Restricted Risk Bayes Estimation for the Mean of the Multivariate Normal Distribution

by

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Abstract

Let \( X = (X_1, \ldots, X_p)^t \) to be an observation from a \( p \)-variate normal distribution with unknown mean vector \( \theta = (\theta_1, \ldots, \theta_p)^t \) and known covariance matrix \( \Sigma \). It is desired to estimate \( \theta \) under the quadratic loss

\[
L(\hat{\theta}, \theta) = (\theta - \hat{\theta})^t Q (\theta - \hat{\theta}).
\]

Suppose prior beliefs concerning \( \theta \) can be approximately modeled by a conjugate prior distribution \( \pi \) which is \( N_p(\mu, A) \), where \( \mu, A \) are known. We find estimators of \( \theta \) which have small Bayes risk and which also satisfy the constraint \( R(\hat{\theta}, \delta) \leq \text{tr}(Q\delta) + c \), \( R(\theta, \delta) \) being the frequentist risk of \( \delta \). Such estimators are good from both the frequentist and Bayesian perspectives.

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Key Words: Risk; Bayes risk; Relative saving risk.
1. Introduction

Let $X = (X_1, \ldots, X_p)^t$ be a $p$-variate normal distribution with known covariance matrix $\Sigma$ and unknown mean vector $\theta = (\theta_1, \ldots, \theta_p)$. It is desired to estimate $\theta$ under quadratic loss

$$L(\theta, \delta) = (\theta - \delta)^t Q (\theta - \delta),$$

where $Q$ is a given positive definite matrix. An estimator will be evaluated by its risk function

$$R(\theta, \delta) = E_{\theta}[L(\theta, \delta(x))].$$

When one has complete knowledge of the prior distribution $\pi$ of $\theta$, one would use a Bayes estimator $\delta^\pi$ which is an estimator minimizing the Bayes risk, i.e., $r(\pi, \delta^\pi) = \min_{\delta} r(\pi, \delta)$, where

$$r(\pi, \delta) = E_{\pi}[R(\theta, \delta)].$$

Unfortunately, the determination of $\pi$ is often very inexact. In a finite amount of time, only subjective approximations to $\pi$ can be constructed and the Bayes estimator can be sensitive to uncertain parts of the prior specification. (See Berger (1980c, 1983) for references.)

On the other hand, the minimax principle tries to protect against the worst possible state of nature. Although a minimax estimator is the "most robust" Bayesian decision rule (in the sense of minimizing $\sup_{\pi} r(\pi, \delta)$), it
often behaves poorly when looked at from the viewpoint of average risk, and can often be inadmissible (c.f. Berger (1980a, c, 1983)).

The situation occurring in practice frequently lies between these two extreme cases. Often, we have some idea about the prior distribution of \( \theta \) but we are not willing to entirely rely on any specific prior. Hence it is important to consider the robustness (with respect to the specification of \( \pi \)) of the estimator selected. Thus, a reasonable goal is to develop a procedure which can incorporate prior knowledge, but is safe with respect to error in the specification of prior knowledge.

One way to do this problem is as follows: First, we restrict the risk; for example

\[
(1.1) \quad R(\theta, \delta) - R(\theta, \delta^0) \leq c
\]

for all \( \theta \in \mathbb{R}^p \), where \( \delta^0(x) = x \) is the MLE and minimax estimator of \( \theta \) and \( c \) is a given nonnegative constant. (Note that \( R(\theta, \delta^0) = \text{tr}(Q \delta) \).) And then, in the class of estimators satisfying the restricted condition (1.1) find an estimator which has minimum Bayes risk. Thus, we impose a constraint on the deviation of the risk of our estimator from the minimax estimator and then, subject to this constraint, we minimize the Bayes risk.

This problem was first proposed by Hodges and Lehmann (1952) and has been considered for various situations in Efron and Morris (1971), Shapiro (1972, 1975), Bickel (1980), Marazzi (1982), and Berger (1982a). It is known that exact mathematical solution of this problem is very messy and, even in the case of a spherically symmetric normal distribution, numerical solution
is very difficult (c.f. Berger (1982a), Marazzi (1982)). For this reason, we will consider various simplifications of the problem.

By using the identity (essentially a variant of Stein's "unbiased estimator of risk")

\[ R(\theta, \delta) - R(\theta, \delta^0) = E_{\theta}[\mathcal{D}\gamma(x)], \]

where \( \gamma(x) = \frac{1}{\sigma^2}(\delta(x)-x) \) and \( \mathcal{D}\gamma(x) \) is an expression involving partial derivatives of \( \gamma(x) \), the condition (1.1) will clearly be satisfied if

\[ \mathcal{D}\gamma(x) \leq c. \]  

Hence we can formulate the following approximate restricted risk Bayes problem:

Select the estimator \( \delta \) which minimizes \( r(\pi, \delta) \) subject to (1.2).

When \( Q \) is a diagonal matrix (always achievable by a linear transformation), Chen (1983) generalizes an idea of Berger (1982a) to show that the optimal estimator in the approximate restricted risk Bayes problem must be a smooth blending of the Bayes estimator \( \delta^\pi \) and estimator arising from the differential equation \( \mathcal{D}\gamma(x) = c \). Unfortunately, in the nonsymmetric multi-dimensional case it is generally impossible to solve the differential equation \( \mathcal{D}\gamma(x) = c \) in closed form. Numerical solution would be a possibility, but in the interest of providing reasonably accessible and understandable estimators we further specialize the problem by considering a particular class of estimators for
which solution of the differential equality is possible. We also restrict
collection to the case where the "approximate" prior that is specified
is \( N_p(\mu, A) \), \( \mu \) being the prior mean vector and \( A \) the positive definite prior
covariance matrix.

