r-OPTIMAL DECISION PROCEDURES FOR SELECTING THE BEST POPULATION IN RANDOMIZED COMPLETE BLOCK DESIGN

by

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ABSTRACT

In randomized complete block design, we face the problem of selecting the best population. If some partial information about the unknown parameters is available, then we wish to determine the optimal decision rule to select the best population.

In this paper, in the class of natural selection rules, we employ the $r$-optimal criterion to determine optimal decision rules that will minimize the maximum expected risk over the class of some partial information. Furthermore, the traditional hypothesis testing is briefly discussed from the view point of ranking and selection.

1. INTRODUCTION

In randomized complete block design (R.C.B.D) with one observation per cell, we can express the observable random variable $X_{i\xi}$ ($i = 1,\ldots,k$, $\xi = 1,\ldots,n$) as
\[ X_{i\ell} = \mu + \tau_i + \beta_\ell + \epsilon_{i\ell}, \quad \sum_{i=1}^{k} \tau_i = 0. \] (1.1)

where \( \mu \) is the overall mean, \( \tau_i \) is the \( i \)-th treatment effect, \( \beta_\ell \) is the \( \ell \)-th block effect, and \( \epsilon_{i\ell} \) is the error component of \((i, \ell)\) cell. We assume that the errors within each block are jointly normally distributed. We also assume that the quality of a treatment is judged by the largeness of \( \tau_i \)'s values. The \( i \)-th population is called the best if \( \tau_i = \max_{1 \leq \ell \leq k} \tau_\ell \). In many practical situations, the goal of the experimenter is to select the best population.

In this paper, we shall use \( \Gamma \)-optimal criterion to determine the sample size of a natural selection procedure so that it will minimize the maximum expected selection risk over the class of some partial information [cf. Gupta and Huang (1976)].

In Section 2 some basic definitions and notations are introduced and basic formulation of the problem is also given. In Section 3, some useful expressions for the probability of correct selection (PCS) is derived, and \( \Gamma \)-optimal sample size is determined. Section 4 deals with a numerical example for illustrative purpose. In Section 5, we discuss the relationship between \( \Delta \) and \( n^* \). Section 6 includes some conclusions and a discussion of the traditional hypothesis testing from the view point of ranking and selection. For general reference of multiple decision procedures, see Gupta and Panchapakesan (1979) and Gupta and Huang (1981).

