ON BROWNIAN SLOW POINTS

by

Burgess Davis
Purdue University

Technical Report #82-1

Department of Statistics
Purdue University

February 1982
On Brownian Slow Points
Burgess Davis

Abstract

It is shown that, for a Wiener process $X_t$, both the quantities
\[ \inf_t \lim_{h \to 0^+} \frac{|X_{t+h} - X_t|}{\sqrt{h}} \text{ and } \sup_t \lim_{h \to 0^+} \frac{(X_{t+h} - X_t)}{\sqrt{h}} \] are almost surely equal to 1.

1. Introduction. Let $W_t$, $t \geq 0$, be standard Brownian motion. In the 1932 paper where they showed that almost every Brownian path is nowhere differentiable ([7]), Paley, Wiener, and Zygmund proved the stronger result that, for each $\varepsilon > 0$,
\[ P(\lim_{h \to 0^+} \frac{|W_{t+h} - W_t|}{h^{1/2} + \varepsilon} = \infty \forall \, t) = 1, \]
and in 1963 A. Dvoretzky ([1]) improved this by establishing
\[ P(\lim_{h \to 0^+} \frac{|W_{t+h} - W_t|}{\sqrt{h}} > c_0 \forall \, t) = 1, \]
for a positive constant $c_0$. The natural question, whether (1.1) holds for all constants, was settled by J.-P. Kahane in 1974 ([2]). The answer is no. Kahane showed
\[ P(\forall \, t: \lim_{h \to 0^+} \frac{|W_{t+h} - W_t|}{\sqrt{h}} < c_1) = 1, \]
for a constant $c_1 < \infty$.

Here, note that $h$ may be allowed to approach 0 from either the left or the right, giving a better result. (The two sided version of (1.1) is of course
weaker than (1.1).) Kahane calls those \( t \) which satisfy
\[
\lim_{h \to 0} \frac{|W_{t+h} - W_t|}{\sqrt{h}} < \infty
\]
slow points.

The law of the iterated logarithm implies that the slow points almost surely have Lebesgue measure 0, but Kahane has proved ([2], [3]) that their Hausdorff dimension a.s. equals 1 and that the Hausdorff dimension of those slow points which are also zeros of \( W_t \) is a.s. \( \frac{3}{2} \), so that the slow points are a fairly thick set. Kahane has very recently given another, simpler, proof of (1.2) and the two sided version of (1.1), together with related results for other Guassian processes, in [4].

Following Kahane, we will call a point \( t \) **slow from the right** if
\[
\lim_{h \to 0^+} \frac{|W_{t+h} - W_t|}{\sqrt{h}} < \infty.
\]

In Section 2 we investigate the question: How slow from the right can a point be? It is shown that

\[
(1.3) \quad \inf_{t} \lim_{h \to 0^+} \frac{|W_{t+h} - W_t|}{\sqrt{h}} = 1 \text{ a.s.}
\]

The proof that the expression to the left of the equality in (1.3) is no smaller than 1 is a refinement of Dvoretzky's proof in [1], while the proof that it is no larger than 1 is not related to Kahane's arguments.

Let \( z \) be the smallest positive 0 of \( M(-\frac{3}{2}, \frac{1}{2}, x^2/2) \), where \( M \) is the confluent hypergeometric function \( z \approx 1.3069 \). In Section 4 it is shown that
\[
(1.4) \quad P(\inf_t \lim_{h \to 0} \frac{|W_{t+h} - W_t|}{\sqrt{|h|}} < z) = 0,
\]

but we cannot prove

\[
P(\inf_t \lim_{h \to 0} \frac{|W_{t+h} - W_t|}{\sqrt{|h|}} = z) = 1.
\]

Nonetheless, this is probably true. Not only does (1.4) hold, but also it is shown in Section 4 that, if \(X_t\) and \(Y_t\) are independent Brownian motions, and if \(D_r^X = \{t: \lim_{h \to 0^+} \frac{|X_{t+h} - X_t|}{\sqrt{h}} < r\}\) and \(D_r^Y\) is defined similarly, then \(D_r^X \cap D_r^Y\) is almost surely empty if \(r < z\) and not empty if \(r > z\).

