ON BAYES AND EMPIRICAL BAYES RULES
FOR SELECTING GOOD POPULATIONS*

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Abstract

This paper deals with the problem of selecting all populations which are close to a control or standard. A general Bayes rule for the above problem is derived. Empirical Bayes rules are derived when the populations are assumed to be uniformly distributed. Under some conditions on the marginal and prior distributions, the rate of convergence of the empirical Bayes risk to the minimum Bayes risk is investigated. The rate of convergence is shown to be $n^{-\delta/3}$ for some $\delta$, $0 < \delta < 2$.

Key words: Bayes rules, empirical Bayes rules, selection procedures, asymptotically optimal, rate of convergence.

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1. Introduction

Empirical Bayes rules have been considered for multiple decision
problems by Deely (1965), Van Ryzin (1970), Van Ryzin and Susarla (1977),
Singh (1977), and Gupta and Hsiao (1983). Most of the papers are concerned
with the selection of the best population where best is usually defined in
terms of the largest or smallest unknown parameter. Gupta and Hsiao
(1983) considered the problem which is concerned with the selection of
populations better than a control. In some practical applications, one may
be interested in selecting populations which are close to a control. We
will consider such a problem in this paper.

In Section 2, we propose a general Bayes rule for selecting good popula-
tions. In Section 3, assuming that the populations are uniformly distributed,
empirical Bayes rules are derived for both the known control parameter and
the unknown control parameter cases. Under some conditions on the marginal
and prior distributions, the rate of convergence of the empirical Bayes
risk to the minimum Bayes risk is investigated. The rate of convergence is
shown to be $n^{-\delta/3}$ for some $\delta$, $0 < \delta < 2$.

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2. A General Bayes Rule for Selecting Good Populations

Let \( \pi_0, \pi_1, \ldots, \pi_k \) be \((k+1)\) independent populations which are characterized by parameters \( \theta_0, \theta_1, \ldots, \theta_k \), respectively. Assume that \( \pi_0 \) is the control population with parameter \( \theta_0 \) which may be known or unknown. When \( \theta_0 \) is unknown, let \( \bar{\theta} = (\theta_0, \theta_1, \ldots, \theta_k) \) and \( \bar{X} = (X_0, X_1, \ldots, X_k) \) where \( X_i \) is an observation from \( \pi_i \), \( i = 0, 1, \ldots, k \). When \( \theta_0 \) is known, no observation from population \( \pi_0 \) is taken, and \( \theta_0, X_0 \) are deleted from \( \bar{\theta} \) and \( \bar{X} \), respectively. When there is no confusion, \( \bar{\theta} \) and \( \bar{X} \) are used to represent either case. We define population \( \pi_i \) to be a good population if \( |\theta_i - \theta_0| < \Delta \) and a bad population if \( |\theta_i - \theta_0| \geq \Delta \), where \( \Delta > 0 \) is a pre-assigned constant. Our goal is to find a Bayes rule which selects all good populations and rejects bad ones. We assume that given \( \theta_i, X_i \) has a probability density function \( f(x_i|\theta_i) \) with respect to a \( \sigma \)-finite measure \( \mu \), for \( i = 0, 1, \ldots, k \), and \( \bar{\theta} \) has a prior distribution

\[
G(\bar{\theta}) = \prod_{i=0}^{k} G_i(\theta_i) \text{ on the parameter space } \Omega.
\]

Let \( G = \{s|s \subseteq \{1, 2, \ldots, k\}\} \) be the action space and let

\[
L(\bar{\theta}, s) = \sum_{i \in s} \{ c_1(\theta_0 - \Delta - \theta_i) I_{\{\theta_i < \theta_0 - \Delta\}}(\theta_i) + c_2(\theta_i - \theta_0 - \Delta) I_{\{\theta_0 - \Delta < \theta_i \}}(\theta_i) + \sum_{i \notin s} \{ c_3(\theta_i - \theta_0 + \Delta) I_{\{\theta_0 - \Delta < \theta_i \}}(\theta_i) + c_4(\theta_0 + \Delta - \theta_i) I_{\{\theta_0 - \Delta > \theta_i\}}(\theta_i) \}
\]

be the loss function defined on \( \Omega \times G \), where \( c_i, i = 1, 2, 3, 4 \) are positive constants and I is the indicator function.