2. BASIC FORMULATION OF THE SELECTION PROBLEM

In R.C.B.D., as (1.1), we assume that \( \epsilon_{\ell} = (\epsilon_{1\ell}, \ldots, \epsilon_{k\ell})' \): error components within \( \ell \)-th block have jointly a multivariate normal distribution with mean vector \( \mathbf{0} = (0, \ldots, 0)' \) and covariance matrix \( \Sigma = \sigma^2 \begin{pmatrix} 1 & \cdots & \lambda \\ \vdots & \ddots & \vdots \\ \lambda & \cdots & 1 \end{pmatrix} \), where \( \sigma^2 \) is unknown and \( \lambda \) is a known constant. Thus, \( (X_{\ell} = X_{1\ell}, \ldots, X_{k\ell})' \) have joint multivariate
normal distribution with mean vector \( \theta_i = (\theta_{1i}, \ldots, \theta_{ki})' \) and
covariance matrix \( \Sigma \), where \( \theta_{ij} = (\mu + \tau_i + \beta_j) \) for all \( i \),
\( 1 \leq i \leq k \), and any \( \ell \), \( 1 \leq \ell \leq n \). For all \( i \), \( 1 \leq i \leq k \), define
\[
\bar{X}_i = \left( \sum_{\ell=1}^{n} X_{i\ell} / n \right). \quad \text{Then } Y_i = (\bar{X}_1, \ldots, \bar{X}_i - \bar{X}_k)' \text{ are jointly suffi-
cient for } (\tau_1 - \tau_i, \ldots, \tau_i - \tau_k)' \text{. Now, if } \tau_i = \max_{\ell \neq i} \tau_{\ell} \text{ then}
\tau_i - \tau_\ell \geq 0 \text{ for all } 1 \leq \ell \leq k. \text{ We consider a class of natural}
selection rules for } i-th \text{ population, } 1 \leq i \leq k, \text{ as:}
\[
\delta^{(i)}(y_n) = \begin{cases} 1 & \text{if } \bar{X}_i \geq \max_{\ell \neq i} \bar{X}_\ell \\ 0 & \text{otherwise} \end{cases} \quad (2.1)
\]
where \( y_n = (\bar{X}_1, \bar{X}_2, \ldots, \bar{X}_k)' \). Some optimal properties have been
studied by several authors (see Gupta and Panchapakesan (1979)).
So the class of natural selection rules can be denoted by:
\[
D = \{ \delta(y_n) | \delta(y_n) = (\delta^{(1)}(y_n), \ldots, \delta^{(k)}(y_n))' \}. \quad (2.2)
\]
The parameter space \( \Omega \) is as follows:
\[
\Omega = \{ \tau = (\tau_1, \ldots, \tau_k)' | \tau_i \in \mathbb{R} \text{ for all } i = 1, \ldots, k \}. \quad (2.3)
\]
Let \( \Delta \) be a given positive constant, and for all \( i, 1 \leq i \leq k \),
\[
\Omega_i = \{ \tau = (\tau_1, \ldots, \tau_k)' | \tau_i \geq \tau_\ell + \Delta \sigma \text{ for all } \ell \neq i \}, \quad (2.4)
\]
\[
\Omega_0 = \{ \tau | \tau_1 = \ldots = \tau_k \}, \quad (2.5)
\]
and \( \Omega_{k+1} = \Omega - \bigcup_{i=0}^{k} \Omega_i \).

Let \( L^{(i)}(\tau; \delta^{(j)}(y_n)) \) represent the loss function for \( \tau \in \Omega_i \),
\( 0 \leq i \leq k+1 \), when the \( j \)-th population, \( 1 \leq j \leq k \), is selected.
Let for \( 1 \leq j \leq k \),
\[
L^{(i)}(\tau; \delta^{(j)}(y_n)) = \begin{cases} (c_0n)\delta^{(j)}(y_n) & \text{if } \tau \in \Omega_0 (i = 0) \\ \frac{\tau_i - \tau_j}{\sigma} \delta^{(j)}(y_n) & \text{if } \tau \in \Omega_i (1 \leq i \leq k) \\ 0 & \text{if } \tau \in \Omega_{k+1} (i = k+1) \end{cases} \quad (2.6)
\]
where \( \lambda \) is some positive increasing function such that \( \lambda(0) = 0 \) and \( \lambda(x) = o(e^{cx^2}) \), \( c > 0 \), and \( c_0 \) represents the sampling cost from each population \((c_0 > 0)\). So, for all \( i \in \Omega_i \), \( 0 \leq i \leq k+1 \), the loss function of \( \delta(x_n) \) is defined as:

\[
L(i; \delta(x_n)) = \frac{k}{j=1} \sum_{j=1}^{k} L(i; \delta(j)(x_n)).
\]

Similarly, we have

\[
R(i; \delta_n) = E\{L(i; \delta(x_n))\}
\]

and for some \( \rho \) (prior distribution) over \( \Omega \), \( \gamma(i; \delta_n) \) is defined as:

\[
\gamma(i; \delta_n) = E\{R(i; \delta_n)\}.
\]

Thus, the Bayes risk of \( \delta_n \) w.r.t. \( \rho \) is defined as

\[
\gamma(\rho; \delta_n) = \sum_{i=0}^{k+1} \gamma(i; \delta_n).
\]

In this selection problem, it is assumed that some partial information is available. So that we can specify \( \pi_i = P_i(\tau \in \Omega_i) \), for all \( i, 0 \leq i \leq k+1 \) and define

\[
\Gamma = \{ \rho \mid \int_{\Omega_i} \rho(\tau) d\rho = \pi_i, \sum_{i=0}^{k+1} \pi_i = 1, 0 \leq i \leq k+1 \}.
\]

If there is no prior information, we can assume that \( \pi_0 = \ldots = \pi_k = (1 - \pi_{k+1})/(k+1) \).

