ANOTHER LOOK AT POISSON PROCESSES

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1. INTRODUCTION. Poisson processes are generally characterized by the independence of increments (for arrival processes) or counts (for point processes). We show that the weaker condition of independence of [no arrivals] or [no points in sets] for non-overlapping intervals (disjoint sets) is sufficient to characterize Poisson processes among processes where events occur one at a time and no point has positive probability of being the site of an event. The reader may refer to an earlier work due to Kallenberg (1973), (1974), where the homogeneous Poisson processes are characterized under different conditions.

2. DEFINITIONS. Let $A$ be a random function on $[0, \infty)$ to the non-negative integers such that $A(0) = 0$ and $A$ is right continuous nondecreasing with unit jumps. We shall refer to such a function as an arrival process. In order for an arrival process to be a Poisson process, the following definitions are most commonly made in literature (see, e.g., Çinlar (1975), Doob (1953), Hoel, Port and Stone (1972), Karlin and Taylor (1975), Parzen (1962), Prabhu (1965), Renyi (1970), and Ross (1983)).

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**Definition 1.** An arrival process \( \{A(t), t \geq 0\} \) is said to be a homogeneous Poisson process if the following two conditions hold:

(i) for any \( n > 1 \) and \( 0 \leq t_0 < t_1 < \ldots < t_n < \infty \), the random variables \( A(t_1) - A(t_0), A(t_2) - A(t_1), \ldots, A(t_n) - A(t_{n-1}) \) are mutually independent;

(ii) for any \( t, s \geq 0 \), the distribution of \( A(t+s) - A(s) \) is independent of \( s \).

**Definition 2.** An arrival process \( \{A(t), t \geq 0\} \) is said to be a possibly non-homogeneous Poisson process if the following two conditions hold:

(i) for any \( n > 1 \) and \( 0 \leq t_0 < t_1 < \ldots < t_n < \infty \), the random variables \( A(t_1) - A(t_0), A(t_2) - A(t_1), \ldots, A(t_n) - A(t_{n-1}) \) are mutually independent;

(ii) \( a(t) \equiv EA(t), t \geq 0 \), is continuous for all \( t \geq 0 \).

Another approach is that of point process (see Karlin and Taylor (1975), p. 31). The intuitive idea is that there is a complete separable metric space \( X \) and on \( X \) there is a random point set \( Q \). If \( X = [0, \infty) \), one way to get such a point process is to take the set \( Q \) of arrivals of an arrival process; although the space need not be \( [0, \infty) \), this approach is almost general. We assume that there is a collection \( C \) of Borel subsets of \( X \) satisfying the following conditions:

(a) The collection \( C \) covers \( X \) and separates the points of \( X \), and if \( C \in C \), \( N(C) \) = number of elements of \( C \cap Q \) is a (possibly infinite) random variable.

(b) For every \( C_1, \ldots, C_m \in C \), there exists a finite collection \( \mathcal{D} \) of disjoint \( D_i \in C, i=1,2,\ldots,n \) such that each \( C_i \) is a union of a subfamily of \( \mathcal{D} \).

(c) For every \( C \) with \( P(N(C)=0)=0 \), there exists a sequence \( \{E_i\} \) of elements of \( C \) such that \( C = \bigcup E_i \) and \( P(N(E_i)=0) > 0 \) for each \( i \).
(d) For every $C \in C$, there is a sequence $\{\pi_j\}$ of finite partitions of $C$ in $C$ such that

$$P(N(C) \neq \sup_j M(\pi_j)) = 0, \quad (1)$$

where $M(\pi)$ denotes the number of cells $K$ of the partition $\pi$ with $N(K) > 0$.

Clearly $N(C) > M(\pi)$ for every partition $\pi$ of $C$. Note that condition (c) implies the nonexistence of a point in $X$ which is almost sure to be in $Q$. Also here we are not requiring that $C$ be a semi-ring; however, in view of condition (b), the collection of sets closed under finite unions of the sets of $C$ forms a ring which we shall denote by $\mathcal{R}$. We introduce the following definitions:

**Definition 3.** A point process $Q$ defined on a space $X$ (along with a collection $C$) as above is said to be a possibly nonhomogeneous spacial Poisson process provided that for any $n \geq 2$ and disjoint sets $C_1, \ldots, C_n \in C$, the random variables $N(C_1), \ldots, N(C_n)$ are mutually independent Poisson random variables.