A natural class of estimators to consider is the class of "compromise"
estimators

\[
(1.3) \quad \delta_\rho(x) = \rho(r) \delta^\pi(x) + (1 - \rho(r)) \delta^0(x) \\
= x - \rho(r) \frac{1}{\sqrt{\frac{1}{r} + A^{-1}}} (x - \mu),
\]

where \( r = (x - \mu)^t (\frac{1}{r} + A)^{-1} (x - \mu) \), and \( \rho \) is a continuous and piecewise differentiable
function taking values in \([0, 1]\). Small \( r \) support the validity of the prior
information (if \( x \sim N_p(\theta, \frac{1}{r}) \) and \( \theta \sim N_p(\mu, A) \), then \( r \sim \chi^2_p \)), and hence suggest
use of \( \delta^\pi(x) = x - \frac{1}{\sqrt{\frac{1}{r} + A^{-1}}} (x - \mu) \) (so \( \rho(r) \) should then be near one), while large
\( r \) suggests that prior information is implausible and that \( \delta^0(x) = x \) should be
used (i.e., \( \rho(r) \) should then be near zero). A "robust Bayes" estimator in this
class was proposed by Berger (1980b). We will find the optimal choice of \( \rho \)
in Section 2 (for the approximate restricted risk problem), a choice which
can offer substantial improvement over the robust Bayes estimator of Berger
(1980b).

In Berger (1982b) a different type of estimator was proposed, one that
incorporated the prior information \( \mu \) and \( A \) and yet was guaranteed to be minimax.
The estimator was based on a "decomposition to subproblems" technique proposed
in Bhattacharya (1966). In Section 3, we modify this estimator by basing it
on the optimal "compromise" estimators of Section 2. The resulting estimator appears to have substantially better performance than the optimal compromise estimator, in the sense of having smaller Bayes risk, while still satisfying the constraint (1.1).

If minimaxity is desired (i.e. \( c = 0 \) in (1.1)), the estimators discussed above have certain limitations (see Section 4) in very nonsymmetric situation. A new estimator, called a "weighted" minimax estimator is proposed in Section 4, and is shown to offer substantial improvement in Bayes risk (while preserving minimaxity) in certain cases.

Section 5 explicitly discusses the comparative performance of the "compromise", "subproblem", and "weighted minimax" estimators for the restricted risk Bayes problem. The overall conclusion is that last two of these estimators are superior, and guidelines are presented concerning the use of each of them.

Instead of using \( \gamma(\pi, \delta) \) to measure the effectiveness of an estimator \( \delta \), it is convenient to use the linearly transformed relative savings risk of Efron and Morris (1972), defined by

\[
\text{RSR}(\pi, \delta) = \frac{\gamma(\pi, \delta) - \gamma(\pi, \delta^\pi)}{\gamma(\pi, \delta_0^\pi) - \gamma(\pi, \delta^\pi)}. 
\]

This measures the proportion of the potential Bayesian improvement over \( \delta_0^\pi \) which is attained by the estimator \( \delta \). Small RSR is desirable from a Bayesian viewpoint.
2. The Optimal Compromise Estimator

Let \( X \) be a \( N_p(\theta, \Sigma) \) and suppose \( \theta \) has (approximately) the conjugate prior distribution \( \pi \) which is \( N_p(\mu, A) \), where \( \mu \) and \( A \) are known. Let 
\[ d_1 \geq d_2 \geq \ldots \geq d_p > 0 \]
be the characteristic roots of the matrix
\[ D = (\Sigma + A)^{-1/2} \Sigma Q (\Sigma + A)^{-1/2} \]
and let 
\[ \tau = \sum_{i=1}^{p} \frac{d_i}{d_1}. \]
We will consider the class of compromise estimators \( \delta_\rho \) defined in (1.3) with \( E^r(\rho^2(r)) < \infty \), and \( E^r(|r_\rho'(r)|) < \infty \).

First, we need the following Stein identity. (c.f. Stein (1981), Hudson (1978), Berger (1982), and Chen (1983)). Let

\[ \delta(x) = x + \frac{1}{\tau} \gamma(x), \]

where \( \gamma_i \) is a continuous and piecewise differentiable function with 
\[ E^X[\gamma_i^2(x)] < \infty \]
and 
\[ E^X[|\gamma_i^{(i)}(x)|] < \infty, \]
where \( \gamma_i^{(i)}(x) = \frac{\partial}{\partial x_i} \gamma_i(x) \) and so on. Then

\[ R(\theta, \delta) - R(\theta, \delta^0) = E_\theta[\mathcal{L}\gamma(x)], \]

where

\[ \mathcal{L}\gamma(x) = \gamma^t(x) \Sigma Q \gamma(x) + 2 \text{tr}(V \Sigma Q) \]

and \( V = (V_{ij}) \) with 
\[ V_{ij} = \frac{\partial \gamma_i}{\partial x_j}. \]
(here trA = trace of A).
Using this identity, we will give a sufficient condition for the compromise estimator $\delta_\rho$ to satisfy the constraint (1.1).