Now if there exists \( n^* \) such that

\[
\sup_{\rho \in \Gamma} \gamma(\rho; \delta_{n^*}) = \inf \{ \sup_{\delta_n \in D} \gamma(\rho; \delta_n) \}.
\]

Then \( \delta_{n^*} \) is called a \( \Gamma \)-optimal decision rule and \( n^* \) is the \( \Gamma \)-optimal decision. In the following discussion, we will determine \( \delta_{n^*} \) for this selection problem.
3. MAIN RESULTS

We can easily show the following lemma

Lemma 3.1. Suppose for any \( \ell, 1 \leq \ell \leq k \), \( \mathbf{x}_\ell = (x_{1\ell}, \ldots, x_{k\ell})' \) follows a multivariate normal distribution with mean vector \( \theta_\ell = (\mu + \tau_1 + \beta_\ell, \ldots, \mu + \tau_k + \beta_\ell)' \) and covariance matrix

\[
\Sigma = \sigma^2 \begin{pmatrix}
1 & \cdots & \lambda \\
\vdots & \ddots & \vdots \\
\lambda & \cdots & 1/k
\end{pmatrix}.
\]

Let \( Y_i = (\bar{x}_{1i} - \bar{x}_1, \ldots, \bar{x}_{ki} - \bar{x}_k)' \) and \( p_1(\tau) = P_r(\bar{x}_i \geq \max_{\ell \neq i} \bar{x}_\ell) \) where

\[
\bar{x}_i = \left( \frac{\sum_{\ell=1}^{n} x_{i\ell}}{n} \right) \text{ for any } i, (i = 1, \ldots, k).
\]

Then

a) \( Y_i \) follows multivariate normal distribution with mean vector \( (\tau_1 - \tau_1, \ldots, \tau_k - \tau_k)' \) and covariance matrix

\[
\frac{2\sigma^2(1-\lambda)}{n} \begin{pmatrix}
1 & \cdots & \frac{1}{k} \\
\vdots & \ddots & \vdots \\
\frac{1}{k} & \cdots & 1
\end{pmatrix} (k-1) \tag{3.1}
\]

b) \( p_1(\tau) = \phi_{k-1} \left( \frac{(\tau_1 - \tau_1)/\sigma}{\sqrt{2(1-\lambda)/n}}, \ldots, \frac{(\tau_k - \tau_k)/\sigma}{\sqrt{2(1-\lambda)/n}} \right) \), \( \tag{3.2} \)

and

c) \( p_1(\tau) = \int_{-\infty}^{\infty} \prod_{i \neq i} \phi \left( \frac{z + \frac{(\tau_i - \tau_i)/\sigma}{\sqrt{2(1-\lambda)/n}}}{\sqrt{(1-\lambda)/n}} \right) dz, \tag{3.3} \)

where \( \phi_{k-1}(\cdot) \) denotes the c.d.f. of \((k-1)\)-variate normal distribution with mean vector \( \hat{\theta} = (0, \ldots, 0)' \) and covariance matrix

\[
\Lambda = \begin{pmatrix}
1 & \cdots & \frac{1}{k} \\
\vdots & \ddots & \vdots \\
\frac{1}{k} & \cdots & 1
\end{pmatrix} (k-1)
\]

and \( \phi(\cdot) \) denotes the c.d.f. of standard normal distribution.
Theorem 3.1. In R.C.B.D., for fixed $\Delta$ and $\lambda$, let
\[ Q_{M}(n) = \sup_{\frac{g}{g \geq \Delta}} (\tau_{g_{K-1}}(\frac{g}{\sqrt{2(1-\lambda)/n}}, \ldots, \frac{g}{2(1-\lambda)/n})) \] (3.4)
and $H(n) = \sup_{\rho \in \Gamma} \gamma(\rho; \delta_n)$. Then there exists $n^*$ such that
\[ H(n^*) = \inf_{n \geq 1} H(n). \]
where $n^* = \begin{cases} <n^*_0 & \text{if } H(<n^*_0>) \leq H([n^*_0]) \\ \lceil n^*_0 \rceil & \text{if } H(<n^*_0>) > H([n^*_0]) \end{cases}$ and $n^*_0$ is a positive real number assumed to exist satisfying the following equations.

a) $Q_{M}^{\prime}(n^*_0) = -(c_0^n)/((1-\pi_0-\pi_{K+1})$)

b) $Q_{M}^{\prime \prime}(n^*_0) > 0$

$x$ ([x]) denotes the smallest (largest) integer which is larger (less) than or equal to $x$.