Note that for such a point process $Q$, the sets $C \in C$ with $\epsilon N(C) < \varepsilon$ form a covering of $X$ for every $\varepsilon > 0$.

**Definition 4.** Let $\lambda$ be a nonnegative finitely additive set function defined on $C$. Then a point process $Q$ defined on a space $X$ as above is said to be a $\lambda$-homogeneous spacial Poisson process if it satisfies definition 3 and if for any disjoint sets $A_1, \ldots, A_m$ and disjoint sets $B_1, \ldots, B_n$ all in $C$,

$$\sum_i \lambda(A_i) \geq \sum_j \lambda(B_j) \Rightarrow P(\sum_i N(A_i) = 0) \leq P(\sum_j N(B_j) = 0). \quad (2)$$

Note that it is possible for $\lambda(B) \neq \lambda(C)$ whenever $B \neq C$, so some condition of the type (2) is needed.
3. **RESULTS.** Let $Q$ be a point process defined on a space $X$ (together with the collection $C$) as in section 2 and let $Z(C)$ be the event $C \cap Q \neq \emptyset$ (empty set) or equivalently that $N(C) > 0$. Then we have

**THEOREM.** If the events $Z(C_i)$ are independent for disjoint $C_i$'s $\in C$ and if for every $\varepsilon > 0$ and every $A \in C$ with $P(Z(A)) < 1$, there exists a finite covering of $A$ by sets $\{C_i\} \in C$, each with $P(Z(C_i)) < \varepsilon$, then $Q$ is a Poisson point process. If in addition $P(Z(C))$ satisfies the condition (2) for some nonnegative finitely additive function $\lambda$ on $C$, then $Q$ is $\lambda$-homogeneous and $E(N(C))$ is proportional to $\lambda(C)$.

Before we prove the theorem, we need the following lemma.

**LEMMA 1.** Let $Y$ be a Poisson random variable with mean $\theta > 0$. Let $Y' = 0$ if $Y = 0$ and $Y' = 1$ if $Y > 1$. Then

$$0 < P(Y \neq Y') < E(Y - Y') < \theta^2/2.$$ (3)

**Proof.**

$$P(Y \neq Y') = \sum_{k=2}^{\infty} P(Y = k)$$

$$< \sum_{k=2}^{\infty} (k-1)P(Y = k)$$

$$= E(Y - Y') = \theta - (1 - \exp(-\theta)) < \theta^2/2. \quad \square$$

**Proof of the Theorem.** Here all the sets will be considered in $C$ unless stated otherwise. The idea of the proof is that if a set $C$ is divided into "small" sets $B_i$'s, the number of points in $C$ is certainly at least the number of $B_i$'s with $N(B_i) \neq 0$; since points do not coincide it will be unlikely that
any \( N(B_i) > 1 \). We modify this approach slightly. For each \( C \) we define
\[
\eta(C) = - \ln [1 - P(Z(C))].
\] Note the independence of the \( Z \)'s for disjoint sets in \( C \) is equivalent to the additivity of \( \eta \) on \( C \). Let \( A_i, i = 1, 2, \ldots, I \), be disjoint elements of \( C \). Since \( \eta(A) = 0 \) implies \( N(A) = 0 \) a.s., without loss of generality we assume that \( \eta(A_i) > 0 \), \( i = 1, 2, \ldots, I \). We consider first the case when \( \eta(A_i) \to \infty \), for all \( i \). We shall prove that \( N(A_i), i = 1, 2, \ldots, I \), are mutually independent having Poisson distributions with means \( \eta(A_i), i = 1, 2, \ldots, I \).

Note that for each \( A_i \), there exists a sequence \( \{ \pi_{ij} \} \) of finite partitions of \( A_i \) such that

