes to infinity. \( v \geq 3. \)

Proof: The \( \tilde{\mu}_i \) are those from Theorem 8.4.3. The \( \mu_{d_i} = \mu_{d*1} \) for \( 2 \leq i \leq v-1 \) and \( \mu_{d1} = \mu_{d*1} - \frac{2}{k-1}. \) Lemmas 8.1.1, 8.1.2 and 8.1.4 are applied to \( e_A(d), e_D(d) \) and \( e_E(d) \) with \( b_0, b_1, b_2 \) and \( A(\lambda_1) \) as in the proof of Theorem 8.4.3. Here \( b_3 = k - \frac{v(p+1)}{k-1} + \frac{2k}{k-1}. \)
Example 8.4.5: \( d^*, \tilde{d} \in \mathcal{D}(4,6,k^*) \) with \( k = v+2 = 6 \). \( e_A(d^*) > .9998 \) and \( e_A(\tilde{d}) > .9814 \). \( e_D(d^*) > .9997 \) and \( e_D(\tilde{d}) > .9493 \). \( e_E(d^*) = 357/358 > .9972 \) and \( e_E(\tilde{d}) = 339/358 > .9469 \).

8.5 Efficiencies for Chapter 6

In Chapter 6 \( k = pv+q \) with \( p \geq 1 \) and \( 1 \leq q \leq v-1 \). For the case \( mq = v \) considered there, \( m = 2 \) with \( q = v/2 \) had \( d^* \) with the eigenvalue structure \( 0 < a < b = \ldots = b \) of Theorem 2.1.1. Examples 6.2.1 and 6.2.2 showed that each case must be looked at separately. Since \( d^* \) is the logical first guess at an optimal design, we present the following theorem.

**Theorem 8.5.1:** The A-, D- and E-efficiencies for \( d^* \in \mathcal{D}(v,b+2, pv+(v/2)) \) as calculated by Method 1 are

\[
e_A(d^*) = \frac{(v-1)(v\lambda_1+k(2p+1) - \frac{v}{2(v-1)})^{-1}}{(v\lambda_1+k(2p+1) - \frac{v}{2})^{-1} + (v-2)(v\lambda_1+k(2p+1) - \frac{v}{2})^{-1}},
\]

\[
e_D(d^*) = e_E(d^*) \left( \frac{v\lambda_1+k(2p+1)}{(v\lambda_1+k(2p+1) - \frac{v}{2(v-1)})} \right)^{v-2},
\]

\[
e_E(d^*) = \frac{v\lambda_1+k(2p+1) - \frac{v}{2}}{v\lambda_1+k(2p+1) - \frac{v}{2(v-1)}},
\]

with each strictly increasing in \( \lambda_1 \) and converging to 1 as \( \lambda_1 \) goes to infinity. \( v \geq 4 \).

**Proof:** In general for \( mq = v \), Method 1 gives \( \mu_d = \frac{v\lambda_1}{k} \) and \( \mu_p = \frac{v-1}{k(v-1)} \). Substitute \( m = 2 \), \( q = v/2 \).
\[ \mu_{d^*1} = \frac{\nu \lambda_1}{k} + \frac{2p+1}{k} - \frac{\nu}{2k} \quad \text{and} \quad \mu_{d^*2} = \ldots = \mu_{d^*,v-1} = \frac{\nu \lambda_1}{k} + \frac{2p+1}{k}. \] Now we apply Lemmas 8.1.1, 8.1.2 and 8.1.4 with \( b_0 = 0, b_1 = b_2 = \frac{\nu}{2(v-1)}, \) \( b_3 = \nu/2 \) and \( A(\lambda_1) = \nu \lambda_1 + k(2p+1) \) and calculus to get the strictly increasing and convergence properties.

**Example 8.5.1:** \( d^*, \hat{d} \in B(8,14+2,12) \) with \( k = \nu+(v/2) = 12. \) Recall that \( \hat{d} \) was the RGD shown to be A-, D- and E-better than \( d^*. \)

\[ e_A(d^*) > 0.999753 \quad \text{and} \quad e_A(\hat{d}) > 0.999756, \quad e_D(d^*) > 0.999140 \quad \text{and} \quad e_D(\hat{d}) > 0.999147, \quad e_E(d^*) = 245/248 > 0.9879 \quad \text{and} \quad e_E(\hat{d}) = 987/992 > 0.9949. \]

It was also shown that for \( m = v \) and \( q = 2 \) \( d^* \) was E-optimal, and this was used to prove \( J_d \)-optimality for \( d^* \) when \( v = 4, m = 2 \) and \( q = 2. \) For \( q = 2 \) in general we present the A- and D-efficiencies.

**Theorem 8.5.2:** The A- and D-efficiencies of \( d^* \in B(v,b+(v/2), \) \( pv+2) \) are

\[ e_A(d^*) = \left( \frac{\nu \lambda_1 + k(mp+1)-2}{(v-1)(v \lambda_1 + k(mp+1)-2)^{v-1+(v-2)(v \lambda_1 + k(mp+1)-2)^{-1}}} \right)^{m-2}, \]

\[ e_D(d^*) = \left( \frac{v \lambda_1 + k(mp+1)-2}{v \lambda_1 + k(mp+1)-2(v-2)} \right)^{m-2} \left( \frac{v \lambda_1 + k(mp+1)}{v \lambda_1 + k(mp+1)-2(v-2)} \right)^{v-m} \]

with both strictly increasing in \( \lambda_1 \) and converging to 1 as \( \lambda_1 \) goes to infinity. \( v \geq 4. \)

**Proof:** Since by Lemma 6.2.2 \( d^* \) is E-optimal, we use

\[ \bar{\nu}_1 = \mu_{d^*1} = \frac{\nu \lambda_1}{k} + (mp+1) - \frac{2}{k} \quad \text{and} \quad \bar{\nu}_2 = \frac{\nu \lambda_1}{k} + mp+1 - \frac{2(m-2)}{(v-2)k}. \] Now calculus and Lemmas 8.1.1 and 8.1.2 are applied to complete the proof.
For \( m q = v-1 \) we saw that \( d^* \) is always E-optimal in Theorem 6.3.1 and for \( m = 1 \) or \( m = v-1 \) \( d^* \) is \( \mathcal{C}_4 \)-optimal in Theorem 6.3.2. For \( 2 \leq m \leq v-2 \) we have the following theorem.