S. Orey and J. Taylor have shown how rapid a point can be by proving (see [5])

\[
sup_t \lim_{h \to 0^+} \frac{|W_{t+h} - W_t|}{\sqrt{2h \log \frac{1}{h}}} = 1 \text{ a.s.}
\]

This is equivalent to

\[
(1.5) \quad \sup_t \lim_{h \to 0^+} \frac{W_{t+h} - W_t}{\sqrt{2h \log \frac{1}{h}}} = 1 \text{ a.s.,}
\]

giving a global upper bound for the lim sup, as \(h \to 0^+\), of \(W_{t+h} - W_t\). In his book ([5], p.148) F. Knight asks for a global lower bound for this lim sup. Precisely, Knight asks for a function \(\phi(h) > 0\) such that, almost surely, for all \(t\)

\[
\lim_{h \to 0^+} (W_{t+h} - W_t)/\phi(h) \geq -1, \text{ and } \lim_{h \to 0^+} (W_{t+h} - W_t)/\phi(h) = -1 \text{ for some } t.
\]

In Section 3 we come pretty close to solving this problem, and do give a lower bound in the sense that (1.5) gives an upper bound. We prove
(1.6) \[ \inf_t \lim_{h \to 0^+} \frac{W_{t+h} - W_{t}}{\sqrt{h}} = -1 \text{ a.s.} \]

We do not know if the \( \lim \) equals \(-1\) for some \( t \).

The two sided version of (1.6) is essentially known and not hard to prove. It is

(1.7) \[ \inf_t \lim_{h \to 0^+} \frac{W_{t+h} - W_{t}}{M_{t+h}} = 0 \text{ a.s.} \]

Note that the existence of times for which \( W_t \) has a local maximum shows that \( 0 \) can not be replaced by a larger number in (1.7). Only strict maxima need be considered, and, these being countable, the proof of (1.7) can be completed by examining the behavior of \( W_t \) around an absolute maximum. See [5] for a treatment of such ideas, which yields sharper results than (1.7).

Define the sets

\[ A_c = \{ \exists t \in [0,1]: |W_{t+h} - W_{t}| < c\sqrt{h} \quad \forall h \in (0,1] \} \]

and

\[ B_c = \{ \exists t \in [0,1]: (W_{t+h} - W_{t}) > c\sqrt{h} \quad \forall h \in (0,1] \} . \]

We will prove that \( P(A_c) = 0 \) if \( c < 1 \) and \( P(A_c) > 0 \) if \( c > 1 \), implying (1.3), and that \( P(B_c) = 0 \) if \( c > 1 \) and \( P(B_c) > 0 \) if \( c < 1 \), implying

\[ \sup_t \lim_{h \to 0^+} (W_{t+h} - W_t) / \sqrt{h} = 1 \text{ a.s., which is equivalent to (1.6).} \]

For the remainder of the paper the qualifier a.s. will usually be omitted.
2. Proof of (1.3). First the following lemma is established. Let \( \wedge \) denote minimum.

**Lemma 2.1.** If the nonnegative random variable \( X \) satisfies

\[
\lim_{n \to \infty} n P(X \geq n) / EX \wedge n = 0,
\]

then \( EX^p < \infty, 0 < p < 1 \).

**PROOF.** Note that the example \( P(X \geq t) = t^{-1}, t \geq 1 \), shows that (2.1) can hold and \( EX = \infty \).

Now for any nonnegative random variable \( Z \) and any \( p > 0 \), \( EZ^p \) is finite or infinite depending on whether \( \sum_{n=1}^{\infty} 2^{np} P(Z \geq 2^n) \) is finite or infinite.

Let \( \gamma_n = 2^n P(X \geq 2^n) \). Then (2.1) implies

\[
\lim_{n \to \infty} \gamma_n / \sum_{i=1}^{n-1} \gamma_i = 0.
\]

For \( \varepsilon > 0 \) let \( N(\varepsilon) = N \) satisfy \( \gamma_n / \sum_{i=1}^{n-1} \gamma_i < \varepsilon \), \( n > N \). Put \( \sum_{i=1}^{N} \gamma_i = y \).

Then \( \gamma_{N+1} < \varepsilon y \) and \( \gamma_{N+k} < \varepsilon (y + \sum_{j=1}^{k-1} \gamma_{N+j}) \). If we put \( a_{N+k} = y(1 + \varepsilon)^k \),

then \( a_{N+1} \geq \varepsilon y \) and \( a_{N+k} \geq \varepsilon (y + \sum_{j=1}^{k-1} a_{N+j}) \), and so, by induction,
\( \gamma_{N+k} \leq \alpha_{N+k} = y(1+\varepsilon)^k, \quad k \geq 1. \) Since \( \varepsilon \) is arbitrary, this implies \( \mathbb{E} P < \infty, \quad p < 1, \) as claimed.