Proof. For any $\delta_n \in D$, we have
\[ \gamma(\rho; \delta_n) = \sum_{i=0}^{k+1} \int_{\Omega_i} R^{(i)}(\tau; \delta_n) d\phi(\tau) \]
\[ = (c_0^n) \sum_{j=1}^{k} \int_{\Omega_j} p_j(\tau) d\phi(\tau) + \sum_{i=1}^{k} \sum_{j=1}^{k} \frac{\tau_i - \tau_j}{\sigma} p_j(\tau) d\phi(\tau). \]

Since

1) $\sup_{\tau \in \Omega_0} p_j(\tau) = \int_{-\infty}^{\infty} e^{-x-1}(z) d\phi(z) = \frac{1}{k}$.

2) By Somerville's paper (1954) for $1 \leq i \leq k$, we have
\[ \sup_{\tau \in \Omega_i} \sum_{j=1}^{k} \frac{\tau_i - \tau_j}{\sigma} p_j(\tau) = \sup_{g_{i} \geq \Delta} \sum_{j \neq i} \sum_{g_{j}} \lambda(g_{i}) p_{j} \]
\[ = \sup_{g_{i} \geq \Delta} (\tau_{g_{K-1}}(\frac{g_{i}}{\sqrt{2(1-\lambda)/n}}, \ldots, \frac{g_{i}}{2(1-\lambda)/n})) = Q_{M}(n). \]
Thus, \( H(n) = \sup_{\rho \in \Gamma} \gamma(\rho; \xi_n) = n \cdot (c_0 \pi_0) + Q_M(n)(1-\pi_0-\pi_{k+1}). \)

Since there exists \( n^*_0 \) such that

\[
\begin{align*}
(a) & \quad Q'_M(n^*_0) = -(c_0 \pi_0)/(1-\pi_0-\pi_{k+1}) \\
(b) & \quad Q''_M(n^*_0) > 0,
\end{align*}
\]

thus, we have
\[
H'(n^*_0) = 0 \text{ and } H''(n^*_0) > 0.
\]

\[<n^*_0> \text{ if } H(<n^*_0>) \leq H([n^*_0])
\]

So \( n^* = \begin{cases} 
[n^*_0] & \text{if } H(<n^*_0>) > H([n^*_0]).
\end{cases} \)

**Lemma 3.2.** (Slepian (1962)) Let \((X_1, \ldots, X_n)\) be multivariate normal with zero mean and positive definite covariance matrix \( \Sigma_1 = \{\rho_{ij}\} \) and \((Y_1, \ldots, Y_n)\) be multivariate normal with zero mean and positive definite covariance matrix \( \Sigma_2 = \{\kappa_{ij}\} \). Let \( \rho_{ij} \geq \kappa_{ij} \) for \( i, j = 1, \ldots, n \) and \( \rho_{ii} = \kappa_{ii}, \; i = 1, \ldots, n \). Then
\[
P_r(X_1 \geq a_1, \ldots, X_n \geq a_n) \geq P_r(Y_1 \geq a_1, \ldots, Y_n \geq a_n).
\]

We need a numerical solution for \( n \) to satisfy the infimum of \( H(n) \).