1. \( \pi_{ij} \subset C \),

2. \( \pi_{i,j+1} \) refines \( \pi_{ij} \),

3. \( \sup_{K \in \pi_{ij}} \eta(K) \to 0 \), as \( j \to \infty \),

4. \( P(N(A_i) \neq \sup_{j} M(\pi_{ij})) = 0 \),

where

\[
M(\pi_{ij}) = \sum_{K \in \pi_{ij}} Y(K),
\]  

and \( Y(K) \) are independent Bernoulli random variables with \( P(Y(K) = 1) = (1 - \exp[-\eta(K)]) \). Here (ii) follows from (b), (iii) follows from condition assumed in the theorem and condition (b), and (iv) follows from condition (d). Now for each \( K \in \pi_{ij} \) define \( W(K) = 0 \) when \( Y(K) = 0 \) and when \( Y(K) \geq 1 \),
let \( W(K) = r \) with probability

\[
\frac{1}{r!} [\eta(K)]^r \exp[-\eta(K)] (1-\exp[-\eta(K)])^{-1}, \quad r = 1, 2, \ldots,
\]

so that \( W(K), \pi_{ij}^{\pi} \) are mutually independent Poisson random variables with means \( \eta(K) \). Now let

\[
L_i(\pi_{ij}) = \sum_{K \in \pi_{ij}} W(K).
\]

Then \( L_i \) has a Poisson distribution with mean \( \eta(A_i) \). Moreover since the \( A_i \)'s are disjoint, the \( L_i \)'s are mutually independent. Also using the lemma we have

\[
P(Y(K) \neq W(K)) \leq \frac{1}{2} [\eta(K)]^2,
\]

so that

\[
\sum_{K \in \pi_{ij}} P(Y(K) \neq W(K)) \leq \frac{1}{2} \sum_{K \in \pi_{ij}} [\eta(K)]^2
\]

\[
\leq \frac{1}{2} [\sup_{K \in \pi_{ij}} \eta(K)] \sum_{K \in \pi_{ij}} \eta(K)
\]

\[
= \frac{1}{2} \eta(A_i) \sup_{K \in \pi_{ij}} \eta(K),
\]

which tends to zero as \( j \to \infty \) in view of (iii). Thus \( P(M(\pi_{ij}) \neq L_i(\pi_{ij})) \) tends to zero as \( j \to \infty \). Also, in view of (iv), \( P(N(A_i) \neq M(\pi_{ij})) \) tends to zero as \( j \to \infty \),
so that it follows that \( P(N(A_i) \neq I_i(\pi_{ij})) \) tends to zero. Consequently \( N(A_i), i=1,2,\ldots, I, \) are mutually independent Poisson random variables with means \( \eta(A_i), i=1,2,\ldots, I. \) Consider now the case with \( \eta(A_i)=\infty \) for some \( i \) or equivalently with \( P(N(A_i)=0)=0. \) Since in view of conditions (b) and (c) there exists a nondecreasing sequence of \( \{E_{ij}\}_{j=1}^{\infty} \) such that each \( E_{ij} \) can be expressed as a finite union of disjoint sets in \( C, A_i = \bigcup E_{ij} \) with \( \eta(E_{ij}) \to \infty; \)

since as \( j \to \infty, N(E_{ij}) \) are Poisson random variables with means \( \eta(E_{ij}) < \infty \) for all \( j, \) it is easy to see that \( P(N(A_i)=\infty)=1. \) In either case we can say that \( N(A_i), i=1,2,\ldots, I, \) are independent Poisson random variables with means \( \eta(A_i), i = 1,2,\ldots, I. \) Finally if condition (2) of definition 4 is satisfied for some nonnegative finitely additive function \( \lambda \) on \( C, \) it follows that the process \( Q \) is \( \lambda \)-homogeneous. The only thing that we still need to prove subject to (2) is that \( EN(C) = \eta(C) \) is proportional to \( \lambda(C). \) In this context we note that since both \( \eta \) and \( \lambda \) are finitely additive, their definitions can be extended to the ring \( \mathcal{R}. \) Also for the last part of the theorem we find it convenient to work in terms of these extended versions of \( \eta \) and \( \lambda \) defined on \( \mathcal{R}. \) The assumed condition (2) in terms of their extended versions can now be rewritten as

\[
\lambda(A) \geq \lambda(B) \Rightarrow \eta(A) \geq \eta(B), \quad \forall \, A, B \in \mathcal{R},
\]

(8)

or equivalently

\[
\eta(A) < \eta(B) \Rightarrow \lambda(A) < \lambda(B), \quad \forall \, A, B \in \mathcal{R}.
\]

(9)

Also the condition of the theorem satisfied by \( \eta(A) \) for all \( A \in C \) with
\(\eta(A) \to \infty\), and the condition (b) on \(C\) imply (*) given below for the extended version of \(\eta\) on \(\mathfrak{R}\).