**Theorem 8.5.3:** The A- and D-efficiencies for \( d^* \in \mathcal{C}(v, b+m, pv+q) \) where \( 2 \leq m \leq v-2 \) and \( m q = v-1 \) are

\[
e_A(d^*) = \frac{(v\lambda_1 + v p(mp+1))^{-1} + (v-2)(v\lambda_1 + k(mp+1) - \frac{q(m-1)}{v-2})^{-1}}{(v\lambda_1 + v p(mp+1))^{-1} + (m-1)(v\lambda_1 + k(mp+1) - q)^{-1} + \frac{v-m-1}{v\lambda_1 + k(mp+1)}},
\]

\[
e_D(d^*) = \left( \frac{v\lambda_1 + k(mp+1) - q}{v\lambda_1 + k(mp+1) - \frac{q(m-1)}{v-2}} \right)^{m-1} \left( \frac{v\lambda_1 + k(mp+1)}{v\lambda_1 + k(mp+1) - \frac{q(m-1)}{v-2}} \right)^{v-m-1},
\]

with both strictly increasing in \( \lambda_1 \) and converging to 1 as \( \lambda_1 \) goes to infinity. \( v \geq 3 \).

**Proof:** Since by Theorem 6.3.1 \( d^* \) is E-optimal, \( \mu_1 = \mu_{d^*1} = \frac{v\lambda_1 + v p(mp+1)}{k} \) and \( \mu_2 = \frac{v\lambda_1 + mp+1 - \frac{q(m-1)}{k} v^{-2}}{k} \). \( \mu_{d^*i} = \frac{v\lambda_1 + mp+1 - \frac{q}{k}}{k} \) for \( 2 \leq i \leq m \) and \( \mu_{d^*i} = \frac{v\lambda_1}{k} + mp+1 \) for \( m+1 \leq i \leq v-1 \). The limit properties are trivial, Lemma 8.1.2 is applied to \( e_D(d^*) \), but \( e_A(d^*) \) is shown to be strictly increasing in \( \lambda_1 \) directly. That completes the proof.

For \( 1 \leq m q \leq v-2 \) only \( m = q = 1 \) gave \( C_{d^*} \) an eigenvalue structure of interest, that of Theorem 1.4.1. However if \( p > v-1 \) we saw \( d^* \) was E-worse than the design \( \hat{d} \) of Theorem 6.4.1.
Theorem 8.5.4: \( \Delta(v, ev+1, pv+1) \) with \( v \geq 3, e \geq 1, p \geq 1 \). The A- and D-efficiencies for \( d^* \) and the E-efficiency for \( d^* \) and \( \hat{d} \) (for \( p \geq v-1 \)) as calculated by Method 1 are

\[
e_A(d^*) = \frac{(v\lambda_1 + vp^2 + \frac{vp}{v-1})^{-1} + (v-2)(v\lambda_1 + kp + \frac{vp}{v-1})^{-1}}{(v-2)(v\lambda_1 + vp^2 + p)^{-1} + (v\lambda_1 + kp + (v-1)p)^{-1}},
\]

\[
e_D(d^*) = \left(\frac{v\lambda_1 + vp^2 + vp}{v\lambda_1 + vp^2 + \frac{vp}{v-1}}\right)^{v-2} \left(\frac{v\lambda_1 + kp}{v\lambda_1 + kp + \frac{vp}{v-1}}\right),
\]

\[
e_E(d^*) = \frac{v\lambda_1 + vp^2 + p}{v\lambda_1 + vp^2 + \frac{vp}{v-1}}, \quad \text{for } p \geq v-1,
\]

with each strictly increasing in \( \lambda_1 \) and converging to 1 as \( \lambda_1 \) goes to infinity.

**Proof:** By Method 1 \( \mu_{d_1} \leq \bar{u}_1 = \frac{v\lambda_1}{k} + \frac{vp}{k} \left( p + \frac{1}{v-1} \right) \) and \( \bar{u}_2 = \frac{v\lambda_1}{k} + \frac{vp}{k} \frac{v-1}{v\lambda_1 + kp + (v-1)p} \). \( \mu_{d_i} = \frac{v\lambda_1}{k} + p \) for \( 1 \leq i \leq v-2 \) and \( \mu_{d^*, v-1} = \frac{v\lambda_1}{k} + \frac{vp(p+1)}{k} \). Recall \( k = pv+1 \). Lemma 8.1.1 is applied to \( e_A(d^*) \) with \( b_0 = 0, b_1 = (v-1)p - \frac{vp}{v-1}, b_2 = (v-1)p + kp - \frac{vp}{v-1} - vp^2, b_3 = (v-1)p \) and \( A(\lambda_1) = v\lambda_1 + kp + (v-1)p \). Lemma 8.1.4 is applied to \( e_E(d^*) \) and \( e_E(\hat{d}) \). Lemma 8.1.3 is applied to \( e_D(d^*) \) with \( a_0 = kp = vp^2 + p, a_1 = vp^2 + \frac{vp}{v-1}, a_2 = kp + \frac{vp}{v-1} = vp^2 + p + \frac{vp}{v-1}, a_3 = vp^2 + vp \) and \( A(\lambda_1) = v\lambda_1 \).
Example 8.5.2: See Example 6.4.1. \( e_A(d^*) = e_D(d^*) = 1364/1365 > .9992 \) since \( v = 3 \) and \( e_E(d^*) = 62/63 > .9841 \) while \( e_E(d) = 1 \) since \( p+1 = v p / (v-1) \).