Since the action space is finite, attention can be restricted to the non-randomized rules for deriving the Bayes rules. For a non-randomized decision function \( \delta: \mathcal{X} \to G \), the corresponding Bayes risk with respect to \( G \) is given by

\[
r(G, \delta) = \int_{\mathcal{X}} \int_{\Omega} L(\bar{\theta}, \delta(x)) f(x|\bar{\theta}) dG(\bar{\theta}) d\mu(x),
\]

where \( \mathcal{X} \) is the sample space and \( f(x|\bar{\theta}) = \Pi_{i} f(x_i|\theta_i) \).
In the sequel we consider the special case where \( c_1 = c_2 = c_3 = c_4 \) is a constant which can be taken to be unity without loss of generality. If \( \phi \) is the empty set, (2.1) can be expressed as

\[
(2.3) \quad L(\theta, s) = L(\theta, \phi) + \sum_{i \in S} \left\{ (\theta_0 - \theta_i) I_{\{\theta_i < \theta_0\}}(\theta_i) + (\theta_i - \theta_0 - \Delta) I_{\{\theta_0 < \theta_i\}}(\theta_i) \right\}.
\]

Hence, for any \( \delta \), we have

\[
(2.4) \quad r(G, \delta) - r(G, \phi)
\]

\[
= \int \sum_{i \in \delta(x)} \left\{ \int_{\Omega} \left( (\theta_0 - \theta_i) f(x|\theta) dG(\theta) + 2 \int_{\{\theta_0 < \theta_i\}} (\theta_i - \theta_0) f(x|\theta) dG(\theta) \right) d\mu(x) \right\}.
\]

From (2.4), it follows that if

\[
(2.5) \quad \int_{\Omega} (\theta_0 - \theta_i) f(x|\theta) dG(\theta) + 2 \int_{\{\theta_0 < \theta_i\}} (\theta_i - \theta_0) f(x|\theta) dG(\theta) < 0,
\]

then the Bayes rule is \( \delta_B(x) \), \( i \in \delta_B(x) \).

Let \( m_i(x_i) = \int f(x_i|\theta_i) dG_i(\theta_i) \) be the marginal distribution of \( X_i \), \( \pi(\theta_i|x_i) \) be the posterior distribution of \( \theta_i \) given \( X_i = x_i \), and \( E(\theta_i|x_i) \) be the expected value of \( \theta_i \) given \( X_i = x_i \). If \( m_i(x_i) > 0 \) for all \( x_i \), then (2.5) is equivalent to

\[
(2.6) \quad (\theta_0 - \Delta) - E(\theta_i|x_i) + 2 \int_{\{\theta_0 < \theta_i\}} (\theta_i - \theta_0) \pi(\theta_i|x_i) d\theta_i < 0
\]

if \( \theta_0 \) is known, or

\[
(2.7) \quad E(\theta_0|x_0) - E(\theta_i|x_i) - \Delta + 2 \int_{\{\theta_0 < \theta_i\}} (\theta_i - \theta_0) \pi(\theta_i|x_i) \pi(\theta_0|x_0) d\theta_i d\theta_0 < 0
\]

if \( \theta_0 \) is unknown.

From the above discussion, we have the following main result:

**Theorem 2.1.** Under the loss function (2.3), the Bayes rule \( \delta_B(x) \) with respect to \( G \) is as follows:
(a) If $\theta_0$ is known, then $i \in \delta_B(x)$ if the inequality (2.6) holds.

(b) If $\theta_0$ is unknown, then $i \in \delta_B(x)$ if the inequality (2.7) holds.

Example:

Suppose that

$$f(x_i | \theta_i) = e^{-\theta_i} \theta_i^{x_i} / (x_i!)$$

and $\theta_i$ has a prior distribution $g_i(\theta_i) = G_i^i(\theta_i)$ which is given by

$$g_i(\theta_i) = \beta_i \theta_i^{a_i-1} e^{-\beta_i \theta_i} I(0, \infty)(\theta_i) / \Gamma(a_i),$$

where $\alpha_i > 0$ and $\beta_i > 0$ are known. Then the Bayes rule $\delta_B(x_i)$ is given by

(a) If $\theta_0$ is known, then $i \in \delta_B(x_i)$ if

$$\frac{x_i + \alpha_i}{1 + \beta_i} \{1 - 2I(\theta_0(1 + \beta_i); x_i + \alpha_i)\} - \theta_0 \{1 - 2I(\theta_0(1 + \beta_i); x_i + \alpha_i)\} < \Delta,$$

where

$$I(a; \alpha) = \frac{a^{a-1}}{\Gamma(\alpha)} e^{-a} dx,$$  $a > 0$,  $\alpha > 0$.