Let \( \tau_r(W) = \tau_r = \inf\{t \geq 1: |X_t| = r\sqrt{\varepsilon}\}. \) Precise information concerning the moments of \( \tau_r \) may be found in Shepp, [8]. For our purposes the following lemma suffices. The notation \( P_{a,b} \) and \( \mathbb{E}_{a,b} \) will signify probability and expectation associated with \( W_t \) given \( W_a = b. \)

Lemma 2.2. If \( r > 1 \) there is a \( p = p(r) < 1 \) such that \( \mathbb{E}_{\tau_r} P \) = \( \infty. \) Furthermore \( \mathbb{E}_{1,0} \tau_1 = \infty. \)

PROOF. First the well known proof of the second statement will be supplied. For a stopping time \( T \) we have

\[ \mathbb{E} W_T^2 = ET \quad \text{if} \quad ET < \infty, \]

and applying this to the Wiener process \( W_{t+1} \) under \( P_{1,0} \) yields

\[ \mathbb{E}_{1,0} \omega_{\tau_1}^2 = \mathbb{E}_{1,0} (\tau_1 - 1) \quad \text{if} \quad \mathbb{E}_{1,0} (\tau_1 - 1) < \infty. \]

Since \( P_{1,0} (W_{\tau_1}^2 = \tau_1) = 1, \) this implies \( \mathbb{E}_{1,0} (\tau_1 - 1) = \infty, \) so \( \mathbb{E}_{1,0} \tau_1 = \infty. \)

To prove the rest of the lemma, fix \( r > 1 \) and define

\[ \gamma_1 = \inf\{t \geq 1: W_t = 0 \quad \text{or} \quad |W_t| \geq r\sqrt{\varepsilon}\}, \]

and in general

\[ \gamma_{2k} = \inf\{t \geq \gamma_{2k-1}: |W_t| \geq \sqrt{\varepsilon}\}, \quad \text{and} \]

\[ \gamma_{2k+1} = \inf\{t \geq \gamma_{2k}: W_t = 0 \quad \text{or} \quad |W_t| \geq r\sqrt{\varepsilon}\}. \]

Note that on \( \{W_{\gamma_{2k-1}} = 0\} \), if \( \lambda > 0, \)

\[ P(\gamma_{2k} - \gamma_{2k-1} > \lambda \gamma_{2k-1} \mid W_{\gamma_{2k-1}}) = P_{1,0} (\tau_1 > 1 + \lambda) \]
using the strong Markov property and Brownian scaling. Furthermore, if 
\[ \varepsilon = P_{1,1}(W_t = 0 \text{ before } |W_t| = r\sqrt{\varepsilon}), \] 
then, on \( \{\gamma_{2k-2} < \tau_r\} \), we have, for \( k \geq 2 \) and \( \lambda > 0 \),

\[
P(\gamma_{2k} - \gamma_{2k-1} > \lambda \gamma_{2k-2} | W_{\gamma_{2k-2}})
\geq P(\gamma_{2k} - \gamma_{2k-1} > \lambda \gamma_{2k-1} | W_{\gamma_{2k-2}} = 0) P(W_{\gamma_{2k-1}} = 0 | W_{\gamma_{2k-2}})
= P_{1,0}(\tau_1 > 1 + \lambda) \varepsilon.
\]

Thus, since \( \{\gamma_{2k-2} < \tau_r\} = \{\gamma_{2k-3} < \gamma_{2k-2}\} \), this gives

\[
E(\gamma_{2k} - \gamma_{2k-1})^p \geq \varepsilon E_{1,0}(\tau_{1-1})^p \cdot E_{\gamma_{2k-2}}^P I(\gamma_{2k-2} < \tau_r)
\geq \varepsilon E_{1,0}(\tau_{1-1})^p E(\gamma_{2k-2} - \gamma_{2k-3})^p
\]

and iteration gives

\[
E(\gamma_{2k} - \gamma_{2k-1})^p \geq \varepsilon E_{1,0}(\tau_{1-1})^p \lambda^k \gamma_{2k} - \gamma_{2k-1})^p
\]

Pick \( p < 1 \) such that \( \varepsilon E_{1,0}(\tau_{1-1})^p > 1 \). This is possible since \( E_{1,0}\tau_1 = \infty \).

Then \( E_{\tau_r}^P \geq E_{\gamma_{2k}}^P > E(\gamma_{2k} - \gamma_{2k-1})^p \to \infty \) as \( k \to \infty \).