Lemma 1. If $\rho^2(r) - 4\rho'(r) \geq 0$ for all $r > 0$, then $R(\theta, \delta_\rho) - R(\theta, \delta^0) \leq C$ for all $\theta$, provided that $\phi_\rho(r) \leq c/d_1$ for all $r > 0$, where

$$\phi_\rho(r) = [\rho^2(r) - 4\rho'(r)] r - 2\tau \rho(r).$$

Proof. Given in the Appendix. ||

Let $D_c$ be the class of compromise estimators satisfying the restricted condition $\phi_\rho(r) \leq c/d_1$ for all $r > 0$, where $\phi_\rho(r)$ is given in (2.5). Chen (1983) showed that the optimal estimator, $\delta_{\rho_c}$, in the class $D_c$ is

$$\delta_{\rho_c}(x) = x - \min(1, \rho_{c,0}(r)) \frac{1}{(\frac{1}{2} + \frac{\tau}{2})} (x - \mu),$$

where

$$\rho_{c,0}(r) = \begin{cases} \frac{2(\tau-2)^+}{r} & \text{if } c = 0 \text{ and } \tau > 2 \\ \frac{c}{2d_1 t} \frac{K_{v+1}(t)}{K_v(t)} & \text{if } c > 0 \text{ and } \tau \geq 2 \\ \frac{c}{2d_1 t} \frac{K_{v-1}(t)}{K_v(t)} & \text{if } c > 0 \text{ and } 1 \leq \tau < 2, \end{cases}$$

$v = |\tau-2|/2$, $t = \frac{1}{2} \sqrt{cr/d_1}$ and $K_v$ is the second kind of modified Bessel function of order $v$. 
The RSR of $\delta_{\rho_c}$ is given in the following theorem.

**Theorem 1.** For the estimator $\delta_{\rho_c}$ defined as in (2.6),

$$\text{RSR}(\pi, \delta_{\rho_c}) = 1 - \frac{2b}{p} f_p(b) + \frac{c}{pd_1} (1 - \psi_p(b)) - \psi_p(b)$$

$$- 2\left[1 - \frac{p}{i=1} \frac{d_i}{pd_1} \right] \int_b^\infty \rho_{c,0}(r) f_p(r) dr,$$

where $b$ is the unique positive solution of the equation $\rho_{c,0}(r) = 1$, and $f_p$ and $\psi_p$ are the density function and cumulative distribution function of the chi-square distribution with $p$ degrees of freedom, respectively.

**Proof.** Given in the Appendix. ||

When $c = 0$, the RSR of $\delta_{\rho_0}$ can be expressed explicitly as follows.

**Corollary 1.** When $c = 0$ and $\tau > 2$,

$$\text{RSR}(\pi, \delta_{\rho_0}) = \frac{1}{p} \left\{ [p-4(\tau-2) + \frac{4(\tau-2)^2}{p-2}][1-\psi_p(2(\tau-2))] 
$$

$$+ 4(\tau-2)[1 - \frac{2(\tau-2)}{p-2}] f_p(2(\tau-2)) \right\}.$$

When $\tau = p$ (i.e., $(\frac{\tau}{2}+A)^{-1/2}(\frac{\tau}{2}Q\frac{\tau}{2}) (\frac{\tau}{2}+A)^{-1/2} = I_p$), then

$$\text{RSR}(\pi, \delta_{\rho_0}) = 1 - \psi_p(2(p-2)) - \frac{4(p-2)}{p} f_p(2(p-2)).$$
Proof. It follows easily from Theorem 1. ||

Remark 1. \( T_{p,p}^0 \leq \text{RSR}(\pi, \delta_0) \leq 1 - \frac{r-2}{p-2} (1-T_{p,p}^0), \)

where

\[
T_{p,p}^0 = \frac{1}{p} \int_{2(p-2)}^{\infty} r(1 - \frac{2(p-2)}{r})^2 f_p(r) dr.
\]

(see Table 4 for values of \( T_{p,p}^0 \))

Remark 2. For fixed \( p \), \( \text{RSR}(\pi, \delta_0) \) is a decreasing function of \( r \) and has minimum value \( T_{p,p}^0 \) when \( r = p \).

3. The Modified Subproblem Estimator

In this section, we use the decomposition to subproblems approach of Bhattacharya (1966) and Berger (1979) to develop an alternative estimator, \( \delta_{MB,c} \), satisfying the constraint (1.1) and yet performance better than \( \delta_{p_c} \). For simplicity of notation in the remainder of the paper, we will assume that \( Q, \frac{1}{2}, \) and \( A \) are diagonal matrix, with diagonal elements \( q_i, \sigma_i^2, \) and \( a_i, \) respectively. (For similar results in the nondiagonal case see Chen (1983).)

