On the Narrowest Tube of a Simple Symmetric Random Walk

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Summary. Let \( X_1, X_2, \ldots \) be i.i.d. random variables with
\[ P(X_1 = +1) = P(X_1 = -1) = 1/2. \]
Put \( S_0 = 0, S_n = X_1 + \ldots + X_n \) (\( n \geq 1 \)).
Our aim is to investigate the a.s. behavior of
\[ U(a_N,N) = \min_{0 \leq j < N-a_N} \max_{0 \leq i < a_N} |S_{j+i} - S_j| \quad \text{and} \quad V(a_N,N) = \min_{0 \leq j < N-a_N} \max_{0 \leq i < a_N} |S_{j+i}|. \]
It is shown that for \( a_N = \lfloor c \log N \rfloor \) both \( U(a_N,N) \) and \( V(a_N,N) \) are a.s.
constant for large \( N \), except for certain values of \( c \), when \( U \) and \( V \)
can take two values for large \( N \).

1. INTRODUCTION

Let \( \{W(t), t \geq 0\} \) be a standard Wiener process. Suppose that
\( a_t \) is a non-decreasing function of \( t \) such that \( 0 < a_t \leq t \) and \( t/a_t \)
is non-decreasing. By investigating the small values of the increments

\[ \zeta(t,a_t) = \sup_{0 \leq s < a_t} |W(t+s) - W(t)|, \]

Csörgő and Révész [5, Theorem 1.7.1] proved the following results.

THEOREM A

\[ \lim_{T \to \infty} \inf_{0 \leq t \leq T-a_T} \inf_{0 \leq s < a_T} \zeta(t,a_t) a_t^{-s} = 1, \]

where
\( \gamma_T = \left( \frac{8(\log \frac{T}{a_T} + \log \log T)}{\pi^2 a_T} \right)^{1/2} \)

If we also have

\( \frac{\log(T/a_T)}{\log \log T} \to \infty \) as \( T \to \infty \),

then

\( \lim_{T \to \infty} \inf_{0 < t \leq T-a_T} \zeta(t, a_T)^{a_T s} = 1 \)

In Csáki and Földes [2] a similar result is given for

\( \xi(t, a_T) = \sup_{0 \leq s \leq a_T} |W(t+s)|, \)

i.e. for the Wiener process instead of the increments.

**THEOREM B**

\( \lim_{T \to \infty} \inf_{0 < t \leq T-a_T} \xi(t, a_T)^{a_T s} = 1, \)

where

\( \beta_T = \left( \frac{4 \log \frac{T}{a_T} + 8 \log \log T}{\pi^2 a_T} \right)^{1/2} \)

If we also have condition (1.4), then
(1.9) \[ \lim_{T \to \infty} \inf_{0 \leq t \leq T-a_T} \varepsilon(t, a_T)^{a_T} \leq 1 \]

Similar problems can also be investigated for partial sums of i.i.d. random variables. Let \( \{X_i, i \geq 1\} \) be a sequence of i.i.d. random variables with mean 0 and variance 1. Put \( \delta_0 = 0, \)

\( S_n = X_1 + \ldots + X_n \quad (n \geq 1). \) Suppose that \( \{a_n, n \geq 1\} \) is a non-decreasing sequence of integers such that \( 0 < a_n \leq n \) and \( n/a_n \) is non-decreasing.

In the case \( \lim_{n \to \infty} a_n / \log n = \infty, \) referring to a small deviation result of Mogul'skii [7] the following theorem is formulated in Csörgö and Révész [5, Theorem 3.3.1.*]

**THEOREM C**

In the case when \( \lim_{N \to \infty} a_N / \log N = \infty \) we have

(1.10) \[ \lim_{N \to \infty} \inf_{0 \leq j \leq N-a_N} \max_{0 \leq i \leq a_N} \gamma_N \left| S_{j+i} - S_j \right|^{a_S} \leq 1, \]

where \( \gamma_N \) is given by (1.3). If the condition (1.4) is also satisfied, then

(1.11) \[ \lim_{N \to \infty} \min_{0 \leq j \leq N-a_N} \max_{0 \leq i \leq a_N} \gamma_N \left| S_{j+i} - S_j \right|^{a_S} \leq 1. \]

It is not difficult to see that an analogue of Theorem B is also true. Our first result states

**THEOREM 1.** In the case when \( \lim_{N \to \infty} a_N / \log N = \infty \) we have
\[(1.12) \liminf_{N \to \infty} \min_{0 \leq j < N-a_N} \max_{0 \leq i \leq a_N} \beta_N |S_{j+1} - S_j|^{a_N} \geq 1,\]

where \(\beta_N\) is given by (1.8). If the condition (1.4) is also satisfied, then

\[(1.13) \lim_{N \to \infty} \min_{0 \leq j < N-a_N} \max_{0 \leq i \leq a_N} \beta_N |S_{j+1} - S_j|^{a_N} \geq 1.\]

Theorem 1 can be proved by the same way as Theorem B was proved in Csáki and Földes [2] using the above mentioned small deviation results of Mogulskii [7]. So we omit this proof.

The more interesting case is \(a_N = [c \log N]\) and is yet unsolved. The following conjecture is formulated in Csörgő and Révész [5]:

**CONJECTURE**

\[(1.14) \lim_{N \to \infty} \min_{0 \leq j < N-a_N} \max_{0 \leq i \leq a_N} |S_{j+1} - S_j|^{a_N} \geq \alpha(c),\]

where \(\alpha(c)\) is a function which uniquely determines the distribution of \(X_1\).