(b) If $\theta_0$ is unknown and $\beta_i = \beta$, $i = 0, 1, \ldots, k$, then $i \in \delta_B(x_i)$ if

$$\frac{x_i + \alpha_i}{1 + \beta_i} \{2I(\frac{1}{2}; x_0 + \alpha_0, x_i + \alpha_i) - 1\} +$$

$$\frac{x_0 + \alpha_0}{1 + \beta} \{1 + 2I(\frac{1}{2}; x_0 + \alpha_0, x_i + \alpha_i) - 4I(\frac{1}{2}; x_0 + \alpha_0 + 1, x_i + \alpha_i)\} < \Delta,$$

where

$$I(z; \alpha, \beta) = \int_0^z \frac{1}{B(\alpha, \beta)} x^{\alpha-1} (1-x)^{\beta-1} dx,$$  $\alpha > 0$,  $\beta > 0$,

and

$$B(\alpha, \beta) = \Gamma(\alpha) \Gamma(\beta) / \Gamma(\alpha + \beta).$$
3. **Empirical Bayes Rules for Uniform Populations**

In this section we will assume that $X_i$ has the probability density function
\[ f(x_i | \theta_i) = \frac{1}{\theta_i} I(0, \theta_i)(x_i), \] where $\theta_i > 0$ is unknown. Suppose that $\theta$ has a prior distribution $G(\theta) = \prod_{i} G_i(\theta_i)$ on $\Omega$ and $G_i$ has a continuous positive probability density function $g_i$. Let $m_i(x_i)$ and $M(x_i)$ be the marginal pdf and cdf of $X_i$, respectively.

### 3.1. Known Control Population

When $\theta_0$ is known, we assume $\theta_0 > \Delta$ and let
\[ (3.1) \quad \Delta_i(x_i) = (\theta_0 - \Delta)m_i(x_i) - \int_{x_i}^{\infty} dG_i(\theta_i) + 2\int_{x_i}^{\infty} (\theta_i - \theta_0)f(x_i | \theta_i) dG_i(\theta_i). \]

From (2.5), we have $i \in \delta_B(x)$ if $\Delta_i(x_i) < 0$.

We can show that
\[ (3.2) \quad \Delta_{1, G_i}(x_i) = (\theta_0 - \Delta - x_i)m_i(x_i) + 1 - 2M_i(\theta_0) + M_i(x_i), \quad \text{if} \quad x_i \leq \theta_0. \]
\[ (3.3) \quad \Delta_{2, G_i}(x_i) = (x_i - \theta_0 - \Delta)m_i(x_i) + 1 - M_i(x_i), \quad \text{if} \quad x_i > \theta_0. \]

Therefore
\[ (3.4) \quad \delta_B(x) = \{i | x_i \leq \theta_0, A_{1, G_i}(x_i) < 0\} \cup \{i | x_i > \theta_0, \Delta_{2, G_i}(x_i) < 0\}. \]

**Remarks:**

1. $\Delta_{G_i}(x_i)$ is strictly decreasing for $0 < x_i < \theta_0 - \Delta$, strictly increasing for $\theta_0 - \Delta < x_i < \theta_0 + \Delta$, and strictly decreasing for $\theta_0 + \Delta < x_i$.

2. If $x_i \geq \theta_0 + \Delta$, then $\Delta_{G_i}(x_i) \geq 0$. Hence $i \notin \delta_B(x)$ if $x_i \geq \theta_0 + \Delta$.

3. If $G_i$ is such that $1 - 2M_i(\theta_0) + M_i(\theta_0 - \Delta) > 0$, then $\delta_B(x) = \phi$.

Otherwise, $i \in \delta_B(x)$ if $0 - \Delta < x_i < \theta_0 + \Delta$, for some positive real numbers $d_1$ and $d_2$. Hence a selection rule of this type is a Bayes rule relative to some prior distribution.
If $G$ is unknown, the Bayes rules are not obtainable. In this case, we consider a sequence $(\Delta_1, \Delta_2, \ldots)$ of independent pairs of random vectors where each $\Delta_i$ is distributed as $G$ on $\Omega$ and $\Delta_i = (X_{i1}, \ldots, X_{ik})$ has conditional density function $f(x|\theta)$ given $\Delta_i = \theta$. The empirical Bayes approach, which was introduced by Robbins (1956), attempts to construct a decision rule concerning $\Delta_{n+1}$ at stage $n+1$ based on $X_1, \ldots, X_{n+1}$. The risk at stage $n+1$ by taking action $\delta_n(x; X_1, \ldots, X_n) = \delta_n(z)$ is given by

$$
(3.5) \quad r_n(G, \delta_n) = \mathbb{E}_{\theta} \left[ \sum_{i \in \delta_n(z)} [f(\theta_0 - \Delta - \theta_1) f(x|\theta) dG(\theta) + 2 \int_{(\theta_0, \infty)} (\theta_i - \theta_0) f(x|\theta) dG(\theta)] dx + r(G, \phi),
\right]
$$

where $E_n$ denotes the expectation with respect to the $n$ independent random vectors $X_1, \ldots, X_n$ each with a common density function

$$
m(x) = \int_\Omega f(x|\theta) dG(\theta) = \prod_{i=1}^k m_i(x_i).
$$

**Definition 3.1.** The sequence of procedures $\{\delta_n\}$ is said to be asymptotically optimal (a.o.) relative to $G$ if $r_n(G, \delta_n) - r(G) = o(1)$ as $n \to \infty$, where $r(G) = \inf_{\delta} r(G, \delta)$.