Suppose (without loss of generality) that \( x_i \) are index so that \( d_1 \geq d_2 \geq \ldots \geq d_p \), where \( d_i = q_i \sigma_i^2/(c_i^2 + a_i) \) and let \( d_{p+1} = 0 \). Then define the \( i \)-th component of \( \delta_{MB,c} \) as

\[
\delta_{1i}^{MB,c}(x) = x_i - \frac{\sigma_i^2}{\sigma_i^2 + a_i} (x_i - u_i) \bigg\lceil \sum_{j=1}^{p} \frac{d_j - d_{j+1}}{d_i} \min(1, \rho_c(j)(r_j)),
\]
where

\[
(3.1) \quad \rho_{c,0}^{(j)}(r_j) = \begin{cases} 
\frac{2(j-2)^+}{r_j} & \text{when } c = 0 \\
\frac{c}{2d_1t_j} \frac{K_{V_j+1}(t_j)}{K_{V_j}(t_j)} & \text{when } c > 0 \\
\frac{c}{2d_1t_j} \frac{K_{V_j-1}(t_j)}{K_{V_j}(t_j)} & \text{when } c > 0 \text{ and } j = 1 \\
\frac{c}{2d_1t_j} \frac{K_{V_j-1}(t_j)}{K_{V_j}(t_j)} & \text{when } c > 0 \text{ and } j \geq 2
\end{cases}
\]

\[v_j = \sum_{i=1}^{j} \frac{(x_i - \mu_i)^2}{\sigma_i^2 + a_i}, \quad t_j = \frac{1}{2} \sqrt{\frac{cr_j}{d_1}}, \quad \text{and } v_j = \frac{1}{2} |j-2|.
\]

Note that this estimator is indeed based on the optimal compromise estimator found in Section 2. Motivation for estimators of this form can be found in Berger (1982b). Note, at least, that if all \(\rho_{c,0}^{(j)}(r_j) \geq 1\), then \(\delta^{MB,c}\) is the conjugate prior Bayes rule.

Using Berger's (1979) decomposition theorem, it is easy to show that \(R(\theta,\delta^{MB,c}) - R(\theta,\delta^0) \leq c\) for all \(\theta\). The RSR for \(\delta^{MB,c}\) is given in the following theorem.

**Theorem 2.**

\[
\text{RSR}(n,\delta^{MB,c}) = \frac{1}{\sum_{i=1}^{p} \frac{d_i}{d_i}} \prod_{j=1}^{p} \prod_{i=1}^{p} \prod_{k=1}^{p} \frac{(d_j - d_{j+1})(d_k - d_{k+1})}{d_i} \frac{T_{c^*}}{j,k}.
\]
where

\[ T_{j,k}^C = E \{ z_i^2 [1 - \min(1, \rho_{c,0}^{(j)}(r_j))] [1 - \min(1, \rho_{c,0}^{(k)}(r_k))] \}, \]

\( z_1, \ldots, z_p \) are i.i.d. \( N(0,1) \), \( r_j = \frac{1}{n} \sum_{n=1}^{j} z_n^2 \), \( c^* = c/d_1 \), and \( \rho_{c,0}^{(j)}(r_j) \) is defined in (3.1).

**Proof.** Given in the Appendix. ||

For \( 1 \leq j, K \leq 10 \), the values of \( T_{j,k}^C \) for some \( c^* \) are given in Table 4.

**Remark 1.** When \( c = 0 \), \( \delta_{MB,0} \) is the same as \( \delta_{MB} \) defined in Berger (1982b).

**Remark 2.** All numerical results indicate that the RSR of \( \delta_{MB,c} \) is smaller than the RSR of \( \delta_{c} \) (c.f. Tables 1, 2, and 3), but we were able to prove this analytically.

**Remark 3.** In Dey and Berger (1983) (See also Efron and Morris (1973b)) the possibility of estimating, separately, groups of coordinates was considered. Dey and Berger (1983) discussed a somewhat inferior version of the estimator \( \delta_{MB,0} \), namely \( \delta_{MB}^* = (\delta_{MB}^*, \ldots, \delta_{MB}^*) \), where

\[
\delta_{MB}^*(x) = X_i - \frac{\sigma_i^2}{\sigma_i^2 + a_i} (x_i - \mu_i) \sum_{j=i}^{p} \frac{d_{j-d_{j+1}}}{d_k} \frac{(j-2)^+}{\sum_{n=1}^{j} (x_n - \mu_n)^2} \frac{\sum_{n=1}^{j} (x_n - \mu_n)}{\sigma_n^2 + a_n},
\]
showing that the "combined" estimator is always better than the "separate" estimator (in the sense of small RSR), where the "combined" estimator is $\delta_{MB}^*$ for the entire parameter vector and the "separate" estimator is $\delta_{MB}^{*1}$ for two groups of coordinates separately, one group consisting of coordinates with larger $d_i$ and the other of coordinates with small $d_i$. When considering $\delta_{MB,0}$, instead of $\delta_{MB}^{*1}$, the "combined" estimator can still be shown to be better than the "separate" estimator. For details see Chen (1983).

4. The Weighted Minimax Estimator

When $d_1$ (and maybe $d_2$) is much larger than the other $d_i$ (i.e., $\tau = \sum_{i=1}^p \frac{d_i}{d_1}$ is less than or only slightly larger than two), then the RSR for the minimax estimators $\delta_{p,0}$ and $\delta_{MB,0}$ are large. In the following, we will propose a new minimax estimator, $\delta_{W,1}$, which has smaller RSR than the previous minimax estimators for this situation.

Let

$$W = \frac{1}{p-2} \left[ (I + (p-2) y_0 D^{-1})^{1/2} - I \right], \quad (4.1)$$

where $y_0$ is the positive solution of the equation

$$\frac{1}{p-2} \sum_{i=1}^p d_i \left[ \sqrt{1 + \frac{(p-2)y}{d_i}} - 1 \right] = y. \quad (4.2)$$

(Lemma 3, given in the Appendix, shows that equation (4.2) has a unique positive
solution.) Then, define

$$\delta^{W,1}(x) = x - \min \left(1, \frac{2(p-2)^+}{r} W \frac{t}{(t+A)^{-1}}(x-\mu), \frac{t}{(t+A)^{-1}}(x-\mu) \right),$$

where $W$ is given in (4.1) and $r = (x-\mu)^t(t+A)^{-1}(x-\mu)$.