As a small step towards the solution of this problem we investigate it for the simple symmetric random walk, i.e. when \(P(X_1 = +1) = P(X_1 = -1) = 1/2\) and show that (1.14) is nearly true in this case. Define

\[(1.15) U(a,N) = \min_{0 \leq j < N-a} \max_{0 \leq j \leq a} |S_{j+1} - S_j|\]
and call it the narrowest tube for the increments. Since \( U(a_N,N) \)
is integer valued, (1.14) would mean that \( U(A_N,N) \) takes only the parti-
cular value \( a(c) \) when \( N \) is large enough. In fact this is true for almost
every \( c \), but there are exceptional values of \( c \) when \( U(a_N,N) \) can take 2
values even for large \( N \).

We investigate also

\[
(1.16) \quad V(a,N) = \min_{0 \leq j < N-a} \max_{0 \leq i < a} |S_{j+i}|
\]

and call it the narrowest tube around zero.

We formulate our main results.

**THEOREM 2.** Assume that \( X_1, X_2, \ldots \) are i.i.d. random variables with
\( P(X_1 = +1) = P(X_i = -1) = 1/2 \). Let \( a_N = [c \log N], c > 0 \) and define
\( a = a(c) > 1 \) as the solution of the equation

\[
(1.17) \quad \cos \frac{\pi}{2a} = e^{-\frac{1}{c}}.
\]

If \( a(c) \) is not an integer, then for almost all \( \omega \), there exists an
\( N_0 = N_0(c,\omega) \) such that

\[
(1.18) \quad U(a_N,N) = [a(c)] \text{ if } N \geq N_0.
\]

If \( a(c) \) is an integer, then for almost all \( \omega \) there exists an
\( N_0 = N_0(c,\omega) \) such that
\( (1.19) \quad \alpha(c) - 1 \leq U(a_N, N) \leq \alpha(c) \quad \text{if} \quad N \geq N_0. \)

Moreover

\( (1.20) \quad P(U(a_N, N) = \alpha(c) - 1 \text{ i.o.}) = 1 \)

and

\( (1.21) \quad P(U(a_N, N) = \alpha(c) \text{ i.o.}) = 1 \)

**THEOREM 3.** Assume that \( X_1, X_2, \ldots \) are i.i.d. random variables with \( P(X_i = +1) = P(X_i = -1) = 1/2. \) Let \( \alpha_N = [c \log N], c > 0 \) and define \( \alpha^* = \alpha^*(c) > 1 \) as the solution of the equation

\[ \cos \left( \frac{\pi}{2\alpha^*} \right) = e^{-\frac{1}{2c}} \]

If \( \alpha^*(c) \) is not an integer, then for almost all \( \omega \) there exists an \( N_0^* = N_0^*(c, \omega) \) such that

\( (1.23) \quad V(a_N, N) = [\alpha^*(c)] \quad \text{if} \quad N \geq N_0^*. \)

If \( \alpha^*(c) \) is an integer, then for almost all \( \omega \) there exists an \( N_0^* = N_0^*(c, \omega) \) such that

\( (1.24) \quad \alpha^*(c) - 1 \leq V(a_N, N) \leq \alpha^*(c) \quad \text{if} \quad N \geq N_0. \)
Moreover

(1.25) \( P(V(a_N, N) = \alpha^*(c) - 1 \, i.o.) = 1 \)

and

(1.26) \( P(V(a_N, N) = \alpha^*(c) \, i.o.) = 1 \).

Theorem 2 and Theorem 3 will be proved in Section 2 and Section 3, respectively.

The basic formula used in our proofs is due to Ellis [6] (see also Takács [9]):

**LEMMA A.** Let \( a \geq 1, \alpha \geq 1, \beta \geq 1, x \) be integers such that \(-\beta < x < \alpha\). Then

\[
P(-\beta < S_k < \alpha, k = i, \ldots, a-1, S_a = x) =
\]

(1.27) \[
\frac{2}{\alpha+\beta} \sum_{k=0}^{\alpha+\beta} (\cos \frac{k\pi}{\alpha+\beta})^a \sin \frac{k\pi\alpha}{\alpha+\beta} \sin \frac{k\pi(\alpha-x)}{\alpha+\beta}
\]

By using the formula

(1.28) \[
\sum_{x=-\beta}^{\alpha} \sin \frac{k\pi(\alpha-x)}{\alpha+\beta} = \frac{1 + \cos \frac{k\pi}{\alpha+\beta}}{\sin \frac{k\pi}{\alpha+\beta}} \left( \frac{1 - (-1)^k}{2} \right)
\]

we obtain
\[ P(-\beta < S_k < \alpha, k = 1, \ldots, a) = \]

\[
\frac{2}{\alpha + \beta} \sum_{k=1}^{\alpha + \beta - 1} \left( \cos \frac{k\pi}{\alpha + \beta} \right)^a \sin \frac{k\pi}{\alpha + \beta} \frac{1 + \cos \frac{k\pi}{\alpha + \beta}}{\sin \frac{k\pi}{\alpha + \beta}} \left( \frac{1 - (-1)^k}{2} \right)
\]  

For large \( a \) the dominating terms of the above sums are for \( k=1 \) and \( k=\alpha + \beta - 1 \) and it is easy to see that for large \( a \) the following inequalities hold:

**COROLLARY.**

\[
K_1 \left( \cos \frac{\pi}{\alpha + \beta} \right)^a \leq P(-\beta < S_k < \alpha, k = 1, \ldots, a) \leq K_2 \left( \cos \frac{\pi}{\alpha + \beta} \right)^a
\]

with some constants \( K_1 \) and \( K_2 \), depending only on \( \alpha + \beta \) and not on \( a \).