In order to find an a.o. sequence of rules, let

$\delta_{1,B}(x) = \{i | x_i \leq \theta_0 + \Delta_1, x_i < 0\}$ and $\delta_{2,B}(x) = \{i | \theta_0 < x_i < \theta_0 + \Delta_1, x_i < 0\}$. From (3.4) and Remark (2), we have

$\delta_{B}(x) = \delta_{1,B}(x) \cup \delta_{2,B}(x)$. For any $i = 1, 2, \ldots, k$ and $i = 1, 2$, let

$\Delta_{k,i,n}(x_i) = \Delta_{k,i}(x_i; X_{1i}, \ldots, X_{ni}), n = 1, 2, \ldots$ be two sequences of real-valued measurable functions, and define $\delta_n$ by

$$
(3.6) \quad \delta_n(x) = \delta_{1,n}(x) \cup \delta_{2,n}(x),
$$

where

$$
\delta_{1,n}(x) = \{i | x_i \leq \theta_0, \Delta_{1,i,n}(x_i) < 0\}
$$

and

$$
\delta_{2,n}(x) = \{i | \theta_0 < x_i < \theta_0 + \Delta_2, \Delta_{1,i,n}(x_i) < 0\}.
$$
Then we have the following theorem:

**Theorem 3.1.** If \( \int_0^\Theta dG_i(\Theta) < \infty \), \( i = 1, 2, \ldots, k \) and \( \Delta_1, i, n(x_i) \Rightarrow \Delta_1, G_i(x_i) \), for almost all \( x_i \leq \Theta_0 \) and \( \Delta_2, i, n(x_i) \Rightarrow \Delta_2, G_i(x_i) \), for almost all \( \Theta_0 < x_i < \Theta_0 + \Delta \), where \( \Rightarrow \) means convergence in probability, then \( \{ \delta_n(x) \} \) defined by (3.6) is a.o. relative to \( G \).

**Proof.** Analogous to the proof of Theorem 2.1 of Gupta and Hsiao (1983), it can be shown that

\[
0 \leq \int_{\Omega} L(\Theta, \delta_n(x)) f(\Theta \mid x) dG(\Theta) - \int_{\Omega} L(\Theta, \delta_B(x)) f(\Theta \mid x) dG(\Theta) \leq 4 \varepsilon \sum_{i=1}^{k} \left( \frac{1}{j} \right) m_j(x_j)
\]

with probability near 1, for large \( n \). Hence

\[
\int_{\Omega} L(\Theta, \delta_n(x)) f(\Theta \mid x) dG(\Theta) \Rightarrow \int_{\Omega} L(\Theta, \delta_B(x)) f(\Theta \mid x) dG(\Theta)
\]

for almost \( x \). By Corollary 1 of Robbins (1964), \( \{ \delta_n(x) \} \) is a.o. relative to \( G \).

From Theorem 3.1, our problem is reduced to finding consistent estimators of \( \Delta_1, G_i(x_i) \) and \( \Delta_2, G_i(x_i) \). Let

(3.7) \[ M_{i,n}(x_i) = \frac{1}{n} \sum_{j=1}^{n} I(-\infty, x_i] \phi_{j}(x_i), \]

then \( M_{i,n}(x_i) \Rightarrow M_i(x_i) \) for all \( x_i > 0 \). Next, let \( \varphi(x) \geq 0 \) be a Borel function satisfying the following conditions:

(3.8) (i) \( \sup_{-\infty < x < \infty} \varphi(x) < \infty \), (ii) \( \int_{-\infty}^{\infty} \varphi(x) dx = 1 \), and (iii) \( \lim_{x \to \infty} x \varphi(x) = 0 \).

Let \( \{ h(n) \} \) be a sequence of positive constants satisfying the following conditions:

(3.9) (i) \( h(n) \to 0 \) as \( n \to \infty \) and (ii) \( nh(n) \to \infty \) as \( n \to \infty \).

If we define

(3.10) \[ m_{i,n}(x) = \frac{1}{nh(n)} \sum_{j=1}^{n} \varphi \left( \frac{x - x_{ji}}{h(n)} \right), \]
then $m_{i\in}(x) \overset{P}{\rightarrow} m_i(x)$ for all $x$ (see Parzen (1962)). For $i = 1, 2, \ldots, k$, let

(3.11) $\Delta_{1,i,n}(x_i) = (\theta_0 - \Delta - x_i) m_{i\in}(x_i) + 1 - 2M_m(\theta_0) + M_i(x_i)$

and

(3.12) $\Delta_{2,i,n}(x_i) = (x_i - \theta_0 - \Delta) m_i(x_i) + 1 - M_i(x_i)$.