8.6 Efficiencies for Chapter 7

In Chapter 7 \( k = p v + q \) with \( p \geq 1 \) and \( 1 \leq q \leq v-1 \). It was noted that for the first case considered, \( m q = v \), that all such designs were E-optimal. The following theorem gives with the A- and D-efficiencies of such designs not already proved A- and D-optimal.

Theorem 8.6.1: The A- and D-efficiencies for \( d^* \in D(v,b-m,p v + q) \) with \( v \geq 3 \), \( b-m \geq 2 \), \( p \geq 1 \), \( m q = v \) and \( 3 \leq m \leq v-1 \) are

\[
e_A(d^*) = \frac{(v \lambda_1 - k (mp+1))^{-1} + (v-2) (v \lambda_1 - k (mp+1) + \frac{q (m-1)}{v-2})^{-1}}{(v-m) (v \lambda_1 - k (mp+1))^{-1} + (m-1) (v \lambda_1 - k (mp+1) + q)^{-1}}
\]

\[
e_D(d^*) = \left( \frac{v \lambda_1 - k (mp+1)}{v \lambda_1 - k (mp+1) + \frac{q (m-1)}{v-2}} \right)^{v-m-1} \left( \frac{v \lambda_1 - k (mp+1) + q}{v \lambda_1 - k (mp+1) + \frac{q (m-1)}{v-2}} \right)^{m-1}
\]

with both strictly increasing in \( \lambda_1 \) and converging to 1 as \( \lambda_1 \) goes to infinity.

Proof: Since \( d^* \) is E-optimal we use \( \bar{\mu}_1 = \frac{v \lambda_1}{k} - (mp+1) \).

\( \bar{\mu}_2 = \frac{v \lambda_1}{k} - (mp+1) + \frac{q (m-1)}{k(v-2)} \)

\( \mu_{d^* i} = \frac{v \lambda_1}{k} - (mp+1) \) for \( 1 \leq i \leq v-m \) and

\( \mu_{d^* i} = \frac{v \lambda_1}{k} - (mp+1) + \frac{q}{k} \) for \( v-m+1 \leq i \leq v-1 \). Lemmas 8.1.1 and 8.1.2 are applied to \( e_A(d^*) \) and \( e_D(d^*) \), respectively, with \( b_0 = 0 \), \( b_1 = q(1- \frac{m-1}{v-2}) \), \( b_2 = q \) or \( b_1 \) (respectively), \( b_3 = q \) and \( A(\lambda_1) = v \lambda_1 - k (mp+1) \).
The proof is complete after calculus is applied to get the limit properties.

For \( m = q = 1 \) only \( m = 1 \) and \( m = q = 1 \) gave eigenvalue structures from Lemma 7.1.1, both structures being \( 0 < a = \ldots = a < b \). For \( m = q = 1 \) (\( q = 1 \)) see the case of \( m = q = 1 \) in Section 6.4 and its corresponding Theorem 8.5.4.

When \( m = 1 \) and \( q = 1 \) \( d^* \) has nonzero eigenvalues \( \frac{\nu \lambda_1}{k} - (p+1) \) with multiplicity \( v-2 \) and \( \frac{\nu \lambda_1}{k} - \frac{\nu p(p+1)}{k} \). There is also a \( \hat{d} \) that is E-better than \( d^* \) for \( p \geq v-2 \), and is given in Theorem 7.3.1.

**Theorem 8.6.2:** \( \varnothing(v, ev-1, pv+v-1) \) with \( v \geq 3, e \geq 1, p \geq 1 \). The \( A^* \) and \( D \)-efficiencies of \( d^* \) and the E-efficiency of \( d^* \) and \( \hat{d} \) (for \( p \geq v-2 \)) as calculated by Method 1 are

\[
e_A(d^*) = \frac{(\nu \lambda_1 - k(p+1) + \frac{p+1}{v-1})^{-1} + (v-2)(\nu \lambda_1 - k(p+1) + \frac{v(p+1)}{v-1})^{-1}}{(v-2)(\nu \lambda_1 - k(p+1))^{-1} + (v \lambda_1 - k(p+1) + (v-1)(p+1))^{-1}},
\]

\[
e_D(d^*) = \left(\frac{\nu \lambda_1 - k(p+1) + (v-1)(p+1)}{\nu \lambda_1 - k(p+1) + \frac{p+1}{v-1}}\right) \left(\frac{\nu \lambda_1 - k(p+1) + \frac{v(p+1)}{v-1}}{\nu \lambda_1 - k(p+1)}\right)^{v-2},
\]

\[
e_E(d^*) = \frac{\nu \lambda_1 - k(p+1)}{\nu \lambda_1 - k(p+1) + \frac{p+1}{v-1}},
\]

\[
e_E(\hat{d}) = \frac{\nu \lambda_1 - k(p+1) + 1}{\nu \lambda_1 - k(p+1) + \frac{p+1}{v-1}}, \text{ for } p \geq v-2,
\]

with each strictly increasing in \( \lambda_1 \) and converging to 1 as \( \lambda_1 \) goes to infinity.
Proof: Method 1 gives \( \bar{\mu}_1 = \frac{v\lambda_1}{k} - \frac{vp(mp+1)}{k} - \frac{v}{v-1} \left( \frac{mp+1}{k} \right) \) in general for \( 1 \leq q \leq v-1 \). Substituting \( m = 1, q = v-1, \bar{\mu}_1 = \frac{v\lambda_1}{k} - \frac{vp(p+1)}{k} \frac{v(v-2)(p+1)}{k(v-1)} = \frac{v\lambda_1}{k} -(p+1) + \frac{p+1}{k(v-1)}. \bar{\mu}_2 = \frac{v\lambda_1}{k} -(p+1) + \frac{v(p+1)}{k(v-1)} \). The eigenvalues of \( d^* \) and \( \hat{d} \) come from Section 7.3. Lemma 8.1.4 is applied to the E-efficiencies while Lemma 8.1.1 is applied to \( e_A(d^*) \) with \( b_0 = 0, b_1 = (v-1)(p+1) - \frac{v(p+1)}{v-1}, b_2 = (v-1)(p+1) - \frac{p+1}{v-1}, b_3 = (v-1)(p+1) \) and \( A(\lambda_1) = v\lambda_1 - k(p+1) \). Lemma 8.1.3 is applied to \( e_D(d^*) \) with \( a_0 = 0, a_1 = \frac{p+1}{v-1}, a_2 = \frac{v(p+1)}{v-1}, a_3 = (v-1)(p+1) \) and \( A(\lambda_1) = v\lambda_1 - k(p+1) \).