In our proofs \( K_1 \) and \( K_2 \) will denote the above constants, but \( K^*, K_1^* \), etc. will denote further constants, whose values are not important for the proof and may change from time to time.

**2. THE TUBE FOR THE INCREMENTS**

In this Section we prove Theorem 2, based on the following 4 lemmas.

**LEMMA 2.1.** If \( \alpha > 1 \) is an integer and

\[
\cos \frac{\pi}{2\alpha} < e^{-\frac{1}{c}}.
\]
then for almost all $\omega$ there exists an $N_0 = N_0(c, \omega)$ such that

$$
(2.2) \quad U(a_N, N) > \alpha \quad \text{for} \quad N > N_0,
$$

where $U(a, N)$ is defined by (1.15) and $a_N = [c \log N]$.

**LEMMA 2.2.** If $\alpha > 2$ is an integer and

$$
(2.3) \quad \cos \frac{\pi}{2\alpha} > e^{-\frac{1}{c}}
$$

then for almost all $\omega$ there exists an $N_0^* = N_0^*(c, \omega)$ such that

$$
(2.4) \quad U(a_N, N) < \alpha \quad \text{for} \quad N > N_0^*,
$$

where $U(a, N)$ is defined by (1.15) and $a_N = [c \log N]$.

**LEMMA 2.3.** Let the events $A_j^a (j = 0, 1, \ldots)$ be defined by

$$
(2.5) \quad A_j^a = \{ \max_{0 \leq i \leq a} |S_{j+i} - S_j| < a(c) \},
$$

where the solution $a(c)$ of (1.17) is an integer. Then

$$
(2.6) \quad P(A_N^a - a_N \text{ i.o.}) = 1,
$$
where \( a_N = [c \log N] \).

**Lemma 2.4.** The following inequality holds true for \( \alpha > 1 \), \( k > 1 \) and \( a \) large enough:

\[
P(U(a, (k+1)a - 1) \geq \alpha)
\geq (1 - K_2 a(\cos \frac{\pi}{2\alpha})^a) - K_2 a(\cos \frac{\pi}{2\alpha})^a,
\]

where \( U(a, N) \) is defined by (1.15) and \( K_2 \) is the constant of (1.30).

**Proof of Lemma 2.1.**

From the inequality (2.1) it follows that there exists an integer \( \rho > 0 \) such that

\[
\cos \frac{\pi}{2\alpha} < e^{-\frac{1+\rho}{C\rho}}.
\]

Define \( N_k = k^\rho \). Then by (1.30),

\[
P(U(a_{N_k}, N_{k+1}) < \alpha) \leq N_{k+1}P(\max_{0 \leq j \leq N_k} |S_j| < \alpha)
\leq K_2 N_{k+1}(\cos \frac{\pi}{2\alpha})^{a_{N_k}} \leq K^{*}(k+1)^{\rho'}(\cos \frac{\pi}{2\alpha})^c \rho \log k
\leq K^{*}k^{-\beta},
\]
where, from (2.8)

(2.9) \[ \beta = \rho + \rho c \log \cos \frac{\pi}{2\alpha} < -1. \]

Hence

(2.10) \[ \sum_k P(U(a_{N_k}, N_{k+1}) < a) < \infty \]

and Lemma 2.1 follows from Borel-Cantelli lemma and from the simple inequality

(2.11) \[ U(a_{N_k}, N_{k+1}) \leq U(a_N, N) \text{ if } N_k \leq N < N_{k+1}. \]

\[ \square \]

PROOF OF LEMMA 2.2.

Since

(2.12) \[ U(a_N, N) \leq \inf_{0 \leq j < \frac{N}{a_N}} \sup_{0 \leq i < \frac{a_N}{N}} |S_{i + ja_N} - S_{ja_N}| \]

and \[ \sup_{0 \leq i < \frac{a_N}{N}} |S_{i + ja_N} - S_{ja_N}| \] are independent for \( j=0,1,2,\ldots \), we obtain

from (1.30)
\[ P(U(a_N, N) \geq \alpha) \leq (P(\sup_{0 \leq i < a_N} |S_i| \geq \alpha))^{\frac{N}{a_N}} \leq \]

\[ (1 - K_1(\cos \frac{\pi}{2\alpha})^{a_N})^{\frac{N}{a_N}} \leq e^{\frac{c \log \cos \frac{\pi}{2\alpha}}{c \log N}} \leq \exp\left\{ -K_1(1 - \frac{N}{\log N} - 1)N \right\}, \]

if \( N \) is large enough. But from (2.3), \( c \log \cos (\pi/(2\alpha)) > -1 \), and hence

\[ \sum_{N=1}^{\infty} P(U(a_N, N) \geq \alpha) < \infty. \]

So Lemma 2.2 follows from Borel-Cantelli lemma.

\[ \square \]

PROOF OF LEMMA 2.3.

From (1.17) and (1.30) we obtain

\[ P(A_N a_N) \geq K_1(\cos \frac{\pi}{2\alpha})^{a_N} \geq \frac{K^*}{N} \]

(2.13)

By choosing \( N_k = [(c + 1)k \log k] \), it can be easily seen that

\[ N_k < N_{k+1} - a_{N_{k+1}} \]

holds for large enough \( k \), hence the events \( A_{N_k} a_{N_k} \) are independent for

\[ k \geq k_0 \] and since
\[ \sum_{k} P(A_{N_k}^{a_{N_k}}) = \infty, \]

(2.6) follows from Borel-Cantelli lemma.