**Theorem 3.2.** Let \( \xi(\cdot) \) be a positive increasing function such that \( \xi(x) = o(e^{cx^2}), \; (c > 0) \),
\[
Q_M(n) = \sup_{g \geq \Delta} \{ \xi(g)(1-\phi_{k-1})(g, \ldots, g) \sqrt{2(1-\lambda)} \sqrt{n} \}
\]

and \( H(n) = n c_0 \pi_0 + (1-\pi_0-\pi_{k+1})Q_M(n) \). Let
\[
n_0 = <(k-1)^2(1-\lambda)(c_0 \pi_0 \sqrt{n} \pi_{k+1})^2 \frac{(\xi(g))^2}{g^2 \; \xi^2(\Delta)/2(1-\lambda)} >. \tag{3.6}
\]

Then
\[
\inf_{n \geq 1} H(n) = \inf_{n \geq n_0} H(n)
\]
where $g_\ast$ is such that \[ \frac{\ell(g_\ast)}{g_\ast} e^{-\frac{g_\ast^2}{4(1-\lambda)}} = \sup_{g \geq \Delta} \left\{ \frac{\ell(g)}{g} e^{-\frac{g^2}{4(1-\lambda)}} \right\}. \]

**Proof.** By Lemma 3.2, we have
\[ 1-\phi_{k-1}(t, \ldots, t) \leq 1-\phi^k(t) \leq (k-1)(1-\phi(t)) \]
and
\[ 1-\phi(t) \leq \frac{1}{k} \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}, \text{ for } t > 0. \]
Thus
\[ Q_M(n) \leq \sup_{g \geq \Delta} \left\{ \frac{\ell(g)(k-1)}{\sqrt{g^2/(2(1-\lambda))}} \right\} \]
\[ \leq (k-1) \frac{\sqrt{(1-\lambda)/\pi}}{\sqrt{n}} \sup_{g \geq \Delta} \left\{ \frac{\ell(g)}{g} e^{-\frac{g^2}{4(1-\lambda)}} \right\}. \]

Since $\ell(g)$ is a positive increasing function such that $\ell(g) = o(e^{c_g^2})$ ($c > 0$), then there exists $g_\ast$ such that
\[ \sup_{g \geq \Delta} \left\{ \frac{\ell(g)}{g} e^{-\frac{g^2}{4(1-\lambda)}} \right\} = \frac{\ell(g_\ast)}{g_\ast} e^{-\frac{g_\ast^2}{4(1-\lambda)}}. \]

By (3.7) and (3.8), as $n \to \infty$, $Q_M(n)$ decreases to 0. We can find
\[ n_0 = \langle \left\{ \frac{(k-1)^2(1-\lambda)}{\pi} \left( \frac{1-\pi_0-\pi_k+1}{c_0\pi_0} \right)^2 \frac{(\ell(g_\ast))^2}{g_\ast^2 e^{2/(1-\lambda)}} \right\} \rangle \]
to satisfy
\[ Q_M(n_0) \leq \frac{c_0\pi_0}{1-\pi_0-\pi_k+1}. \]

Now for any $n \geq n_0$, $Q_M(n) - Q_M(n+1) \leq Q_M(n_0) \leq \frac{c_0\pi_0}{1-\pi_0-\pi_k+1}$, and $H(n+1) - H(n) = c_0\pi_0 - (1-\pi_0-\pi_k+1)(Q_M(n) - Q_M(n+1)) \geq 0$. In other words, $H(n)$ is an increasing function of $n$. Thus,
\[ \inf_{n \geq 1} H(n) = \inf_{n \leq n_0} H(n). \]

Under a finite domain of \( n \), we can solve for the infimum of \( n^* \) numerically by the following algorithm.

1. Determine \( n_0 \) such that (3.6) holds.

2. Determine a non-empty set \( C \), where
   \[ C = \left\{ n' \left| \begin{array}{l}
Q_M(n') \leq Q_M(n'-1) - c_0 \pi_0 / (1 - \pi_0 - \pi_{k+1}) \\
Q_M(n') \leq Q_M(n'+1) + c_0 \pi_0 / (1 - \pi_0 - \pi_{k+1})
\end{array} \right., \quad n' \leq n_0 \left. \right\} \]

3. If \( C \) is a singleton consisting of \( n' \), then \( n^* = n' \); if \( C \) has a cardinality \( \geq 2 \), then choose \( n^* \) such that
   \[ H(n^*) = \inf_{n \in C} H(n). \]

An example is considered in Section 4.