Example 8.6.1: For \( d^*, \hat{d} \in \{3,3-1,5\} \) presented below, 
\( e_A(d^*) = e_D(d^*) = 84/85 > .9882 \) and \( e_E(d^*) = 14/15 = .9333 \) while \( e_E(\hat{d}) = 1 \) since \( p+1 = v-1 \).

\[
\begin{array}{cc}
1 & 1 \\
2 & 2 \\
d^*: & 3 & 3 & \hat{d}: & 3 & 3 \\
1 & 1 & 1 & 2 \\
2 & 3 & 1 & 3
\end{array}
\]

For \( 1 \leq mq \leq v-2 \) the only case giving an eigenvalue structure of Theorems 1.4.1, 1.4.4 or 2.1.1 was \( m = q = 1 \) and that was proven \( J_4 \)-optimal. Therefore no efficiencies are presented for \( 2 \leq mq \leq v-2 \).
CHAPTER 9
TREND-FREE BLOCK DESIGNS

9.1 Preliminaries

Bradley and Yeh (1980) introduced the theory of trend-free block designs. In this setting an experimenter is interested in applying $v$ treatments to the plots of $b$ blocks of size $k \leq v$, but there is thought to be a common polynomial trend over the $k$ plots of each block.

The additive model for a block design with polynomial trends taken into consideration is

$$y_{jt} = u + \sum_{i=1}^{v} \delta_{djt} x_{i} + \beta_{j} + \sum_{\alpha=1}^{p} \theta_{\alpha} \phi_{\alpha}(t) + \epsilon_{jt} \quad (9.1.1)$$

where $u$ is the overall mean, $x_{i}$ is the treatment effect ($1 \leq i \leq v$), $\beta_{j}$ is the block effect ($1 \leq j \leq b$), $y_{jt}$ is the observation for plot $t$ in block $j$ ($1 \leq t \leq k$), $\theta_{\alpha}$ is the regression coefficient for $\phi_{\alpha}(t)$, $\phi_{\alpha}(t)$ is the value of the orthogonal polynomial of degree $\alpha$ at $t$ ($1 \leq \alpha \leq p$) and the $\epsilon_{jt}$ are assumed to be i.i.d. zero-mean random errors.

If we label the $bk$ plots with ordered pairs $(j,t)$ then $\delta_{djt}^i = 1$ and not 0 only if treatment $i$ is applied to plot $(j,t)$.
\[ P \sum_{\alpha=1}^{p} \phi_\alpha(t) \] is the overall trend effect for plot \( t \).

For the orthogonal polynomials we have \( \sum_{t=1}^{k} \phi_\alpha(t) = 0 \) and \( \sum_{t=1}^{k} \phi_\alpha^2(t) = 1 \) for \( 1 \leq \alpha \leq p \) and \( \sum_{t=1}^{k} \phi_\alpha(t)\phi_\beta(t) = 0 \) for \( 1 \leq \alpha < \beta \leq p \).

Bradley and Yeh (1980) derived the appropriate C-matrix and gave a condition for a design to be trend-free (TF) of degree p. The details of the derivations can be found there.

The C-matrix will be designated as \( C_d \) as before, but it has a new form:

\[ C_d = r_d^\delta - k^{-1} r_d \gamma_d \gamma_d^t - b^{-1} \Delta_d \phi_p \phi_p^t \Delta_d. \]

\( \Delta_d \) is the \( k \) by \( v \) matrix with \( r_{dit} = \sum_{j=1}^{b} \delta_{djt} \) in row \( t \) and column \( i \), where \( r_{dit} \) is the number of times treatment \( i \) is applied to plot \( t \) in the design. Of course the column sums of \( \Delta_d \) are \( r_{di} = \sum_{t=1}^{k} r_{dit} \) and the row sums are \( b = \sum_{i=1}^{v} r_{dit} \).

\( \phi_p \) is the \( k \) by \( p \) matrix with \( \phi_\alpha(t) \) in row \( t \) and column \( \alpha \). If we define \( \phi_\alpha'(t) = (\phi_\alpha(1), \ldots, \phi_\alpha(k)) \) for \( 1 \leq \alpha \leq p \), then \( \phi_p = [\phi_1', \phi_2', \ldots, \phi_p'] \). The reader should not confuse \( \phi_p \) with the symbols \( \phi \) or \( \phi_p \) for optimality criteria.

**Lemma 9.1.1:** (Bradley and Yeh (1980)) Under the model at (9.1.1) a design \( d \) is a trend-free block design for degree \( p \) (TFBD(p)) if and only if
\[
\sum_{j=1}^{b} \sum_{t=1}^{k} \delta_{djt}^{i} \phi_{\alpha}(t) = 0, \quad \alpha = 1, \ldots, p; \quad i = 1, \ldots, v;
\]  
(9.1.2)

or equivalently
\[
\sum_{\alpha=1}^{p} \sum_{i=1}^{v} \sum_{j=1}^{b} \sum_{t=1}^{k} \delta_{djt}^{i} \phi_{\alpha}(t)^2 = 0
\]  
(9.1.3)

or equivalently
\[
\Delta_{+d}^{i} \phi_{p} = C_{v,p}^{i}
\]  
(9.1.4)

Remark 9.1.1: If \(d\) is a TFBD(\(p\)) then \(\Delta_{+d}^{i} \phi_{\alpha} = C_{v,1}\) for \(1 \leq \alpha \leq p\).