Since:

\[ U(a_{N,N}) \leq \max_{0 \leq i \leq a_N} |S_i - S_{i-a_N}|, \]

Lemma 2.3 gives also

\[ P(U(a_{N,N}) \leq \alpha(c) - 1 \text{ i.o.}) = 1. \]

PROOF OF LEMMA 2.4.

Let the events \( A_j = A_j^a \) \((j=0,1,\ldots)\) be defined by (2.5) and assume that \( a \) is large enough so that (1.30) holds true. Then

\[ \{ U(a,(k+1)a - 1 \geq \alpha) = \bar{A}_0 \bar{A}_1 \ldots \bar{A}_{ka-1} \}

Introduce the following notations:

\[ p_k = P(\bar{A}_0 \bar{A}_1 \ldots \bar{A}_{ka-1}), \quad k = 1,2,\ldots \]

\[ D_k = \bar{A}_{(k-1)a} \ldots \bar{A}_{ka-1}, \quad k = 1,2,\ldots \]
For \( k = 1 \), we have

\[
p_1 = P(\bar{A}_0 \bar{A}_1 \ldots \bar{A}_{a-1}) \geq 1 - aP(A_0) \geq 1 - 2aP(A_0)
\]

and (2.7) follows from (1.30).

Furthermore

\[
p_1 = P(\bar{A}_0 \bar{A}_1 \ldots \bar{A}_{a-1}) = P(\bar{A}_0 \bar{A}_1 \ldots \bar{A}_{a-1} D_2) + P(\bar{A}_0 \bar{A}_1 \ldots \bar{A}_{a-1} \bar{D}_2)
\]

\[
\leq p_2 + P(\bar{D}_2).
\]

Hence from

\[
P(\bar{D}_2) \leq \sum_{i=a}^{2a-1} P(A_i) = aP(A_0),
\]

\[
p_2 \geq p_1 - P(\bar{D}_2) \geq 1 - 2aP(A_0) \geq (1 - aP(A_0))^2 - aP(A_0)
\]

and (2.7) for \( k = 2 \) follows again from (1.30).

For \( k \geq 3 \) we have

\[
p_{k-1} = P(\bar{A}_0 \bar{A}_1 \ldots \bar{A}_{(k-1)a-1}) =
\]

\[
= P(\bar{A}_0 \bar{A}_1 \ldots \bar{A}_{(k-1)a-1} D_k) + P(\bar{A}_0 \ldots \bar{A}_{(k-1)a-1} \bar{D}_k)
\]

\[
\leq p_k + P(\bar{D}_k)p_{k-2}
\]
Using this inequality with \( k-1 \) replaced by \( k-2 \),

\[
\begin{align*}
 p_{k-1} &\leq p_k + p(\overline{B}_k)(p_{k-1} + p(\overline{B}_{k-1})p_{k-3}) \\
 &\leq p_k + p_{k-1}p(\overline{B}_k) + p(\overline{B}_k)p(\overline{B}_{k-1}).
\end{align*}
\]

Since

\[
\begin{align*}
 p(\overline{B}_k) = p(\overline{B}_{k-1}) \leq aP(A_0),
\end{align*}
\]

we obtain

\[
\begin{align*}
 (2.19) \quad p_k \geq (1 - aP(A_0))p_{k-1} - a^2p^2(A_0)
\end{align*}
\]

and by induction it is easy to see that

\[
\begin{align*}
 (2.20) \quad p_k \geq (1 - aP(A_0))^k - aP(A_0)
\end{align*}
\]

Applying (1.30) again, we get (2.7).

\( \square \)

Now we are ready to prove Theorem 2. Lemma 2.1 and Lemma 2.2 clearly imply (1.18) and (1.19), while Lemma 2.1 and Lemma 2.3 imply (1.20). It remains to prove (1.21).
By putting \( a = a_N = [c \log N], k = [N/a_N], \alpha = \alpha(c) \) into the inequality (2.7), one can easily see that

\[
\lim\inf_{N \to \infty} P(\{c \log N, N \geq \alpha(c)\}) > 0,
\]

where \( \alpha(c) \), the solution of (1.17) is an integer. Consequently

\[
P(\{c \log N, N \geq \alpha(c) \text{ i.o.}\}) > 0.
\]

We cannot claim however that the above probability is equal to 1, since

\[
\{c \log N, N \geq \alpha(c) \text{ i.o.}\}
\]

is not a tail event. A small modification of the argument given above, however gives the desired result. Consider

\[
\{ \min_{a_N+1 \leq j \leq N-a_N} \max_{0 \leq i \leq a_N} |S_{j+i} - S_j| \geq \alpha(c) \text{ i.o.}\}
\]

which is already a tail event since \( a_N \to \infty \) as \( N \to \infty \), therefore its probability is either 0 or 1. But similarly to the above argument, one can verify also that the probability of (2.24) is bounded away from 0 and hence
(2.25) \( P( \min_{a_{N+1} \leq j \leq N-a_N} \max_{0 \leq i \leq a_N} |S_{j+1} - S_j| \geq \alpha(c) \text{ i.o.}) = 1. \)

On the other hand it follows from Theorem C that for almost all \( \omega, \)

(2.26) \( \min_{0 \leq j \leq a} \max_{0 \leq i \leq a} |S_{j+i} - S_j| \geq \alpha(c) \)

if \( a \) is large enough, therefore we have also

(2.27) \( P(U([c \log N], N) \geq \alpha(c) \text{ i.o.}) = 1 \)

which together with Lemma 2.2 yields (1.21). This completes the proof of Theorem 2.