Then

$\Delta_{1,i,n}(x_i) \overset{P}{\rightarrow} \Delta_{1,G_i}(x_i)$ for all $x_i \leq \theta_0$

and

$\Delta_{2,i,n}(x_i) \overset{P}{\rightarrow} \Delta_{2,G_i}(x_i)$ for all $\theta_0 < x_i < \theta_0 + \Delta$.

Thus the sequence of procedures $\{\delta_n\}$ defined by

$\delta_n(x) = \{i|x_i \leq \theta_0, \Delta_{1,i,n}(x_i) < 0\} \cup \{i|\theta_0 < x_i < \theta_0 + \Delta, \Delta_{2,i,n}(x_i) < 0\}$,

is a.o. relative to $G$.

3.2. Unknown Control Population

In this subsection we consider the case of the unknown parameter $\theta_0$ of the control population $\pi_0$. As indicated in Section 2, the notations $\theta$, $\omega$, $x$, $x_*$, $G(\omega)$ and $f(x|\omega)$ should be interpreted accordingly. For example, the observation at stage $n$ is denoted by $x_n = (x_{n0}, x_{n1}, \ldots, x_{nk})$. Under the loss function (2.3), the Bayes rule $\delta_B(x)$ is given as follows:

It can be shown that

(3.13) $\Delta_{G_0,G_i}(x_0,x_i) = m_i(x_i)(1-M_0(x_0)) + (1 + M_i(x_i)) m_0(x_0) +$

$\int_0^{\theta_0} m_i(x_i) m_0(x_0) \frac{M_i(\theta_0)}{\theta_0} dG_0(\theta_0)$

$= \Delta_{1,G_0,G_i}(x_0,x_i)$ (say), if $0 < x_i \leq x_0$

and
\[ (3.14) \quad \Delta G_0, G_i(x_0, x_i) = (1 - M_i(x_i))m_0(x_0) + (1 + M_0(x_0) - 2M_0(x_i))m_i(x_i) + \\
\quad (x_i - x_0 - \Delta)m_i(x_i)m_0(x_0) + 2M_i(x_i)m_0(x_i) - \int_{x_i}^{\infty} \frac{M_i(\theta_0)}{\theta_0} dG_0(\theta_0) \\
= \Delta_2, G_0, G_i(x_0, x_i) \text{ (say), if } 0 < x_0 < x_i. \]

Thus
\[ (3.15) \quad \delta_B(x) = \delta_{1, B}(x) \cup \delta_{2, B}(x) \]

where
\[ \delta_{1, B}(x) = \{i | 0 < x_i \leq x_0, \Delta_1, G_0, G_i(x_0, x_i) < 0\} \]

and
\[ \delta_{2, B}(x) = \{i | 0 < x_0 < x_i, \Delta_2, G_0, G_i(x_0, x_i) < 0\}. \]

Similar to Theorem 3.1, we have the following result.

**Theorem 3.2.** If \( \int_0^\infty \theta dG_i(\theta) < \infty, i = 0, 1, \ldots, k \) and for all \( 1 \leq i \leq k, \)

\[ \Delta_1, i, n(x_0, x_i) \xrightarrow{p} \Delta_1, G_0, G_i(x_0, x_i) \text{ for } x_i \leq x_0 \text{ and} \]

\[ \Delta_2, i, n(x_0, x_i) \xrightarrow{p} \Delta_2, G_0, G_i(x_0, x_i) \text{ for } x_0 < x_i, \]

the sequence \( \{\delta_n\} \) defined by

\[ \delta_n(x) = \{i | x_i \leq x_0, \Delta_1, i, n(x_0, x_i) < 0\} \cup \{i | x_0 < x_i, \Delta_2, i, n(x_0, x_i) < 0\}, \]

is a.o. relative to \( G. \)

Now our problem is to find a consistent estimator of

\[ \int_a^\infty \frac{M_i(\theta_0)}{\theta_0} dG_0(\theta_0) \text{ for } x_0 \leq a. \]