As partial motivation for considering such an estimator, note that the optimal minimax estimator in the class $D_c$ (see Section 2) of compromise estimators is $\delta_{\rho_0}(x) = x - \min \left(1, \frac{2(\tau-2)^+}{r} \frac{t}{(t+A)^{-1}}(x-\mu), \frac{t}{(t+A)^{-1}}(x-\mu) \right)$, which has the somewhat unappealing property of giving the same weight to each component of $\frac{t}{(t+A)^{-1}}(x-\mu)$. An obvious modification is to give different weights to different components (usually, the bigger $d_i$, the less the weight). This will be seen to work quite well.

**Theorem 3.** $\delta^{W,1}$ is a minimax estimator and

$$\text{RSR}(\pi, \delta^{W,1}) = 1 - \frac{\text{tr}(WD)}{\text{tr}D} \left(1-T^0_{\rho, \rho}\right).$$

**Proof.** Given in the Appendix. ||

**Remark 1.** It can be shown that $\frac{\text{tr}(WD)}{\text{tr}D} \geq \frac{\tau-2}{p-2}$, so that

$$\text{RSR}(\pi, \delta^{W,1}) \leq 1 - \frac{\tau-2}{p-2} \left(1-T^0_{\rho, \rho}\right).$$

**5. Comparisons and Conclusions**

Analytic comparisons among $\delta_{\rho_0}^{MB,c}$, $\delta^{MB,c}$ and $\delta^{W,1}$ are difficult. Indeed the following result was the only explicit analytic result obtained.
Theorem 4. When (i) \( d_1 > d_2 = \ldots = d_p \) or (ii) \( d_1 = d_2 > d_3 = \ldots = d_p \), then
\[
\text{RSR}(\pi, \delta^W_1) \leq \text{RSR}(\pi, \delta^\text{MB,0}) .
\]

Proof. Given in the Appendix. ||

Of course, Theorem 1 (and Corollary 1), Theorem 2, and Theorem 3, together with Table 4, allow explicit computation of the RSR of any of the three estimators and hence selection of the best one for the given situation. To give a general feeling for the performance of the estimators, however, we give three examples of application, two in which \( c = 0 \) (i.e., minimaxity is desired) so that \( \delta^0 \), \( \delta^\text{MB,0} \), and \( \delta^W_1 \) are compared, and one in which \( c > 0 \) so that only \( \delta^0 \) and \( \delta^\text{MB,c} \) are compared.

Example 1. (Randomized Block Design) Assume that
\[
Y_{ij} = \alpha_i + \beta_j + e_{ij}, \ i = 1, \ldots, p, \ j = 1, \ldots, n,
\]
where the \( \beta \)'s are i.i.d. \( N(0, \tau^2) \) and the \( e_{ij} \)'s are i.i.d. \( N_p(0, \sigma^2) \), independent of the \( \beta \)'s. Also assume that \( \sigma^2 \) and \( \tau^2 \) are known. Under the sum of squares error loss function, we want to estimate \( \alpha = (\alpha_1, \ldots, \alpha_p)^t \). Letting \( Y_j = (Y_{1j}, \ldots, Y_{pj})^t, j = 1, \ldots, n \), it is clear that the \( Y_j \) are i.i.d. \( N_p(\alpha, \Sigma) \), where
\[
\Sigma = \sigma^2 I_p + \tau^2 I_p I_p^t ,
\]
and \( l_p = (1,1,...,1)^t \). Therefore, the UMVU estimator of \( \alpha \) is

\[
\bar{Y} = n^{-1} \sum_{j=1}^{n} Y_j,
\]

which is \( N_p(\alpha, n^{-1}\Sigma) \). Suppose \( \alpha \) has (approximately) the conjugate prior distribution \( N_p(0,aI) \). First, we will transform to the diagonal case. Let \( H \) be a \((p \times p)\) orthogonal matrix with first row \( \sqrt{1/\rho}(1,1,...,1) \), and let \( Z_j = HY_j \), \( j = 1,...,n \). Then \( \bar{Z} = n^{-1} \sum_{j=1}^{n} Z_j \) is \( N_p(\alpha^*, \Sigma^*) \), where \( \alpha^* = Ha \) and \( \Sigma^* = n^{-1}H\Sigma H^t = \text{diag} \left( (\sigma^2 + p \tau^2)/n, \sigma^2/n, ..., \sigma^2/n \right) \). Also, \( \alpha^* \sim N_p(0,A^*) \), where \( A^* = H(aI)H^t = aI \).