3. THE TUBE AROUND ZERO

In this Section we prove our Theorem 3, based on the following 4 lemmas.

LEMMA 3.1. If \( \alpha > 1 \) is an integer and

(3.1) \( \cos \frac{\pi}{2 \alpha} < e^{-\frac{1}{2c}} \)

then for almost \( \omega \) there exists an \( N_0 = N_0(c, \omega) \) such that
(3.2) \[ V(a_N, N) \geq \alpha \quad \text{for} \quad N \geq N_0 \]

where \( V(a, N) \) is defined by (1.16) and \( a_N = [c \log N] \).

**LEMMA 3.2.** If \( \alpha > 2 \) is an integer and

\[
(3.3) \quad \cos \frac{\pi}{2^a} > e^{-\frac{1}{2c}}
\]

then for almost all \( \omega \) there exists an \( N_0^* = N_0^*(c, \omega) \) such that

\[
(3.4) \quad V(a_N, N) < \alpha \quad \text{for} \quad N \geq N_0^*
\]

where \( V(a, N) \) is defined by (1.16) and \( a_N = [c \log N] \).

**LEMMA 3.3.** Let the events \( A_j^a (j = 0, 1, \ldots) \) be defined by

\[
(3.5) \quad A_j^a = \{ \max_{0 \leq t \leq a} |S_{j+i} < \alpha^*(c)\}
\]

where the solution \( \alpha^*(c) \) of (1.22) is an integer. Then

\[
(3.6) \quad P(A_N^a_{N-a_N} \text{ i.o.}) = 1,
\]

where \( a_N = [c \log N] \).

**LEMMA 3.4.** The following inequality holds true for \( \alpha > 1, n > a \) and
large enough:

\[ P(\hat{\nu}(a,n+a) \geq \alpha) \geq \]

\[ \geq \prod_{j=a+1}^{n} \left( 1 - \frac{K^*}{\sqrt{j}} \left( \cos \frac{\pi}{2\alpha} \right) \right) \tag{3.7} \]

with some constant \( K^* \) (which may depend on \( \alpha \), but not on \( a \) and \( n \)),

where

\[ \hat{\nu}(a,n+a) = \min_{a<j<n} \max_{0<i<a} |S_{j+i}| \tag{3.8} \]

PROOF OF LEMMA 3.1.

Define

\[ A_j = A_j^\alpha = \{-\alpha < S_{j+i} < \alpha, \ i = 0,1,\ldots,a\} \ (j \geq 0). \tag{3.9} \]

Then

\[ P(A_j) = \sum_{z=-\alpha+1}^{\alpha-1} P(A_j|S_j = z))(S_j = z). \tag{3.10} \]

From Stirling's formula we obtain

\[ P(S_j = z) \leq \frac{K_0}{\sqrt{j}} \quad \text{for } -\alpha < z < \alpha, \ j > 0 \tag{3.11} \]

with certain constant \( K_0 \), depending only on \( \alpha \). Hence
(3.12) \[ P(A_j) \leq \frac{K_0}{\sqrt{\alpha}} \sum_{z=-\alpha+1}^{\alpha-1} P(A_j | S_j = z). \]

Applying (1.30), we get

(3.13) \[ P(A_j) \leq \frac{K^*}{\sqrt{\alpha}} \left( \cos \frac{\pi}{2\alpha} \right)^a. \]

Therefore

\[
P(V(a_{N,N} < \alpha)) \leq \sum_{j=0}^{N-a_N} P(A_j) \leq K^*(\cos \frac{\pi}{2\alpha})^a N \left( 1 + \sum_{j=1}^{N-a_N} \frac{1}{\sqrt{j}} \right) \leq K^* \sqrt{N-a_N} \left( \cos \frac{\pi}{2\alpha} \right)^a N.
\]

Considering a subsequence \( N_k = k^0 k = 1, 2, \ldots \) with integer \( \rho > 0 \), we clearly have for \( N_k \leq N < N_{k+1} \)

(3.15) \[ V(a_{N,N}) \geq V(a_{N_k,N_{k+1}}) \]

hence it is enough to prove that under the conditions of our Lemma 3.1 for almost all \( \omega \) there exists \( k^* = k^*(c, \omega) \) such that

(3.16) \[ V(a_{N_k,N_{k+1}}) \geq \alpha \quad \text{for} \quad k \geq k^*. \]
From (3.14) we obtain

$$\sum_{k=1}^{\infty} P(V(a_{N_k}, N_{k+1}) < a) \leq \sum_{k=1}^{\infty} K^*(\alpha) \sqrt{N_{k+1}} \left( \cos \frac{\pi}{2\alpha} \right)^{a_{N_k}} \leq \sum_{k=1}^{\infty} K^*(\alpha) \sqrt{N_{k+1}} \left( \cos \frac{\pi}{2\alpha} \right) c \log N_k \leq \sum_{k=1}^{\infty} K^*(\alpha) (k+1)^{\rho/2} k^{\rho} c \log \cos \frac{\pi}{2\alpha}
(3.17)$$

The last number (3.17) clearly converges if

$$\frac{\rho}{2} + \rho \ c \log \cos \frac{\pi}{2\alpha} < -1$$

which is equivalent to

$$\cos \frac{\pi}{2\alpha} < e^{-\frac{2+\rho}{2\rho c}}. \quad (3.18)$$

If (3.1) holds then there exists a big enough integer $\rho$ which satisfies (3.18) implying the convergence of (3.17) and hence our lemma.