Yeh and Bradley (1983) presented results for the existence and construction of trend-free block designs (TFBD's). For situations where \(v, b, k, p\) and \(r\) do not allow a TFBD(\(p\)) to exist, Yeh and Bradley (1984) introduced two measures of closeness to the trend-free property. They are paraphrased here as definitions.

Definition 9.1.1: (their Definition 1) A block design under model (9.1.1) is said to be a nearly trend-free block design of Type A for incidence matrix \(\gamma\) and degree \(p\) (NTFBD(\(A, p; \gamma\))) if (9.1.3) is minimized among all designs with the same \(\gamma\).

Definition 9.1.2: (their Definition 2) A block design under model (9.1.1) is said to be a nearly trend-free block design of Type B for incidence matrix \(\gamma\) and degree \(p\) (NTFBD(\(B, p; \gamma\))) if among all designs with the same \(\gamma\) the following holds:

\[
(i) \quad \sum_{\alpha=1}^{p-1} \sum_{i=1}^{v} \sum_{j=1}^{b} \sum_{t=1}^{k} \delta_{djt}^{i} \phi_{\alpha}(t)^2 \text{ is minimum, and}
\]
(ii) \[ \sum_{j=1}^{b} \sum_{i=1}^{v} \sum_{t=1}^{k} \delta_{j} \delta_{i} \delta_{t} \phi(p(t))^{2} \] is minimum among all \( d \) satisfying (i).

**Remark 9.1.2:** If \( d \) is a NTFBD(\( A, p; \gamma \)) then it is a NTFBD(\( B, p; \gamma \)), but the converse does not necessarily hold. (9.1.3) is just the trace of \( A_{+d}^{T} \phi_{p}^{T} \phi_{p} A_{+d} \). If (9.1.3) equals zero then \( d \) is a TFBD(\( p \)). If the minimum of (i) in Definition 9.1.2 is 0 then \( d \) is a TFBD(\( p-1 \)) and a NTFBD(\( B, p; \gamma \)).

Yeh and Bradley (1984) also proved the following theorem which is improved upon in Section 9.2.

**Theorem 9.1.1:** (their Theorem 1) If \( d^0 \) is a BIBD or CBD for model (9.1.1) under a polynomial trend of degree \( p \) having incidence matrix \( \gamma_{0} \), and \( d^0 \) is a TFBD(\( p-1 \)) and a NTFBD(\( B, p; \gamma_{0} \)) then \( d^0 \) is A- and D-optimal among all \( d \) with \( \gamma_{d} = \gamma_{0} \) and that are a TFBD(\( p-1 \)).

Note that in Definitions 9.1.1 and 9.1.2 we restricted ourselves to \( d \) with the same \( \gamma \). In Theorem 9.1.1 the class of designs over which \( d^0 \) is optimal is a subclass of the class in the definitions.

### 9.2 Extended Results

We introduce some notation first. Since we are keeping \( v, b \) and \( k \) fixed for all discussions we shall use \( \mathcal{D} \) instead of \( \mathcal{D}(v, b, k) \) for the class of all connected designs with each block containing at least two treatments. Now let \( \mathcal{D}(\gamma_{0}) = \{ d : \gamma_{d} = \gamma_{0} \} \) and \( \mathcal{D}(\Lambda_{0}, \rho_{0}) = \{ d : \Lambda_{d} = \Lambda_{0} \text{ and } \rho_{d} = \rho_{0} \} \) where \( \Lambda_{d} = \Lambda_{0} \gamma_{d} \gamma_{0}' \) and was defined in Chapter 1. Also let TF(\( p \)) stand for "trend-free for degree \( p \)."
We shall now extend the scope of Definitions 9.1.1 and 9.1.2 and Theorem 9.1.1.

**Definition 9.2.1:** A block design $d_0$ with $\Lambda_{d_0} = \Lambda_0$ and $r_{d_0} = r_0$ under model (9.1.1) is said to be a nearly trend-free block design of Type A for the pair $(\Lambda_0, r_0)$ and degree $p$ ($\text{NTBDF}(A, p; \Lambda_0, r_0)$) if (9.1.3) is minimized over $B(\Lambda_0, r_0)$.

**Definition 9.2.2:** A block design $d_0$ with $\Lambda_{d_0} = \Lambda_0$ and $r_{d_0} = r_0$ under model (9.1.1) is said to be a nearly trend-free block design of Type B for the pair $(\Lambda_0, r_0)$ and degree $p$ ($\text{NTBDF}(B, p; \Lambda_0, r_0)$) if the following holds:

(i) $\sum_{\alpha=1}^{p-1} \sum_{i=1}^{v} \sum_{j=1}^{b} \sum_{t=1}^{k} \delta_{djt}^{i} \phi_{\alpha}(t)^2$ is a minimum over $B(\Lambda_0, r_0)$ and

(ii) $\sum_{i=1}^{v} \sum_{j=1}^{b} \sum_{t=1}^{k} \delta_{djt}^{i} \phi_{p}(t)^2$ is minimum among all $d \in B(\Lambda_0, r_0)$ satisfying (i).