**Theorem 3.3.** Let \( M_{in}(x) \) and \( m_{in}(x) \) be defined by (3.7) and (3.10), respectively. Then

\[ \int_a^\infty \frac{M_{in}(\theta_0)}{\theta_0} dG_{in}(\theta_0) \xrightarrow{p} \int_a^\infty \frac{M_i(\theta_0)}{\theta_0} dG_0(\theta_0) \text{ for } x_0 \leq a, \]

where \( G_{in}(\theta_0) = M_{in}(\theta_0) - \theta_0 m_{in}(\theta_0). \)
Proof. \[
\int_a^\infty \frac{m_{i\cdot n}(\theta_0)}{\theta_0} \, dG_{0\cdot n}(\theta_0) - \int_a^\infty \frac{m_i(\theta_0)}{\theta_0} \, dG_{0\cdot n}(\theta_0) \]
\[
\leq \int_a^\infty \frac{|m_{i\cdot n}(\theta_0) - m_i(\theta_0)|}{\theta_0} \, dG_{0\cdot n}(\theta_0) \]
\[
\leq \frac{1}{a} \sup_{-\infty < \theta_0 < \infty} |m_{i\cdot n}(x) - m_i(x)| \leq \epsilon
\]
with probability near 1, for large n, by Glivenko-Cantelli Theorem. Since \(M_i(\theta_0)\) is bounded continuous and \(G_{0\cdot n}(\theta_0) \overset{P}{\rightarrow} G_0(\theta_0)\), we have
\[
\int_a^\infty \frac{m_i(\theta_0)}{\theta_0} \, dG_{0\cdot n}(\theta_0) \overset{P}{\rightarrow} \int_a^\infty \frac{m_i(\theta_0)}{\theta_0} \, dG_0(\theta_0).
\]

Thus
\[
\int_a^\infty \frac{m_{i\cdot n}(\theta_0)}{\theta_0} \, dG_{0\cdot n}(\theta_0) - \int_a^\infty \frac{m_i(\theta_0)}{\theta_0} \, dG_{0\cdot n}(\theta_0) \]
\[
\leq \int_a^\infty \frac{m_{i\cdot n}(\theta_0)}{\theta_0} \, dG_{0\cdot n}(\theta_0) - \int_a^\infty \frac{m_i(\theta_0)}{\theta_0} \, dG_{0\cdot n}(\theta_0) \]
\[
+ \int_a^\infty \frac{m_i(\theta_0)}{\theta_0} \, dG_{0\cdot n}(\theta_0) - \int_a^\infty \frac{m_i(\theta_0)}{\theta_0} \, dG_0(\theta_0) \]
\[
\leq \epsilon \quad \text{with probability near 1, for large n.}
\]

From Theorem 3.3, if we define
\[
\Delta_{1,i,n}(x_0,x_i) = m_{i\cdot n}(x_i)(1-M_{0\cdot n}(x_0)) + m_{0\cdot n}(x_0)(1+M_{i\cdot n}(x_i))
\]
\[
+ (x_0 - x_i - \Delta)m_{i\cdot n}(x_i)m_{0\cdot n}(x_0) - 2\int_{x_0}^\infty \frac{m_{i\cdot n}(\theta_0)}{\theta_0} \, dG_{0\cdot n}(\theta_0),
\]

and
\[
\Delta_{2,i,n}(x_0,x_i) = m_{0\cdot n}(x_0)(1-M_{i\cdot n}(x_i)) + m_{i\cdot n}(x_i)(1+M_{0\cdot n}(x_0)-2M_{0\cdot n}(x_i))
\]
\[
+ (x_i - x_0 - \Delta)m_{0\cdot n}(x_0)m_{i\cdot n}(x_i) + 2m_{i\cdot n}(x_i)m_{0\cdot n}(x_i) - 2\int_{x_i}^\infty \frac{m_{i\cdot n}(\theta_0)}{\theta_0} \, dG_{0\cdot n}(\theta_0),
\]
then we have
\[
\Delta_{x,i,n}(x_0,x_i) \overset{P}{\rightarrow} \Delta_{x,G_0,G_i}(x_0,x_i), \quad x = 1,2.
\]
Therefore, the sequence of rules \( \delta_n \) defined by
\[
\delta_n(x) = \{i \mid x_i \leq x_0, \Delta_1, i, n(x_0, x_i) < 0\} \cup \{i \mid x_0 < x_i, \Delta_2, i, n(x_0, x_i) < 0\},
\]
is a.o. relative to \( G \) by Theorem 3.2.

3.3. Rate of Convergence of the Empirical Bayes Rules

In this section we will consider the rate of convergence of the empirical Bayes rule derived in Section 3.1.

**Definition 3.2.** The sequence of procedures \( \delta_n \) is said to be asymptotically optimal of order \( \alpha_n \) relative to \( G \) if \( r_n(G, \delta_n) - r(G) = O(\alpha_n) \) as \( n \to \infty \), where
\[
\lim_{n \to \infty} \alpha_n = 0.
\]

The main result (Theorem 3.8) of this section is based on a series of lemmas.