PROOF OF LEMMA 3.2.

To prove Lemma 3.2, we need the following result. Let
(3.19) \[ T_n = \min\{j: \ 0 \leq j, S_{n+j} = 0\} \]

We define a sequence of stopping times as follows. Let \( a \) be a positive integer, and let

\[ \eta_0(a) = 0, \quad \xi_k(a) = \eta_{k-1}(a) + a, \quad \eta_k(a) = T_{\xi_k(a)} \quad k=1,2. \]

Denote \( \alpha_k(a) = \eta_k(a) = \xi_k(a) \)

Then clearly \( \{\alpha_k(a)\} \ k=1,2,... \) is a sequence of independent identically distributed random variables having the same distribution as \( T_a \).

Let \( v_N \) be the largest integer for which

\[ \sum_{i=1}^{v_N} \alpha_i(a_N) + (v_N + 1)a_N \leq N \tag{3.10} \]

For \( v_N \) we proved the following result (see Csáki and Földes [3] and also for more general recurrent random walk in Csáki and Földes [4]).

**Lemma 3.3.** If \( a_N < \frac{1}{2}N \) then there exists a small enough \( C_1 \) and a big enough \( C_2 \) such that for any \( 0 < \epsilon < 1 \)

\[ P(v_N \leq C_1(\frac{N}{a_N})^{\frac{1-\epsilon}{2}}) \leq C_2(\frac{a_N \epsilon}{N})^\frac{\epsilon}{2}. \tag{3.11} \]
Now we are ready to prove Lemma 3.2. From the inequality
\[ V(a^N_N, N) \leq \inf_{0 \leq k < \nu_N} \sup_{0 \leq i < a_N} |S_{\eta_k(a_N) + i}| \] we obtain the following estimation. Let \( A_j \) be defined by (3.9). Then

(3.22) \[ P(V(a^N_N, N) \geq \alpha) = P\left( \bigcap_{j=0}^{N-a_N} A_j \right) \leq \]

\[ \nu_N \]

\[ P\left( \bigcap_{k=0}^{\nu_N} A_{\eta_k(a_N)} \right) \leq \]

\[ \nu_N \]

\[ P\left( \bigcap_{k=0}^{\nu_N} A_{\eta_k(a_N)}, \nu_N < C_1\left(\frac{N}{a_N}\right)^2 \right) + \]

\[ \nu_N \]

\[ P\left( \bigcap_{k=0}^{\nu_N} A_{\eta_k(a_N)}, \nu_N \geq C_1\left(\frac{N}{a_N}\right)^2 \right) \leq \]

\[ C_2\left(\frac{a_N^{-\epsilon}}{N^2}\right) + (1 - P(A_0)) \]

by Lemma 3.3.

Consequently applying (1.30) we have for \( a_N = \lceil c \log N \rceil \)

(3.23) \[ P(V(a^N_N, N) \geq \alpha) \leq C_2\left(\frac{c \log N}{N}\right)^{\frac{\epsilon}{2}} + \]

\[ + \left(1 - K_1\left(\cos \frac{\pi}{2a}\lceil c \log N \rceil\right)\right)^{\frac{1-\epsilon}{2}} \]
\[ \leq C_2\left(\frac{c \log N}{N}\right)^2 + \exp\left\{-K_1^*(\cos \frac{\pi}{2\alpha})\right\} \left(\frac{c \log N}{\log N}\right)^{1-\epsilon} \]

Choosing a sequence \(N_k = k^\rho\), \(k = 1, 2, \ldots\) similarly to the proof of Lemma 3.1, it is enough to show that

\[ \sum_{k=1}^{\infty} P(V(a_{N_k+1}, N_k) \geq \alpha) < +\infty. \]

\[
\sum_{k=1}^{\infty} P(V(a_{N_k+1}, N_k) \geq \alpha) \leq \sum_{k=1}^{\infty} \left( C_2\left(\frac{c \rho \log (k+1)}{k^\rho}\right)^2 \right) + \\
+ \exp\left\{-K_1^*(\cos \frac{\pi}{2\alpha})\left(\frac{c \rho \log (k+1)}{c \rho \log (k+1)}\right)^{1-\epsilon}\right\}
\]

\[ = \sum_{k=1}^{\infty} \left( C_2\left(\frac{c \rho \log (k+1)}{k^\rho}\right)^2 + \exp\left\{-K_1^*(k+1)\right\} \frac{c \rho \log \cos \frac{\pi}{2\alpha}}{(c \rho \log (k+1))^{1-\epsilon}} \right) \]

which is clearly convergent if

\[ c \rho \log \cos \frac{\pi}{2\alpha} + \rho \frac{1-\epsilon}{2} > 0 \]

and

\[ \frac{\rho \epsilon}{2} > 1 \]
Under the condition of our Lemma we may choose a small enough 
\( \varepsilon > 0 \) such that

\[
(3.27) \quad \cos \frac{\pi}{2\alpha} > e^{-\frac{1-\varepsilon}{2c}}
\]

and to this \( \varepsilon \) we might choose an integer \( \rho > 0 \) such that (3.26)
should hold. For this choice of \( \rho \) and \( \varepsilon \) (3.25) is valid and this proves
(3.24) implying the lemma.

\[ \square \]

PROOF OF LEMMA 3.3.