4. Numerical example for the existence of \( n^* \)

We consider a special case of the loss function, namely, \( \varepsilon(g) = c' g^\alpha \), \( c' > 0 \), \( g \geq \Delta \), \( \alpha \geq 1 \), then \( g(g) e^{-g^2/4(1-\lambda)} = c' g^{\alpha-1} e^{-g^2/4(1-\lambda)} \) has the maximum point at \( g_* = \max(\sqrt{2/(1-\lambda)}(\alpha-1); \Delta) \). Thus \( n_0 \) can be expressed as

\[ n_0 = <(k-1) \left( \frac{1-\lambda}{\pi} \right) \left( \frac{c'}{c_0} \cdot \frac{1-\pi_0-\pi_{k+1}}{\pi_0} \right)^2 g_*^2 / 2(1-\lambda) \times. \]

Let

\[ M^\alpha_{k-1}(x) = \sup_{t \geq x} t^\alpha (1 - \phi_{k-1}(t, \ldots, t)), \]

then
\[ Q_M(n) = c' \left( \frac{2(1-\lambda)}{n} \right)^{\alpha} M_k^{\alpha}(\frac{\Delta}{\sqrt{2(1-\lambda)}}). \]

Now, \( C \) is a set of all \( n' \leq n_0 \) such that

\[ \begin{align*}
(1) \quad & M_k^{\alpha} \left( \frac{\Delta}{\sqrt{2(1-\lambda)}} \right) \left( \frac{\lambda}{n'} \right)^{\alpha/2} M_k^{\alpha} \left( \frac{\Delta}{\sqrt{2(1-\lambda)}} \right) / (n'+1)^{\alpha/2} \leq \\
& \frac{c_0 \pi_0^{1-\pi_0 - \pi_k+1}}{c' \left( (1-\lambda) \right)^{\alpha/2}} \\
(2) \quad & M_k^{\alpha} \left( \frac{\Delta}{\sqrt{2(1-\lambda)}} \right) \left( \frac{\lambda}{n'} \right)^{\alpha/2} M_k^{\alpha} \left( \frac{\Delta}{\sqrt{2(1-\lambda)}} \right) / (n'-1)^{\alpha/2} \leq \\
& \frac{c_0 \pi_0^{1-\pi_0 - \pi_k+1}}{c' \left( (1-\lambda) \right)^{\alpha/2}}.
\end{align*} \]

By using the table 3.1 of Somerville's paper (1954), we can compute \( H(n^*) = \inf H(n) \) directly. Some \( \tau \)-optimal sample sizes are given in Tables I and II for \( \lambda = 0.0, 0.5, \pi_0 = 0.05, 0.10, 0.15, \alpha = 1.0, 2.0, \Delta \leq 0.05, \pi_k+1 \neq 0.0 \) and \( c'/c_0 = 15, 30, 45, 60 \).

5. Sensitivity analysis between \( \Delta \) and \( n^* \)

In this section, we discuss some relationships between \( \Delta \) and \( n^* \). Since \( \Delta \) and \( n^* \) depend on \( \lambda, \alpha, k, c'/c_0, \pi_0, \pi_k+1 \), we fix \( \lambda = 0.5, \alpha = 1.0, k = 4 \) and \( \pi_k+1 = 0.0 \). Let \( c'/c_0 \) change from 15 to 30 and \( \pi_0 \) change from 0.10 to 0.15. With different values of \( c'/c_0 \) and \( \pi_0 \), we get a clear idea of the relationship between \( \Delta \) and \( n^* \). The results are shown in Table III and Fig. 1. We observe that the relation in Fig. 1.b is more stable than in Fig. 1.a and the relation in Fig. 1.d is more stable than in Fig. 1.c. Thus for fixed \( c'/c_0 \), the larger \( \pi_0 \) corresponds to more stable relationship between \( \Delta \) and \( n^* \). Similarly, Fig. 1.a is more stable than Fig. 1.c and Fig. 1.b is more stable than Fig. 1.d; this means that for fixed \( \pi_0 \), the smaller \( c'/c_0 \) corresponds to more stable relationship between \( \Delta \) and \( n^* \).
6. Discussion