**Remark 9.2.1:** As in Section 9.1, if $d$ is a NTFBD($A, p; \Lambda_0, r_0$) then it is a NTFBD($B, p; \Lambda_0, r_0$). The $d^0$ and $d_3$ in Example 9.2.1 illustrate that the converse is not necessarily true; $d^0$ is a NTFBD($B, 2, \Lambda_0, r_0$) but $d_3$ is "more free" than $d^0$ in the sense of Definition 9.2.1. If the minimum of (i) of definition 9.2.2 is 0, then $d^0$ is a TFBBD($p-1$) as well as a NTFBD($B, p; \Lambda_0, r_0$). The $d^0$ in each of Examples 9.2.1 and 9.2.2 is such a $d^0$. Finally (9.1.3) is still the trace of $\Lambda_{d}^{\top} \phi_{p} \phi_{p}^{\top} \Lambda_{d}$ and Lemma 9.1.1 still holds.
The scope of Definitions 9.1.1 and 9.1.2 were widened because the optimality and trend-free properties of a design \( d \) depend on \( \eta_d \) only through \( \Lambda_d \) and \( \tau_d \). Therefore taking a specific \( \Lambda_0 \) and \( \tau_0 \) allows us to include any \( \eta_d \) with \( \Lambda_d = \Lambda_0 \) and \( \tau_d = \tau_0 \). This is particularly useful when we consider \( \Lambda_0 \) and \( \tau_0 \) for a BIBD.

**Theorem 9.2.1:** If \( d^0 \) is a BIBD (or CBD) in \( \mathcal{D}(\Lambda_0, \tau_0) \) for model (9.1.1) under a polynomial trend of degree \( p \), and \( d^0 \) is a TFBD(p-1) and a NTFBD(B, p; \( \Lambda_0, \tau_0 \)) then \( d^0 \) is \( J_4 \)-optimal over all TF(p-1) designs \( d \) in \( \mathcal{D}(\Lambda_0, \tau_0) \).

**Proof:** This is essentially the proof of Theorem 9.1.1. For all TF(p-1) designs \( d \in \mathcal{D}(\Lambda_0, \tau_0) \) we know \( \Delta_d^+ \Phi_p \tau - 1 = 0 \) so \( \Delta_d^+ \Phi_p = \Delta_d^+ \Phi_p = \Delta_d^+ \Phi_p \). We also know \( \tau_d^0 \Delta_d^{-1} \Lambda_0 = \tau_d^0 \Delta_d^{-1} \Lambda_0 \) is completely symmetric, the matrix \( b^{-1} \Delta_d^+ \Phi_p \Phi_p \Delta_d \) has exactly one nonzero eigenvalue \( \gamma_d \), and the eigenvalues of \( C_d = \tau_d^0 \Delta_d^{-1} \Lambda_0 - b \Delta_d^+ \Phi_p \Phi_p \Delta_d \) are

\[
0 < \frac{\nu \lambda_1}{k} - \gamma_d, \frac{\nu \lambda_2}{k}, ..., \frac{\nu \lambda_{\nu}}{k} \]

as pointed out by Yeh and Bradley (1984). Since \( d^0 \) is one of these designs and is also a NTFBD(B, p; \( \Lambda_0, \tau_0 \)), then

\[
\gamma_d^0 = \gamma_d = \text{tr}(b^{-1} \Delta_d^+ \Phi_p \Phi_p \Delta_d) = \sum_{i=1}^{\nu} \sum_{j=1}^{k} \sum_{t=1}^{\delta_d} b_{ij} t \phi(t)^2.
\]

This makes the eigenvalue structure of \( C_d^0 \) that of Theorem 2.1.1, \( d^0 \) E-optimal over the TF(p-1) designs in \( \mathcal{D}(\Lambda_0, \tau_0) \), and \( C_d^0 \) of maximum trace for the TF(p-1) designs in \( \mathcal{D}(\Lambda_0, \tau_0) \). So applying Theorem 2.1.1 we conclude the proof.
This optimality of \( d^0 \) of Theorem 9.2.1 is still over a very small subclass of \( \mathcal{D} \). Three questions naturally arise. Is such a \( d^0 \) \( \mathcal{J}_4 \)-optimal over all designs in \( \mathcal{D} \)? Is it optimal over all designs that are trend-free for at least degree \( p-1 \)? Is it optimal over all designs in \( \mathcal{D}(\Lambda_0, r_0) \) but which do not even have to be TF(\( p-1 \))? 

As the following two examples illustrate, in the BIBD case the answers to the last two questions, and therefore the first, can be no. The orthogonal polynomials used for the examples are those in Appendix II of Anderson and McLean (1974).

**Example 9.2.1:** \( \mathcal{D} = \mathcal{D}(6,10,3) \) and \( p = 2 \). Below are \( d^0 \) of Theorem 9.2.1, an E-better non-BIBD \( d_1 \) which is TF(1), a D-better non-BIBD \( d_2 \) which is TF(1) and a \( \mathcal{J}_4 \)-better BIBD \( d_3 \) with \( \Lambda_0 \) and \( r_{d_3} = r_0 \), but which is not even TF(1).

\[
\begin{array}{cccccccc}
1 & 1 & 6 & 5 & 4 & 2 & 2 & 5 \\
\hline
2 & 5 & 3 & 4 & 1 & 3 & 4 & 6 & 4 & 3 \\
3 & 2 & 1 & 6 & 4 & 6 & 2 & 5 & 5 \\
1 & 3 & 6 & 5 & 4 & 2 & 2 & 5 & 1 & 6 \\
\end{array}
\]

\[
\begin{array}{cccccccc}
1 & 1 & 6 & 5 & 4 & 2 & 2 & 5 \\
\hline
2 & 5 & 3 & 4 & 1 & 3 & 4 & 6 & 4 & 3 \\
3 & 2 & 1 & 6 & 4 & 6 & 2 & 5 & 5 \\
1 & 2 & 3 & 4 & 5 & 6 & 6 & 2 & 5 & 4 \\
2 & 3 & 4 & 5 & 6 & 1 & 1 & 3 & 3 & 6 \\
4 & 5 & 6 & 1 & 2 & 3 & 2 & 4 & 6 & 5 \\
\end{array}
\]
Example 9.2.2: $\omega = \omega(6,15,4)$ and $p = 3$. Below we have $d^0$ of Theorem 9.2.1, a $\omega$-better non-BIBD $d_1$ which is TF(2), and an A- and D-better non-BIBD $d_2$ which is TF(3).

\[
\begin{align*}
&1 1 3 4 4 2 2 5 3 \quad 6 \\
&d_3: 2 2 1 1 6 3 4 6 5 5 \\
&3 5 6 5 1 4 6 2 4 3
\end{align*}
\]

There are three easy to follow principles that worked for finding the counterexamples to the $\omega_4$-optimality of $d^0$ over all or part of $\omega$ beyond that of Theorem 9.2.1.