From (1.30), (3.10) and the local central limit theorem one can
obtain that

\[
(3.28) \quad P\left( A_j^a \right) \geq \frac{K^*}{\sqrt{\rho}} \left( \cos \frac{\pi}{2\alpha} \right)^a
\]

if \( a \) and \( j \) are large enough, where \( A_j^a \) is defined by (3.9). Hence
from (1.22),

\[
(3.29) \quad P\left( A_N^{a_n} \right) \geq \frac{K^*}{N}
\]

and we may choose a subsequence \( N_k = [(c+1)k \log k] \) such that

\[
(3.30) \quad N_k < N_{k+1} - a_{N_{k+1}}
\]
for \( k \) large and

\[
(3.31) \quad \sum_k^a N_k \ P(\frac{A_n^{N_k}}{N_k} - \frac{a}{N_k}) = \infty.
\]

But this is not enough, because in this case the events in the brackets are not independent, even for large \( k \). We apply the following version of Borel-Cantelli lemma due to Erdős and Rényi (see Rényi [8]):

If \( \sum_k P(B_k) = \infty \) and

\[
(3.32) \quad \lim_{n \to \infty} \inf \frac{\sum_{k=1}^{n} \sum_{j=1}^{n} P(B_k B_j)}{(\sum_{k=1}^{n} P(B_k))^2} \leq 1,
\]

then \( P(B_i \ i.o.) = 1 \). To verify (3.32) for \( B_k = \frac{a}{N_k} - \frac{a}{N_k} \), consider two events \( a_{j_1}^{a_1} \) and \( a_{j_2}^{a_2} \), where \( j_1 + a_1 < j_2 \). Then the probability of joint occurrence of these two events can be written as

\[
(3.33) \quad P(a_{j_1}^{a_1} a_{j_2}^{a_2}) = \\
= \sum_{x_1 = -a+1}^{a-1} \sum_{x_2 = -a+1}^{a-1} P(a_{j_1}^{a_1} | S_{j_1} + a_1 = x_1) P(S_{j_1} + a_1 = x_1) \times \\
\times P(a_{j_2}^{a_2} | S_{j_2} = x_2) P(S_{j_2} = x_2 | S_{j_1} + a_1 = x_1),
\]

while
\[ P(A_j^1)P(A_j^2) = \]

\[ \prod_{x_1 = -\alpha + 1}^{\alpha - 1} \prod_{x_2 = -\alpha + 1}^{\alpha - 1} P(A_j^1 | S_j^1 + a_1 = x_1) P(S_j^1 + a_1 = x_1) \times P(A_j^2 | S_j^2 = x_2) P(S_j^2 = x_2). \]

From the local limit theorem (or from Stirling's formula),

\[ P(S_j^2 = x_2) \sim \frac{\sqrt{2}}{\sqrt{\pi j_2}} \text{ for } -\alpha < x_2 < \alpha \]

and

\[ P(S_j^2 = x_2 | S_j^1 + a_1 = x_1) \sim \frac{\sqrt{2}}{\sqrt{\pi (j_2 - j_1 - a_1)}} \]

\[ \text{for } -\alpha < x_1 < \alpha, \]

\[ -\alpha < x_2 < \alpha, \]

provided \( j_2 \) and \( j_2 - j_1 - a_1 \) are large enough and \( j_2 \) and \( j_1 \) have the same parity. Hence it can be seen that for all \( \varepsilon > 0 \),

\[ P(A_{j_1}^1A_{j_2}^2) \leq (1 + \varepsilon) \sqrt{\frac{j_2}{j_2 - j_1 - a_1}} P(A_{j_1}^1)P(A_{j_2}^2) \]

\[ (3.37) \]
provided $j_2 - j_1 - a_1$ is large enough. Since $N_{k+1} - a_{N_{k+1}} - N_k \to \infty$ as $k \to \infty$, we have for large enough $k$ and $k < \xi$,

$$(3.38) \quad P(B_k B_{\xi}) \leq (1+\epsilon) \sqrt{\frac{N_{\xi} - a_{N_{\xi}}}{N_k - a_{N_k} - N_k}} P(B_k)P(B_{\xi}).$$

Now from (3.38) one can verify (3.32) similarly to Csáki-Csörgő-Földes-Révész [1, Lemma 3.4.]

PROOF OF LEMMA 3.4.

Assume that $a$ is large enough, so that the inequality (1.30) holds true. Let $A_j$ be defined by (3.9). We show that

$$(3.39) \quad P(A_{a+1} \ldots A_n) \geq P(A_{a+1}) \ldots P(A_n),$$

provided $a$ is large enough and $n > a$.

We start from the following identity:

$$P(A_{a+1} \ldots A_n) = P(A_{a+1} \ldots A_{n-1}) - P(A_n) + \sum_{k=a+1}^{n-1} P(A_{a+1} \ldots A_{k-1} A_k A_n).$$

(3.40)

The next step is to show that
(3.41) \[ P(\overline{a}_{a+1} \ldots \overline{a}_{k-1} \overline{A}_k A_n) \geq P(\overline{a}_{a+1} \ldots \overline{a}_{k-1} \overline{A}_k) P(A_n) \]

We distinguish three cases.

Case (i). \( n-a < k < n-1 \).