Let \( \delta^*(z) = H\delta(y) \). Then

\[
L(\alpha, \delta) = (\alpha - \delta)^t(\alpha - \delta)
\]

\[
= (\alpha^* - \delta^*)^t(\alpha^* - \delta^*)
\]

\[
= L(\alpha^*, \delta^*).
\]

Thus, \( Q^* = I \), which implies that

\[
D = (\Sigma^* + A^*)^{-1/2} \Sigma^* Q^* \Sigma^* (\Sigma^* + A^*)^{-1/2}
\]

\[
= n^{-2} \text{diag} \left( \frac{(\sigma^2 + p \tau^2)^2}{a + (\sigma^2 + p \tau^2)/n}, \frac{\sigma^4}{a + \sigma^2/n}, ..., \frac{\sigma^4}{a + \sigma^2/n} \right).
\]

For \( p = 6, n = 3, \) and \( a = \tau^2 = 1 \), the RSR for the estimators \( \delta^o_0 \), \( \delta^{MB}_0 \), \( \delta^W_0 \) and \( \delta^*, \delta^{MB*}, \delta^{W*} \)
and for some different $\sigma$ are given in the following table:

<table>
<thead>
<tr>
<th>$\sigma$</th>
<th>$\sqrt{1/2}$</th>
<th>1</th>
<th>$\sqrt{2}$</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{RSR}(\pi, \delta_{\rho_0})$</td>
<td>1.000</td>
<td>1.000</td>
<td>0.935</td>
<td>0.447</td>
<td>0.167</td>
</tr>
<tr>
<td>$\text{RSR}(\pi, \delta_{MB,0})$</td>
<td>0.831</td>
<td>0.600</td>
<td>0.332</td>
<td>0.156</td>
<td>0.073</td>
</tr>
<tr>
<td>$\text{RSR}(\pi, \delta_{W,1})$</td>
<td>0.687</td>
<td>0.419</td>
<td>0.193</td>
<td>0.087</td>
<td>0.051</td>
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</table>

Example 2. Suppose $d = (d_1, d_2, \ldots, d_p)$ is the diagonal elements of the matrix $D = (\Sigma + A)^{-1/2} \Sigma Q \Sigma (\Sigma + A)^{-1/2}$, where $\Sigma$, $A$, and $Q$ are diagonal matrices. The RSR of the estimators $\delta_{\rho_0}$, $\delta_{MB,0}$, and $\delta_{W,1}$, for various $d$, are given in Table 2.

<table>
<thead>
<tr>
<th>$d$</th>
<th>RSR$(\pi, \delta_{\rho_0})$</th>
<th>RSR$(\pi, \delta_{MB,0})$</th>
<th>RSR$(\pi, \delta_{W,1})$</th>
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<tbody>
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<td>$(1.0,0.5,0.1)$</td>
<td>1.000</td>
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<td>.535</td>
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<tr>
<td>$(1.0,1.0,1.0,0.5)$</td>
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<td>.190</td>
<td>.164</td>
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<tr>
<td>$(1.0,1.0,0.5,0.5)$</td>
<td>.368</td>
<td>.325</td>
<td>.179</td>
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<td>.328</td>
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Remark 1. When \( \frac{d_p}{d_2} \) or \( \frac{d_p}{d_3} \) is close to one, \( \delta^{W,1} \) appears to be better than \( \delta^{MB,0} \) (see also Theorem 4).

Remark 2. When \( p < 6 \), \( \delta^{W,1} \) appears to be superior to \( \delta^{MB,0} \), except when \( \frac{d_{i+1}}{d_i} \) is very small (say, less than or equal to 0.1) for some \( 3 \leq i \leq p-1 \).

Remark 3. When \( p \geq 6 \) and at least four of the \( \frac{d_j}{d_i} \) are moderately close to one, then \( \delta^{MB,0} \) is often better and rarely much worse than \( \delta^{W,1} \). Hence \( \delta^{MB,0} \) is reasonable to use in this situation.

Remark 4. We can use the fact that a convex combination of minimax estimators is also minimax to develop an estimator improving on both \( \delta^{MB,0} \) and \( \delta^{W,1} \). (See Chen (1983) for details.) The improvement obtained did not seem to be substantial enough to justify the added complexity, however.

Example 3. Suppose \( p = 6 \), \( c = 0.1 \). The RSR of the estimators \( \delta_{\rho_C} \) and \( \delta^{MB,c} \) for various \( (d_1, \ldots, d_p) \) are given in Table 3.

<table>
<thead>
<tr>
<th>( d )</th>
<th>RSR(( \pi, \delta_{\rho_C} ))</th>
<th>RSR(( \pi, \delta^{MB,c} ))</th>
</tr>
</thead>
<tbody>
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</table>

Acknowledgment. The author wishes to thank Professor James Berger for his helpful suggestions and patient corrections.
Appendix

Proof of Lemma 1.

Using the identity (2.3), we have

\[(A.1) \quad R(\theta, \delta_\rho) - R(\theta, \delta^0) = E_\theta[(\rho^2(r) - 4\rho'(r))(x-\mu)^t(\frac{\delta+A}{2})^{-1/2} \cdot D(\frac{\delta+A}{2})^{-1/2}(x-\mu) - 2\rho(r)trD].\]

but

\[(A.2) \quad (x-\mu)^t(\frac{\delta+A}{2})^{-1/2} D(\frac{\delta+A}{2})^{-1/2}(x-\mu) \leq d_1 r.\]

Together with (A.1) and (A.2), it follows that

\[
R(\theta, \delta_\rho) - R(\theta, \delta^0) \leq d_1 E_\theta[\left(\rho^2(r) - 4\rho'(r)r - 2\tau \rho(r)\right] < c. \quad ||
\]

Lemma 2. Let \(X \sim N_p(\mu, M)\), \(M\) be a \((p \times p)\) positive definite matrix and \(\rho\) be a function of \(r\), where \(r = (x-\mu)^t B^{-1}(x-\mu)\). If \(E^r(\rho^2(r)) < \infty\), then

\[
E^x(\rho(r))(x-\mu)^t B^{-1/2} MB^{-1/2}(x-\mu)) = \left(\text{tr} M / p\right) E^r(\rho(r)).
\]