For every \(\varepsilon > 0\) and every \(A \in \mathfrak{R}\) with \(\eta(A) < \infty\), there exists a finite partition of \(A\) in \(\mathfrak{R}\), say \(\{D_i\}\) with \(A = \bigcup D_i\) and \(\eta(D_i) < \varepsilon\) for every \(D_i\). (*)

The proportionality of \(\eta\) and \(\lambda\) now follows from lemma 3, which in turn needs lemma 2 given below.

**Lemma 2.** Let two nonnegative finitely additive set functions \(\eta\) and \(\lambda\) be defined on sets of a ring \(\mathfrak{R}\) and satisfy the condition (8). Let \(\eta\) satisfy the conditions (*) and (c) given above. Also to avoid trivial cases we assume that there exists a set \(B \in \mathfrak{R}\) such that \(0 < \eta(B) < \infty\). Then we have

(i) \(\lambda(C) = 0 \iff \eta(C) = 0\),

(ii) \(\lambda\) too satisfies the condition (*), and

(iii) for sets \(A_1, A_2 \in \mathfrak{R}\)

\[
\lambda(A_1) < \lambda(A_2) < \infty \Rightarrow \eta(A_1) < \eta(A_2).
\]

Proof. (i) That \(\lambda(C) = 0 \Rightarrow \eta(C) = 0\) follows by using (8) with \(A = \emptyset\) and \(B = C\). Conversely let \(\eta(C) = 0\). Since \(0 < \eta(B) < \infty\) so that \(\eta(C) < \eta(B)\), using (8) we have \(\lambda(B) > 0\). Again in view of (*), for an arbitrary \(n\), there exists a finite partition of \(B\) in \(\mathfrak{R}\), say \(\{D_i\}\) such that \(\bigcup D_i = B\) and \(\eta(D_i) < [\eta(B)/n]\) for all \(i\). Evidently the \#\(D_i\)'s: \(\eta(D_i) > 0\) > \(n\). However, since for such \(D_i\)'s, \(0 = \eta(C) < \eta(D_i)\), we have \(\lambda(C) < \lambda(D_i)\) so that

\[
\{i : \eta(D_i) > 0\} \subseteq \{i : \lambda(D_i) > 0\}.
\]
Consequently since
\[
\#\{D_i: \eta(D_i) > 0\} \lambda(C) < \sum_{D_i: \eta(D_i) > 0} \lambda(D_i) \leq \sum_i \lambda(D_i) = \lambda(B),
\]
we have \(\lambda(C) < \lambda(B)/n\) for all \(n \geq 1\), so that \(\lambda(C) = 0\).

(ii) Let \(C \in \mathcal{R}\) be a set with \(\eta(C)\) and \(\lambda(C)\) both finite. Since when
\(\lambda(C) = 0\) (*) is trivially satisfied, in view of (i) we let both \(\eta(C)\) and \(\lambda(C)\)
be positive. Now for every \(n\), using (*), there exists a finite partition of
\(C\) in \(\mathcal{R}\), say \(\{D_i\}\) with \(C = \cup D_i\) and \(\eta(D_i) \leq [\eta(C)/n]\) for all \(i\). Let \(D^*\) be a \(D_i\) with
\(\eta(D^*) = \min \{\eta(D_i): \eta(D_i) > 0\}\). Then \(0 < \eta(D^*) \leq \eta(C)/n\). Since \(\eta(D_i) = 0 \Rightarrow \lambda(D_i) = 0\),
we also have \(\lambda(D^*) > 0\); in fact we can choose \(D^*\) to be such that
\(\lambda(D^*) = \min \{\lambda(D_i): \lambda(D_i) > 0\}\), so that \(0 < \lambda(D^*) \leq \lambda(C)/n\). Now take a new finite partition
of \(C\) in \(\mathcal{R}\) say \(\{D_{\ell}\}\) such that \(\eta(D_{\ell}) < \eta(D^*)\). From (*) it follows that
\[
\lambda(D_{\ell}) < \lambda(D^*) \leq \lambda(C)/n, \quad \forall \ell,
\]
which proves (*) for \(\lambda\). Next consider the case where \(C \in \mathcal{R}\), \(\eta(C) = \infty\) but \(\lambda(C) < \infty\).
In view of condition (c), there is a sequence \(C_k \uparrow C\) with \(C_k \in \mathcal{R}\), \(0 < \eta(C_k) \leq \infty\)
and \(\eta(C_k) \not< \infty\), as \(k \to \infty\). Likewise \(\lambda(C_k) \uparrow \lambda(C)\). Choose \(k_0\) large enough such
that \(\lambda(C \setminus C_{k_0}) < \varepsilon\). Now treat the set \(C_{k_0}\) as above since both \(\lambda(C_{k_0})\) and \(\eta(C_{k_0})\)
are finite. The finite partition of \(C_{k_0}\) so obtained together with the set
\((C \setminus C_{k_0}) \in \mathcal{R}\), yields the desired partition of the set \(C\).