The first was used to get $d_1$ in each of Examples 9.2.1 and 9.2.2. After finding $d^0$ we knew the "smallest" $B_d$ for equireplicated designs that are TF(p-1), and this was fixed. Using $B_d$ as a guide, without
changing the fact that the treatments are equi-replicated, adjust the way the treatments are paired to make $\Lambda_d$ less smooth so that the sum $k^{-1}\Lambda_d + B_d$ is more smooth than it was for $d_0$, therefore making $C_d$ more smooth. For Example 9.2.2 we present $k^{-1}\Lambda_d^0$, $B_d^0 = B_d^1$ and $k^{-1}\Lambda_d^1$, respectively.

$$
\begin{bmatrix}
30 & 18 & 18 & 18 & 18 & 18 \\
30 & 18 & 18 & 18 & 18 \\
30 & 18 & 18 & 18 & 18 \\
\end{bmatrix}
$$

$$
k^{-1}\Lambda_d^0 = \frac{1}{12}
$$

$$
\begin{bmatrix}
4 & 4 & 4 & -4 & -4 & -4 \\
4 & 4 & -4 & -4 & -4 \\
4 & -4 & -4 & -4 & -4 \\
\end{bmatrix}
$$

$$
B_d^0 = \frac{1}{12}
$$

$$
\begin{bmatrix}
4 & 4 & 4 & -4 & -4 & -4 \\
4 & 4 & -4 & -4 & -4 \\
4 & -4 & -4 & -4 & -4 \\
\end{bmatrix}
$$

$$
\begin{bmatrix}
30 & 15 & 15 & 21 & 21 & 18 \\
30 & 15 & 15 & 21 & 21 \\
\end{bmatrix}
$$

$$
k^{-1}\Lambda_d^1 = \frac{1}{12}
$$

$$
\begin{bmatrix}
30 & 15 & 15 & 21 & 21 & 18 \\
30 & 15 & 15 & 21 & 21 \\
\end{bmatrix}
$$
The second principle was used to get $d_2$ in each of Examples 9.2.1 and 9.2.2. Here one finds different replications for some treatments that do not affect the TF(p-1) property but make a design nearer to TF(p). $r'_{d_2}$ equals $(4, 5, 5, 5, 5, 6)$ and $(8, 8, 8, 12, 12, 12)$ instead of $516'$ and $1016'$ in Examples 9.2.1 and 9.2.2, respectively.

The third principle was used to get $d_3$ in Example 9.2.1. Keeping the $\gamma$ of the BIBD generated $d^0$, just change the $r_{dit'}$, $1 \leq t \leq k$, for each $i$ so that $(r_{dit'}, ..., r_{dik})^\alpha$ is made small for each $i$ and all $1 \leq \alpha \leq p$. 
CHAPTER 10
SUMMARY

10.1 Optimality of Designs

In Table 10.1.1 are listed the classes $\mathcal{A}(v,b,k)$ for which a design $d^*$ was proved optimal in Chapters 3 through 7. The number of treatments ($v$), the number of blocks ($b$), and block size or sizes ($k$) for which optimality was proved appear in the first three columns, respectively. Then the optimality criteria for which $d^*$ was proved optimal are listed. "gW_1" stands for the generalized type 1 criteria of Cheng (1978) (see Theorem 1.4.1), "J_4" stands for the criteria of Theorem 2.1.1, and "E" stands for the E-optimality criterion. Finally the theorem or lemma where the result was proved is given.

Note that $b_0$ is the number of blocks allowing a BBD of Kiefer (Definition 1.2.1) to be constructed.

10.2 Efficiencies

In Table 10.2.1 are presented the classes $\mathcal{A}(v,b,k)$ where efficiencies were obtained for some designs of interest. The table is set up just as Table 10.1.1 with one minor exception: The design for which the efficiency was obtained appears with the number of the respective theorem.
Table 10.1.1: Classes with optimalities