Proof. Let \(\Lambda\) be the \((p \times p)\) orthogonal matrix such that \(H = \Lambda M \Lambda^+\) is the diagonal
matrix with diagonal elements $h_1 \geq h_2 \geq \ldots \geq h_p$, and let

$$z = \lambda B^{-1/2}(x-\mu).$$

Then $Z$ is $N_p(0,I)$ and

$$r = (x-\mu)^tB^{-1}(x-\mu) = z^t z$$

has a chi-square distribution with $p$ degrees of freedom. But

$$[(x-\mu)^tB^{-1/2}MB^{-1/2}(x-\mu)]^2 \leq h_1^2 r^2,$$

which implies that

$$E^X[(x-\mu)^tB^{-1/2}MB^{-1/2}(x-\mu)] < \infty.$$

Therefore, by the Cauchy-Schwarz inequality

$$E^X[\rho(r)(x-\mu)^tB^{-1/2}MB^{-1/2}(x-\mu)]$$

exists, and

(A.3) \hspace{1cm} E^X[\rho(r)(x-\mu)^tB^{-1/2}MB^{-1/2}(x-\mu)] = E^X[\rho(r)z^tHz]

= \sum_{i=1}^{p} h_i E^Z(\rho(r)z_i^2).
Since \( z_1, z_2, \ldots, z_p \) are i.i.d.,

\[
(A.4) \quad E^Z(r) Z_i^2 = \frac{1}{p} E^Z(r_p(r)).
\]

The desired result follows from (A.3), (A.4) and the fact that \( \text{tr} H = \text{tr} M \).

**Proof of Theorem 1.**

It is clear that

\[
\begin{aligned}
& r(\pi, \sigma) - r(\pi, \sigma^\pi) \\
& = E^X \left[ 1 - \min(1, \rho_{c, 0}(r)) \right]^2 (x - \mu)^t (\mathbb{I} + A)^{-1} \mathbb{I} q^t (\mathbb{I} + A)^{-1} (x - \mu) \right].
\end{aligned}
\]

By Lemma 2,

\[
\begin{aligned}
& r(\pi, \sigma) - r(\pi, \sigma^\pi) \\
& = E^r \left[ \max(0, 1 - \rho_{c, 0}(r)) \right]^2 r \\
& = \int_b^\infty (1 - \rho_{c, 0}(r))^2 r f_p(r) dr.
\end{aligned}
\]

Using the fact that \( \rho_{c, 0}(r) = c/d_1 \) for \( r \geq b \) and noting that

\[
\int_b^\infty r f_p(r) dr = 2b f_p(b) + p(1 - \psi_p(b))
\]
and

\[ r(\pi, \sigma^0) - r(\pi, \sigma^\pi) = \sum_{i=1}^{p} d_i, \]

the desired result follows. ||

Proof of Theorem 2.

Let \( z_i = (a_i^2 + \alpha_i)^{-1/2}(x_i - \mu_i) \), and observe that the \( z_i \) are i.i.d. \( N(0,1) \) and that

\[
\begin{align*}
& r(\pi, \sigma^{MB,c}) - r(\pi, \sigma^\pi) \\
= & \mathbb{E}^X \left\{ \frac{\prod_{i=1}^{p} x_i}{\prod_{j=1}^{p} x_j} \left[ \prod_{j=i}^{p} \frac{d_j - d_{j+1}}{d_i} \min \left( 1, \rho(j), \rho(k) \right) \right] \right\} \\
= & \mathbb{E}^Z \left\{ z_i^2 \left[ \prod_{j=i}^{p} \frac{d_j - d_{j+1}}{d_i} \min \left( 1, \rho(j), \rho(k) \right) \right] \right\} \\
= & \sum_{i=1}^{p} \frac{d_i}{d_i} \left( d_i - d_{i+1} \right) \left( d_k - d_{k+1} \right) \\
& \times \mathbb{E}^Z \left[ z_i^2 \left( \min(1, \rho(j)) \right) (1 - \min(1, \rho(k))) \right].
\end{align*}
\]

The desired result follows. ||

Lemma 3. When \( p \geq 3 \), the equation (4.1) has a unique positive solution.
Proof.

Let

\[ h(y) = y - \frac{1}{p-2} \left[ \sum_{i=1}^{p} d_i \left( \sqrt{1 + \frac{(p-2)y}{d_i}} - 1 \right) \right]. \]

Clearly, \( h(0) = 0 \) and \( h(y) > 0 \) for sufficiently large \( y \). On the other hand, for a sufficiently small positive number \( \varepsilon \), \( \sqrt{1 + \varepsilon} \approx 1 + \frac{\varepsilon}{2} \). Thus we have

\[ h(\varepsilon) \approx \varepsilon - \frac{1}{p-2} \left[ \sum_{i=1}^{p} d_i \left( 1 + \frac{(p-2)\varepsilon}{2d_i} - 1 \right) \right] \]

\[ = (1 - \frac{p-2}{2})\varepsilon < 0. \]

Therefore, \( h \) has at least one positive root.