(iii) Let \(A_1, A_2 \in \mathcal{R}\), \(\lambda(A_1) < \lambda(A_2) < \infty\). With \(c = \lambda(A_2) - \lambda(A_1)\), in view of
(*) there exists a finite partition of \(A_2\) in \(\mathcal{R}\), say \(\{D_i\}\), such that \(A_2 = \cup D_i\).
\( \lambda(D_i) < \varepsilon \) for all \( i \). Without loss of generality, let \( D_1 \) be such that \( \lambda(D_1) > 0 \) and hence \( \eta(D_1) > 0 \). Consequently \( \lambda(A_2 \setminus D_1) > \lambda(A_1) \) so that using (8) we have \( \eta(A_2 \setminus D_1) \geq \eta(A_1) \). Hence

\[
\eta(A_2) = \eta(A_2 \setminus D_1) + \eta(D_1) > \eta(A_2 \setminus D_1) \geq \eta(A_1).
\]

\[ \square \]

**Lemma 3.** Let \( \eta \) and \( \lambda \) satisfy the conditions of lemma 2. Define \( \psi(x) = y \) if for some \( \lambda \in \mathbb{R} \), \( \lambda(A) = x \) and \( \eta(A) = y \).

1. \( \psi \) is uniformly continuous over a dense set in \([0, \alpha)\) and hence can be extended uniquely to a strictly monotone \( \psi \) with its domain = \([0, \alpha)\), where \( \alpha = \sup_{A \in \mathcal{A}(\lambda)} \lambda(A) \).

2. The extended \( \psi \) satisfies

\[
\psi(x_1 + x_2) = \psi(x_1) + \psi(x_2), \quad 0 \leq x_1, x_2 < \infty, x_1 + x_2 < \alpha, \tag{13}
\]

so that \( \psi(x) = \beta x \), for some \( \beta > 0 \) and hence \( \eta \) and \( \lambda \) are proportional.

**Proof.** (1) Clearly the domain of the function \( \psi \), say \( D(\psi) \), is dense in \([0, \alpha)\).

It is sufficient to prove that for every \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that if \( 0 < x_1 < x_2 < x_1 + \delta, \ x_1, x_2 \in D(\psi) \), we have \( \psi(x_2) - \psi(x_1) < \varepsilon \). In view of condition (*), there exists \( A \in \mathcal{A} \) with \( 0 < \eta(A) < \varepsilon \), for a given \( \varepsilon > 0 \). Take \( \delta = \frac{1}{2} \lambda(A) \). Now for \( 0 < x_1 < x_2 < \delta, \ x_1, x_2 \in D(\psi) \), let \( E_1 \) and \( E_2 \in \mathcal{A} \) be such that \( x_1 = \lambda(E_1) \) and \( x_2 = \lambda(E_2) \). Then

\[
0 < \lambda(E_1) < \lambda(E_2) < \delta < \lambda(A),
\]
from which, using lemma 2 (iii), follows

\[ 0 < \eta(E_1) < \eta(E_2) < \eta(A) < \varepsilon, \]

so that \( \eta(E_2) - \eta(E_1) = \psi(x_2) - \psi(x_1) < \varepsilon \). Again for the case with

\[ 0 < x_1 < x_2 < x_1 + \delta, \ x_2 > \delta, \ x_1, x_2 \in \mathcal{B}(\psi), \] as before let \( x_1 = \lambda(E_1) \) and \( x_2 = \lambda(E_2) \)

for some \( E_1, E_2 \in \mathcal{R} \). Then using lemma 2 (ii), there exists a finite partition