**Lemma 3.4.** Let \( \Delta_1, G_1(x_i), \Delta_2, G_1(x_i), \Delta_1, i, n(x_i) \) and \( \Delta_2, i, n(x_i) \) be defined by (3.2), (3.3), (3.11) and (3.12), respectively. Then we have
\[
0 \leq r_n(G, \delta_n) - r(G)
\]
\[
\leq \sum_{i=1}^k \theta_0 \int_{\Delta_1, i, n(x_i) - \Delta_1, G_i(x_i)} |\delta| dx_i + \sum_{i=1}^k \theta_0^{\Delta} \int_{\Delta_2, i, n(x_i) - \Delta_2, G_i(x_i)} |\delta| dx_i, \quad \theta > 0.
\]

**Proof.** The proof is similar to that of Lemma 3 of Van Ryzin and Susarla (1977) and hence omitted.
Lemma 3.5. Assume that \( h(n) \) satisfies the condition (3.9) and that \( \varphi(x) \) satisfies the following:

(i) \( \varphi(x) = 0 \) if \( x \notin (0,a) \) for some finite \( a > 0 \)

(ii) \( \int_0^a \varphi(x) dx = 1 \)

(iii) \( \sup_x |\varphi(x)| < \infty \).

Then, for \( m_{in}(x_i) \) defined by (3.10), we have

\[
|E m_{in}(x_i) - m_i(x_i)| \leq h(n) f_{\varepsilon}(x_i) \int_0^a |u \varphi(u)| du
\]

for large \( n \) where \( f_{\varepsilon}(x_i) = \sup_{0 \leq y \leq \varepsilon} |m_i'(x_i + y)|, \varepsilon > 0 \).

Proof. \( E m_{in}(x_i) - m_i(x_i) \)

\[
= \frac{1}{h(n)} \int \varphi \left( \frac{y-x_i}{h(n)} \right) m_i(y) dy - m_i(x_i)
\]

\[
= \int_0^a \varphi(u)[m_i(x_i + uh(n)) - m_i(x_i)] du
\]

\[
= \int_0^a \varphi(u)[uh(n)m_i'(x_i + \eta_n(x_i,u))] du
\]

where \( 0 < \eta_n(x_i,u) < uh(n) \).

For \( \varepsilon > 0 \), let \( n \) be large enough so that \( ah(n) \leq \varepsilon \), then

\[
|E m_{in}(x_i) - m_i(x_i)| \leq h(n) f_{\varepsilon}(x_i) \int_0^a |u \varphi(u)| du.
\]

Lemma 3.6. Under the assumptions of Lemma 3.5, we have

\[
\text{Var} m_{in}(x_i) \leq \frac{1}{nh(n)} \int_0^a \varphi^2(u) du
\]

Proof. \( \text{Var} m_{in}(x_i) = \text{Var} \left( \frac{1}{nh(n)} \sum_{j=1}^n \varphi \left( \frac{x_{ij} - x_i}{h(n)} \right) \right) \)

\[
\leq \frac{1}{nh(n)} \int_0^a \varphi^2(u)m_i(x_i + uh(n)) du
\]
\[
\leq \frac{1}{nh(n)} \int_0^a \varphi^2(u) \, du, \text{ since } m_i(x_i) \text{ is non-increasing.}
\]

Remark: From Lemma 3.5 and Lemma 3.6, we have

\[m_{i,n}(x_i) \leq m_1(x_i) \text{ if } f_{e}(x_i) < \infty.\]

Lemma 3.7. Under the assumptions of Lemma 3.5, we have

(a) \( \text{Var} \Delta_{1,i,n}(x_i) \leq M(\theta_0 - \Delta - x_i)^2 m_i(x_i)(nh(n))^{-1} + \frac{5}{n} \),

(b) \( \text{Var} \Delta_{2,i,n}(x_i) \leq M(x_i - \theta_0 - \Delta)^2 m_i(x_i)(nh(n))^{-1} + \frac{1}{2n} \),

where \( M = 2 \int_0^a \varphi^2(u) \, du \).

Proof. Since \( \text{Var}(X+Y) \leq 2(\text{Var}(X) + \text{Var}(Y)) \) for any random variables \( X \) and \( Y \), it follows from Lemma 3.6 that

\[
\text{Var} \Delta_{1,i,n}(x_i) \leq 2((\theta_0 - \Delta - x_i)^2 \text{Var} m_{i,n}(x_i) + \text{Var}(M_{i,n}(x_i) - 2M_{i,n}(\theta_0)))
\]

\[
\leq 2((\theta_0 - \Delta - x_i)^2 m_i(x_i)(nh(n))^{-1} + \int_0^a \varphi^2(u) \, du + \frac{5}{2n}),
\]

which proves (a). Similarly we have the result (b).