But

\[ h'(y) = 1 - \frac{1}{2} \sum_{i=1}^{p} \left( 1 + \frac{(p-2)y}{d_i} \right)^{-1/2} \]

and

\[ h''(y) = \frac{p-2}{4} \sum_{i=1}^{p} \frac{1}{d_i} \left[ 1 + \frac{(p-2)y}{d_i} \right]^{-3/2} > 0 \]

for all \( y > 0 \). Thus, \( h \) is concave upward. Hence \( h \) has a unique positive root.

This completes the proof. \(||\)
Proof of Theorem 3.

Let

$$\gamma(x) = -\min(1, \frac{2(p-2)}{r}) \left(\frac{1}{r} + A\right)^{-1}(x-\mu).$$

Using the identity (2.3), when $r \leq 2(p-2)$, we have

$$\varnothing \gamma(x) = (x-\mu)^t \left(\frac{1}{r} + A\right)^{-1} W^2 D (x-\mu) - 2\text{tr}(WD)$$

$$\leq r \left( \frac{1}{r} \text{ch}_{\max}(W^2 D) \right) - 2\text{tr}(WD)$$

$$\leq 2(p-2) \left( \frac{1}{r} \text{ch}_{\max}(W^2 D) \right) - 2\text{tr}(WD).$$

Since $\text{ch}_{\max}(W^2 D) \leq \text{ch}_{\max}(W^2 D + 2\frac{r}{p-2} WD) = \frac{1}{p-2} \text{tr}(WD)$, it follows that $\varnothing \gamma(x) \leq 0$.

On the other hand, when $r > 2(p-2)$,

$$\varnothing \gamma(x) = \left[ \frac{2(p-2)}{r} \right]^2 (x-\mu)^t W^2 D \left(\frac{1}{r} + A\right)^{-1}(x-\mu)$$

$$+ \left[ \frac{4(p-2)}{r^2} \right] (x-\mu)^t WD \left(\frac{1}{r} + A\right)^{-1}(x-\mu) - \frac{4(p-2)}{r} \text{tr}(WD)$$

$$= \left[ \frac{4(p-2)}{r^2} \right] (x-\mu)^t [(p-2) W^2 D + 2WD] \left(\frac{1}{r} + A\right)^{-1}(x-\mu)$$

$$- \frac{4(p-2)}{r} \text{tr}(WD)$$

$$\leq \left[ \frac{4(p-2)}{r} \right] \left\{ \frac{1}{r} \text{ch}_{\max} [(p-2) W^2 D + 2WD] - \text{tr}(WD) \right\}$$

$$\leq 0.$$ 

Thus $\delta$ is a minimax estimator of $\theta$. 
For the calculation of $\text{RSR}(\pi, \delta^W, 1)$, note first that

$$r(\pi, \delta^W, 1) - r(\pi, \delta^n) = E\{(x-\mu)^t [I - \min(1, \frac{2(p-2)}{r}) W] W D(\delta + A)^{-1} (x-\mu)\}.$$ 

Lemma 2 implies that

$$r(\pi, \delta^W, 1) - r(\pi, \delta^n) = p \text{ tr}D - 2 \text{ tr}(WD) E^r \left[ \min(1, \frac{2(p-2)}{r}) r \right] + \text{ tr}(W^2 D) E^r \left[ (\min(1, \frac{2(p-2)}{r}))^2 r \right].$$

Noting that

$$\text{ tr}(W^2 D) = \text{ tr}(WD),$$

the desired result follows. \boxed{} 

Proof of Theorem 4.

(i) Without loss of generality, we will assume that $d_1 = 1$. Let $d_2 = d_3 = \ldots = d_p = t$. Then, by Theorem 2,

$$r(\pi, \delta^{MB, 0}) - r(\pi, \delta^n) = (1-t)^2 + 2t(1-t) T_{1,p}^0 + t^2 T_{p,p}^0.$$
Since \( \sum_{i=1}^{p} d_i = 1 + (p-1)t \) and \( T_{1,p}^0 \geq T_{p,p}^0 \).

\[ (A.5) \quad r(\pi, \delta_{MB}^0) - r(\pi, \delta_{\pi}^0) \geq \sum_{i=1}^{p} d_i - (pt + t(1-t))(1-T_{p,p}^0). \]

On the other hand,

\[ (A.6) \quad r(\pi, \delta_{W,1}^0) - r(\pi, \delta_{\pi}^0) = \sum_{i=1}^{p} d_i - y(1-T_{p,p}). \]

Now, if we can show that

\[ (A.7) \quad y_0 \geq pt + t(1+t), \]

then the desired result follows from (A.5) and (A.6).

From the proof in Lemma 3, we know that showing that (A.7) holds is equivalent to showing that

\[ (A.8) \quad h(x) \leq 0, \]

where

\[ h(x) = x + \frac{1}{p-2} \left[ 1 + (p-1)t - \sqrt{1+(p-2)x} - (p-1)t\sqrt{1 + \frac{(p-2)x}{t}} \right] \]

and \( x = pt + t(1-t) \). After a little algebra, the inequality (A.8) is equivalent to
(A.9) \[
\frac{-t(1-t)^2}{(p-2)^2} \left[ (p-2)^3 \, t \,(4-t) + (p-2)^2 \, t \,(t^2-7t+14) \\
+ (p-2) \, (-2t^2+6t+3) + (1+t) \right] \leq 0.
\]

But it is clear that (A.9) holds for $0 < t < 1$, completing the proof of part (i). The proof of part (ii) is similar to that of part (i).
Table 4. Values of $T_{j,k}^{c^*}$

$c^* = 0$

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Bibliography