\( \{D_i\} \) of sets in \( \mathcal{R} \) such that \( E_2 = \bigcup D_i \) and \( \lambda(D_i) < \delta \), for all \( i \). Thus there exists a set \( C \), which is union of some of the \( D_i \)'s such that \( \delta < \lambda(C) < 2\delta = \lambda(A) \) holds, so that

\[ \lambda(E_2 \setminus C) = \lambda(E_2) - \lambda(C) = x_2 - \lambda(C) < x_2 - \delta < x_1 = \lambda(E_1). \] (14)

Consequently,

\[ \psi(x_1) > \psi(\lambda(E_2 \setminus C)) = \eta(E_2 \setminus C) \]

\[ = \eta(E_2) - \eta(C) \]

\[ > \psi(x_2) - \psi(\lambda(A)) \]

\[ = \psi(x_2) - \eta(A) \]

\[ > \psi(x_2) - \varepsilon. \]

Here the strict inequalities follow from (8) and lemma 2 (iii). The same reasoning applies to justify the strict monotonicity of the extended \( \psi \) to the domain \([0, a)\).
Let $\varepsilon > 0$ and $\delta$ be as defined in the proof of part (i) above. From part (i) it follows that for $0 \leq x_1, x_2 < \infty$, $x_1 + x_2 < \alpha$, there exists $C \in \mathcal{R}$, such that

$$|\lambda(C) - (x_1 + x_2)| < \delta/2. \quad (15)$$

Then in view of lemma 2 (ii), there exists a finite partition \{E_i\} of sets in $\mathcal{R}$ with $C = \bigcup E_i$ and $\lambda(E_i) < \delta/2$ for all $i$. Consequently there exist disjoint sets $C_1$ and $C_2$, both unions of different $E_i$'s such that $C = C_1 \cup C_2$ and

$$|\lambda(C_1) - x_1| < \delta/2, \quad |\lambda(C_2) - x_2| < \delta. \quad (16)$$

Using part (i) above, (15), (16) and the fact that $\eta(C) = \eta(C_1) + \eta(C_2)$, it follows that $|\eta(C) - \psi(x_1 + x_2)| < \varepsilon$, $|\eta(C_1) - \psi(x_1)| < \varepsilon$ and $|\eta(C_2) - \psi(x_2)| < \varepsilon$, so that

$$|\psi(x_1 + x_2) - \psi(x_1) - \psi(x_2)| < 3\varepsilon. \quad (17)$$

The number $\varepsilon$ being arbitrary, we have proved (13) and the lemma. \qed

We observe that the usual methods of extending measures to a $\sigma$-field can be carried out also in this case, so that we have

\textbf{COROLLARY 1.} Under the assumptions of the theorem and the added assumption:

$$P(N(C) = 0) = \prod_{i=1}^{\infty} P(N(A_i) = 0) \quad (***)$$

whenever $A_i, C \in \mathcal{C}, C = \bigcap_{i=1}^{\infty} A_i$, $A_i$ disjoint.
\( N(B_1) \) are independent Poisson random variables if \( B_1 \)'s are disjoint elements of the \( \sigma \)-field generated by \( C \).

Here the condition (***) is merely the \( \sigma \)-additivity of \( \eta \).

**COROLLARY 2.** If \( \{ A(t), t \geq 0 \} \) is an arrival process on \([0, \infty)\) such that

(i) If \( 0 \leq t_0 < t_1 < \ldots < t_n \) the events \( A(t_i) = A(t_{i-1}) \) are independent, and

(ii) for each \( t > 0 \) and \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that \( P(A(t+\delta) = A((t-\delta) \vee 0) > 1 - \varepsilon, \)

then \( A \) is a Poisson arrival process. If (i) holds and \( P(A(t+s) = A(t)) \) is independent of \( t \), \( A \) is a homogeneous Poisson arrival process.

**NOTE.** After this paper was written our attention was drawn to a recent paper due to Brown (1984) appeared in *The American Mathematical Monthly*, Vol. 91, pp. 116-123, which deals with the same topic as ours. However, the main point of our paper is that we only assume independence property for the emptiness of disjoint sets rather than for the numbers of points in them, and we do not even assume this for all measurable sets, but only for "sufficiently many" of them. Also the special case of arrival processes with \( X = [0, \infty) \) was already dealt with in a Ph.D. dissertation of Huang (1983).

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