In this case we have

\[ P(\overline{a}_{a+1} \ldots \overline{a}_{k-1} \overline{A}_k A_n) = \]

(3.42) \[ \sum_{x=-\alpha+1}^{\alpha-1} P(\overline{a}_{a+1} \ldots \overline{a}_{k-1} \overline{A}_k A_n | S_{k+a} = x) P(S_{k+a} = x) = \]

\[ = \sum_{x=-\alpha+1}^{\alpha-1} P(\overline{a}_{a+1} \ldots \overline{a}_{k-1} \overline{A}_k | S_{k+a} = x) P(\max_{k+a \leq \overline{i} \leq n+a} |S_{\overline{i}}| < \alpha |S_{k+a} = x) P(S_{k+a} = x) \]

But from (1.30),

(3.43) \[ P(\max_{k+a \leq \overline{i} \leq n+a} |S_{\overline{i}}| < \alpha |S_{k+a} = x) \geq K_1 (\cos \frac{\pi}{2\alpha})^{n-k} \]

for \(-\alpha < x < \alpha\), \(P(S_{k+a} = x) > 0\), and from (3.13),

(3.44) \[ P(A_n) \leq \frac{K^*}{\sqrt{n}} (\cos \frac{\pi}{2\alpha})^a. \]

Since \(n-k \leq a\), we have for large enough \(n\),

(3.45) \[ P(\max_{k+a \leq \overline{i} \leq n+a} |S_{\overline{i}}| < \alpha |S_{k+a} = x) \geq P(A_n) \]
and this together with (3.42) yields (3.41).

Case (ii). \( n-a-\sqrt{a} \leq k < n-a \).

In this case we have

\[
P(A_{a+1} \ldots A_{a+k-1} A_k A_n) = \\
= \sum_{x=-a+1}^{\alpha-1} P(A_{a+1} \ldots A_{a+k-1} A_k A_n | S_{k+a} = x) P(S_{k+a} = x)
\]

(3.46)

\[
= \sum_{x=-a+1}^{\alpha-1} P(A_{a+1} \ldots A_{a+k-1} A_k | S_{k+a} = x) P(A_n | S_{k+a} = x) P(S_{k+a} = x).
\]

But from (1.30),

\[
P(A_n | S_{k+a} = x) = \sum_{y=-\alpha+1}^{\alpha-1} P(A_n | S_n = y) P(S_n = y | S_{k+a} = x)
\]

(3.47)

\[
\geq K_1 (\cos \frac{\pi}{2a})^a \sum_{y=-\alpha+1}^{\alpha-1} P(S_n = y | S_{k+a} = x)
\]

\[
= K_1 (\cos \frac{\pi}{2a})^a P(|S_n| < \alpha | S_{k+a} = x)
\]

From the local limit theorem

(3.48) \[ P(|S_n| < \alpha | S_{k+a} = x) \geq \frac{K_1^*}{\sqrt{n-k-a}} \geq \frac{K_1^*}{\sqrt{a}} \]

for \(-\alpha < x < \alpha\) and \(P(S_{k+a} = x) > 0\), hence by (3.44) for large enough \(a\),
(3.49) \[ P(A_n|S_{k+a} = x) \geq P(A_n) \]

and (3.46) yields (3.41).

Case (iii). \( a+1 \leq k < n-a-\sqrt{a} \).

Compare

(3.50) \[ P(A_n) = \sum_{y=-\alpha+1}^{\alpha-1} P(A_n|S_n = y)P(S_n = y) \]

and

(3.51) \[ P(A_n|S_{k+a} = x) = \sum_{y=-\alpha+1}^{\alpha-1} P(A_n|S_n = y)P(S_n = y|S_{k+a} = x) \]

Then either both \( P(S_n = y) \) and \( P(S_n = y|S_{k+a} = x) \) are zero or by the local limit theorem (or Stirling's formula)

(3.52) \[ P(S_n = y) \sim \frac{\sqrt{\pi}}{\sqrt{n}} \quad (n \to \infty) \]

and

(3.53) \[ P(S_n = y|S_{k+a} = x) \sim \frac{\sqrt{\pi}}{\sqrt{n-k-a}} \quad (n-k-a \to \infty) \]

Therefore

(3.54) \[ P(S_n = y) \leq P(S_n = y|S_{k+a} = x) \]
for $-\alpha < x < \alpha$, $-\alpha < y < \alpha$ and a large enough, which gives also (see (3.50) and (3.51))

\[(3.55) \quad P(A_n) \leq P(A_n | S_{k+a} = x)\]

implying (3.41) as in case (ii).

(3.40), (3.41) and a simple induction argument yield (3.39) and this together with (3.13) proves (3.7).

Now we can complete the proof of Theorem 3 similarly to the proof of Theorem 2. Lemma 3.1 and Lemma 3.2 imply (1.23) and (1.24), while Lemma 3.1 and Lemma 3.3 imply (1.25). So we have to prove (1.26).

By putting $a = a_N = [c \log N]$, $n = N - a_N$ into (3.7) one can easily see that

\[(3.56) \quad \lim_{N \to \infty} \inf P(\hat{V}(a_N, N) \geq \alpha^*(c)) > 0\]

where $\alpha^*(c)$, the solution of (1.22) is an integer. Consequently

\[(3.57) \quad P(\hat{V}(a_N, N) \geq \alpha^*(c) \text{ i.o.}) > 0.\]

But the event

\[(3.58) \quad \hat{V}(a_N, N) \geq \alpha^*(c) \text{ i.o.}\]
is a tail event (for $S_n$), therefore its probability is either 0 or 1. Moreover

$$\min_{0 \leq j < a} \max_{0 \leq i \leq a} |S_{j+i}| \geq \alpha^*(c) \text{ a.s.}$$

for large enough $a$ follows from Theorem 1. Hence we have also

$$P(V([c \log N], N) \geq \alpha^*(c) \text{ i.o.}) = 1$$

and this with Lemma 3.2 implies (1.26). The proof of Theorem 3 is complete.
REFERENCES