In the special case of $k = 2$, we have

$\Omega = \{ \bar{\tau} = (\tau_1; \tau_2)' | \tau_i \in \mathbb{R}, i = 1, 2\},$

$\Omega_0 = \{ \bar{\tau} | \tau_1 = \tau_2 \},$

$\Omega_1 = \{ \bar{\tau} | \tau_1 \geq \tau_2 + \Delta \sigma \},$

$\Omega_2 = \{ \bar{\tau} | \tau_2 \geq \tau_1 + \Delta \sigma \},$

and $\Omega_3 = \Omega - \bigcup_{i=0}^{2} \Omega_i$.

If we do not know any prior information about the parameters we can take $P_r(\bar{\tau} \in \Omega_0) = P_r(\bar{\tau} \in \Omega_1) = \frac{1}{2}$. Then this is reduced to the traditional problem of testing

$(*) \quad H_0: \tau_1 = \tau_2 \quad vs \quad H_1: \tau_1 \geq \tau_2 + \Delta \sigma.$

It should be pointed out that both the type I and type II errors are controlled simultaneously.

TABLE III.

Relationship between $\Delta$ and $n^*$

<table>
<thead>
<tr>
<th>$c'/c_0$</th>
<th>$\pi_0$</th>
<th>$n^*$</th>
<th>$\Delta &lt; 0.25$</th>
<th>0.275</th>
<th>0.30</th>
<th>0.35</th>
<th>0.40</th>
<th>0.45</th>
<th>0.50</th>
</tr>
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<tbody>
<tr>
<td>15</td>
<td>0.10</td>
<td>$n_1^*$</td>
<td>8</td>
<td>8</td>
<td>9</td>
<td>11</td>
<td>12</td>
<td>12</td>
<td>11</td>
</tr>
<tr>
<td>15</td>
<td>0.15</td>
<td>$n_2^*$</td>
<td>6</td>
<td>6</td>
<td>8</td>
<td>8</td>
<td>9</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>30</td>
<td>0.10</td>
<td>$n_3^*$</td>
<td>13</td>
<td>15</td>
<td>17</td>
<td>17</td>
<td>18</td>
<td>17</td>
<td>16</td>
</tr>
<tr>
<td>30</td>
<td>0.15</td>
<td>$n_4^*$</td>
<td>9</td>
<td>9</td>
<td>12</td>
<td>13</td>
<td>13</td>
<td>12</td>
<td>12</td>
</tr>
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TABLE I (α = 1.0)
Γ-optimal Sample Size for R.C.B.D. Problem

<table>
<thead>
<tr>
<th>k</th>
<th>$\lambda$</th>
<th>$(λ = 0.0)$</th>
<th>$(λ = 0.5)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(c'/c_0)</td>
<td>0.05</td>
<td>0.10</td>
</tr>
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Table II ($\alpha = 2.0$)
$r$-optimal Sample Size for R.C.B.D. Problem

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Figure 1

Graphical relationship between $\Delta$ and $n^*$

Fig 1.a under $c'/c_0 = 15$, $\eta_0 = 0.10$.

Fig 1.b under $c'/c_0 = 15$, $\eta_0 = 0.15$.

Fig 1.c under $c'/c_0 = 30$, $\eta_0 = 0.10$.

Fig 1.d under $c'/c_0 = 30$, $\eta_0 = 0.15$. 
ACKNOWLEDGEMENT

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BIBLIOGRAPHY


Selection procedures, randomized complete block design, $\Gamma$-optimal decision rule, best population.

In randomized complete block design, we face the problem of selecting the best population. If some partial information about the unknown parameters is available, then we wish to determine the optimal decision rule to select the best population.

In this paper, in the class of natural selection rules, we employ the $\Gamma$-optimal criterion to determine optimal decision rules that will minimize the maximum expected risk over the class of some partial information. Furthermore, the traditional hypothesis testing is briefly discussed from the viewpoint of ranking and selection.