Theorem 3.8. Assume the conditions of Lemma 3.5 and the following for \( 0 < \delta < 2; \)

(i) \( \int_0^{\theta_{0} + \Delta} |\Delta_{1,G_i}(x_i)|^{1-\delta} dx_i < \infty \) and \( \int_0^{\theta_{0} + \Delta} |\Delta_{1,G_i}(x_i)|^{1-\delta} |\theta_0 - \Delta - x_i|^\delta m_i^{\delta/2}(x_i) dx_i < \infty \),

(ii) \( \int_{\theta_0}^{\theta_{0} + \Delta} |\Delta_{2,G_i}(x_i)|^{1-\delta} dx_i < \infty \) and \( \int_{\theta_0}^{\theta_{0} + \Delta} |\Delta_{2,G_i}(x_i)|^{1-\delta} |x_i - \theta_0 - \Delta|^\delta m_i^{\delta/2}(x_i) dx_i < \infty \),

(iii) \( \int_0^{\theta_0} |\Delta_{1,G_i}(x_i)|^{1-\delta} |\theta_0 - \Delta - x_i|^\delta \varphi^2(x_i) dx_i < \infty \).
\[ \int_{0}^{\theta_0+\Delta} |x_i - \theta_0 - \Delta|^{\delta} f^{\delta}(x_i) \, dx_i < \infty. \]

Then we have

\[ r_n(G, \delta_n) - r(G) = O(\max((1/\delta_n)^{\delta/2}, (\delta_n)^{\delta})) \text{ as } n \to \infty. \]

Proof. For \(0 < \delta < 2\), by Hölder inequality and Lemma 3.4, we have

\[ 0 \leq r_n(G, \delta_n) - r(G) \]

\[ \leq \sum_{i=1}^{k} (\max(1,2^{\delta-1})\int_{0}^{\theta_0} |\Delta_1, G_i(x_i)|^{1-\delta} (\text{Var} \; \Delta_1, i, n(x_i))^{\delta/2} \, dx_i + \]

\[ \int_{0}^{\theta_0} |\Delta_1, G_i(x_i)|^{1-\delta} |(\theta_0 - \Delta - x_i)(\text{Em}_{i, n}(x_i) - m_i(x_i))|^{\delta} \, dx_i |) + \]

\[ + \sum_{i=1}^{k} (\max(1,2^{\delta-1})\int_{0}^{\theta_0+\Delta} |\Delta_2, G_i(x_i)|^{1-\delta} (\text{Var} \; \Delta_2, i, n(x_i))^{\delta/2} \, dx_i + \]

\[ \int_{0}^{\theta_0+\Delta} |\Delta_2, G_i(x_i)|^{1-\delta} |(x_i - \theta_0 - \Delta)(\text{Em}_{i, n}(x_i) - m_i(x_i))|^{\delta} \, dx_i |). \]

Since \((a+b)^{\delta/2} \leq a^{\delta/2} + b^{\delta/2}\) for \(a > 0, b > 0\) and \(0 < \delta < 2\), it follows from Lemma 3.7, (i) and (ii) that

\[ \int_{0}^{\theta_0} |\Delta_1, G_i(x_i)|^{1-\delta} (\text{Var} \; \Delta_1, i, n(x_i))^{\delta/2} \, dx_i = O((\delta_n)^{-\delta/2}) \]

and

\[ \int_{0}^{\theta_0+\Delta} |\Delta_2, G_i(x_i)|^{1-\delta} (\text{Var} \; \Delta_2, i, n(x_i))^{\delta/2} \, dx_i = O((\delta_n)^{-\delta/2}). \]

By Lemma 3.5 (iii) and (iv),

\[ \int_{0}^{\theta_0} |\Delta_1, G_i(x_i)|^{1-\delta} |\theta_0 - \Delta - x_i|^{\delta} |E \; m_{i, n}(x_i) - m_i(x_i)|^{\delta} \, dx_i = O((\delta_n)^{\delta}) \]

and
\[
\int_{\theta_0}^{\theta_0 + \Delta} |\Delta_2, G_i(x_i)|^{1-\delta} |(x_i - \theta_0 - \Delta)(E m_i n(x_i) - m_i(x_i))|^\delta dx_i = O((h(n))^{\delta}).
\]

Hence

\[
r_n(G, \delta_n) - r(G) = O(\max((nh(n))^{-\delta/2}, (h(n))^{\delta})) \text{ as } n \to \infty.
\]

Corollary 3.9. Assume the conditions of Theorem 3.8. If we take \( h(n) = n^{-\alpha} \), \( 0 < \alpha < 1 \), then the optimal choice of \( \alpha \) is 1/3 and \( r_n(G, \delta_n) - r(G) = O(n^{-\delta/3}) \) as \( n \to \infty \).

Remark: If the prior distribution \( G_i \) is such that both \( g_i(x)/x \) and \( m_i(x) \) are bounded on \( (0, \theta_0 + \Delta + \varepsilon) \), it is easy to check that the conditions of Theorem 3.8 are satisfied for \( 0 < \delta \leq 1 \).

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