<table>
<thead>
<tr>
<th>Values of ( v )</th>
<th>Number of Blocks</th>
<th>Block Sizes</th>
<th>Optimality Criteria</th>
<th>Theorem or Lemma</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \geq 3 )</td>
<td>( \frac{ev(v-1)}{2} + 1 )</td>
<td>(2, ..., 2)</td>
<td>( g_3 )</td>
<td>T3.2.1</td>
</tr>
<tr>
<td>( \geq 3 )</td>
<td>( ev-1 )</td>
<td>(v-1, ..., v-1)</td>
<td>( g_3 )</td>
<td>T3.2.2</td>
</tr>
<tr>
<td>( \geq 3 )</td>
<td>( ev+1 )</td>
<td>(v-1, ..., v-1)</td>
<td>( J_4 )</td>
<td>T3.3.1</td>
</tr>
<tr>
<td>3, 4, 5, 6</td>
<td>( \frac{ev(v-1)}{2} - 1 )</td>
<td>(2, ..., 2)</td>
<td>( J_4 )</td>
<td>T3.3.2</td>
</tr>
<tr>
<td>( \geq 3 )</td>
<td>( b_0 + 1 )</td>
<td>(v, ..., v, v-1)</td>
<td>( J_4 )</td>
<td>T4.1.1</td>
</tr>
<tr>
<td>( \geq 3 )</td>
<td>( b_0 )</td>
<td>(pv-1, pv, ..., pv)</td>
<td>( J_4 )</td>
<td>T5.3.1</td>
</tr>
<tr>
<td>( \geq 3 )</td>
<td>( ev )</td>
<td>(pv, pv+1, ..., pv+1)</td>
<td>( J_4 )</td>
<td>T5.3.2</td>
</tr>
<tr>
<td>( \geq 3 )</td>
<td>( b_0 + v/2 )</td>
<td>(pv+2, ..., pv+2)</td>
<td>E</td>
<td>L6.2.2</td>
</tr>
<tr>
<td>4</td>
<td>( 6v+2 )</td>
<td>(4v+2, ..., 4v+2)</td>
<td>( J_4 )</td>
<td>T6.2.1</td>
</tr>
<tr>
<td>( \geq 3 )</td>
<td>( b_0 + m )</td>
<td>(pv + ( \frac{v-1}{m} ), ..., pv + ( \frac{v-1}{m} ))</td>
<td>E</td>
<td>T6.3.1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>and ( 1 \leq m \leq v-1 )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \geq 3 )</td>
<td>( ev+1 )</td>
<td>(pv+v-1, ..., pv+v-1)</td>
<td>( J_4 )</td>
<td>T6.3.2</td>
</tr>
<tr>
<td>( \geq 3 )</td>
<td>( ev+v-1 )</td>
<td>(pv+1, ..., pv+1)</td>
<td>( J_4 )</td>
<td>T6.3.2</td>
</tr>
<tr>
<td>( \geq 3 )</td>
<td>( ev-1 )</td>
<td>(pv+1, ..., pv+1)</td>
<td>( J_4 )</td>
<td>T7.4.1</td>
</tr>
<tr>
<td>Values of $v$</td>
<td>Number of Blocks</td>
<td>Block Sizes</td>
<td>Optimality Criteria</td>
<td>Theorem and Design</td>
</tr>
<tr>
<td>--------------</td>
<td>------------------</td>
<td>-------------</td>
<td>---------------------</td>
<td>-------------------</td>
</tr>
<tr>
<td>$\geq 3$</td>
<td>$b_0 + l \cdot \frac{ev(v-1)}{2} - 1$</td>
<td>$(2, \ldots, 2v - 1)$</td>
<td>A, D, E</td>
<td>8.2.1, d*</td>
</tr>
<tr>
<td>$\geq 7$</td>
<td>$b_0 + l \cdot (v-1)$ and $(v-1)/3 \leq k &lt; v-1$</td>
<td>A, D, E</td>
<td>8.3.1, d*</td>
<td></td>
</tr>
<tr>
<td>$4, 5, 6$</td>
<td>$b_0 + l \cdot (v-1)$ and $2 \leq k &lt; v-1$</td>
<td>A, D, E</td>
<td>8.3.1, d*</td>
<td></td>
</tr>
<tr>
<td>$\geq 7$</td>
<td>$b_0 + l \cdot (\sqrt{v+1})/2$</td>
<td>A, D, E</td>
<td>8.3.2, d*</td>
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</tr>
<tr>
<td>$\geq 3$</td>
<td>$b_0 + l \cdot (pv, \ldots, pv, v-1)$ and $p \geq 2$</td>
<td>A, D, E E*</td>
<td>8.3.3, d*</td>
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<tr>
<td>$\geq 3$</td>
<td>$b_0 + l \cdot (pv+q, \ldots, pv+q, v-1)$ and $q \leq v-1$</td>
<td>A, D, E E*</td>
<td>8.3.4, d*</td>
<td></td>
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<tr>
<td>$\geq 3$</td>
<td>$ev+1 \cdot (pv+1, \ldots, pv+1, v-1)$</td>
<td>A, D, E</td>
<td>8.3.5, d*</td>
<td></td>
</tr>
<tr>
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<td>A, D, E</td>
<td>8.3.5, d*</td>
<td></td>
</tr>
<tr>
<td>$\geq 3$</td>
<td>$b_0 \cdot (v, \ldots, v, v-1)$ and $3 \leq k \leq v-1$</td>
<td>A, D, E</td>
<td>8.4.1, d*</td>
<td></td>
</tr>
<tr>
<td>$\geq 3$</td>
<td>$ev \cdot (pv+1, \ldots, pv+1, pv)$</td>
<td>A, D, E</td>
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<td></td>
</tr>
<tr>
<td>$\geq 3$</td>
<td>$b_0 \cdot (pv+q, \ldots, pv+q, pv+q-1)$ and $2 \leq q \leq v-1$</td>
<td>A, D, E</td>
<td>8.4.3, d*</td>
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</tr>
<tr>
<td>$\geq 4$</td>
<td>$b_0 + 2 \cdot (pv+v/2, \ldots, pv+v/2)$</td>
<td>A, D, E</td>
<td>8.5.1, d*</td>
<td></td>
</tr>
<tr>
<td>$\geq 4$</td>
<td>$b_0 + v/2 \cdot (pv+v/2, \ldots, pv+v/2)$</td>
<td>A, D</td>
<td>8.5.2, d*</td>
<td></td>
</tr>
<tr>
<td>$\geq 3$</td>
<td>$b_0 + m \cdot (pv+ \frac{v-1}{m}, \ldots, pv+ \frac{v-1}{m})$ and $2 \leq m \leq v-2$</td>
<td>A, D</td>
<td>8.5.3, d*</td>
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<tr>
<td>$\geq 3$</td>
<td>$ev+1 \cdot (pv+1, \ldots, pv+1)$</td>
<td>A, D, E E(p $\geq v-1$)</td>
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</tr>
<tr>
<td>$\geq 3$</td>
<td>$b_0 - m \geq 2 \cdot (pv+v/m, \ldots, pv+v/m)$ and $3 \leq m \leq v-1$</td>
<td>A, D</td>
<td>8.6.1, d*</td>
<td></td>
</tr>
<tr>
<td>$\geq 3$</td>
<td>$ev-1 \cdot (pv+v-1, \ldots, pv+v-1)$</td>
<td>A, D, E E(p $\geq v-2$)</td>
<td>8.6.2, d*</td>
<td></td>
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</table>
BIBLIOGRAPHY