Next put \( M_w = M = \max_{0 \leq t \leq 1} |W_t| \), and \( T_r = T_r = \inf\{t > 1: |W_t| = M + r\sqrt{\varepsilon}\} \).

Lemma 2.3. If \( c < 1 \), \( ET_c < \infty \).

PROOF. For \( t \geq 1 \) the equality (2.2) gives

\[
ET_c^t = EW_{\gamma_{T_c}}^t
\leq E(M + c\sqrt{T_c^\gamma_{T_c}})E^2
= EM^2 + 2cEM\sqrt{T_c^\gamma_{T_c}} + c^2ET_c^t.
\]
Thus $(1-c^2)E_{c^*} \geq EM^2 + 2cEM \sqrt{ET_{c^*} t} \leq EM^2 + 2c(EM^2)^{\frac{3}{2}}(ET_{c^*} t)^{\frac{3}{2}}$.

Since $EM^2$ is finite, $ET_{c^*} t$ must stay bounded as $t \to \infty$, so $ET_{c^*} < \infty$.

As has been mentioned, the following theorem implies (1.3).

Theorem 2.1. If $c < 1$, $P(A_c) = 0$, and if $c > 1$, $P(A_c) > 0$.

PROOF. Fix $c < 1$, and for a subintervals $[a,b] = I$ of $[0,1]$ let

$\Delta_I = \{ \exists t \in I: |X_{t+h} - X_t| < c\sqrt{h}, 0 < h \leq 1 \}$. Note that, if

$M_I = \max_{a \leq t \leq b} |W_t - W_a|$, then $\Delta_I \subset \{ |W_{a+h} - W_a| < M_I + c\sqrt{h}, b-a \leq h \leq 1 \}$, by a geometrical argument. Thus, conditioning on $W_a$ and changing scale, we have

$P(\Delta_I) \leq P(T_{c} \geq (b-a)^{-1})$,

and especially, if $I$ has length $n^{-1}$, $P(\Delta_I) \leq P(T_{c} \geq n)$. Divide $[0,1]$ into intervals $I_k$ of length $n^{-1}$. Then $P(A_c) \leq \sum P(\Delta_{I_k}) \leq nP(T_{c} \geq n)$. Lemma 2.3 gives $ET_{c} < \infty$, so $nP(T_{c} \geq n) \to 0$, proving $P(A_c) = 0$.

Now fix $c > 1$ and put $\Gamma_n = \{ \exists t \in [0,1]: |W_{t+h} - W_t| < c\sqrt{h}, n^{-1} \leq h \leq 1 \}$. Note that $\Gamma_n \subseteq \Gamma_m$ if $n \geq m$. We will show that $\lim_{n \to \infty} P(\Gamma_n) > 0$, implying

$P(\bigcap_{n=1}^{\infty} \Gamma_n) = P(A_c) > 0$. Put $v_{0,n} = v_0 = 0$, and, if $i \geq 1$,

$v_{i,n} = v_i = (v_{i-1} + 1)^{\inf(t \geq v_{i-1} + n^{-1}: |W_t - W_{v_{i-1}}| \geq \sqrt{\varepsilon(v_{i-1})})}$.

Then

(2.3) \hspace{1cm} P(v_{i+1} - v_i = 1|W_{v_i}) = P(T_{c} \geq n), \text{ and}

(2.4) \hspace{1cm} E(v_{i+1} - v_i|W_{v_i}) = n^{-1}E_{\tau_{c}} \tilde{\tau}_{c} n.$
Of course \( v_k \geq 1 \) if \( v_i - v_{i-1} = 1 \) for some \( i \leq k \). Thus

\[
\varphi_{n,m} = \mathbb{P}(v_{i+1} - v_i = 1 \text{ for some } i \leq m \text{ such that } v_i \leq 1)
\]

\[
= \sum_{i=1}^{m} \mathbb{P}(\tau_c \geq n) \mathbb{P}(v_i \leq 1)
\]

\[
\geq m \mathbb{P}(\tau_c \geq n) \mathbb{P}(v_m \leq 1).
\]

Let \( \{n_k\}_{k=1}^{\infty} \) be a sequence of integers approaching infinity such that \( n_k \mathbb{P}(\tau_c \geq n_k)/E_{\tau_c \cap n_k} \overset{\alpha}{\to} > 0 \) for all \( k \), such a choice being possible by Lemmas 2.1 and 2.2. We also assume

\[
E_{\tau_c \cap n_k}/n_k \leq 1/6.
\]

Let the integer \( m_k \) satisfy

\[
1/3 \leq (m_k/n_k)E_{\tau_c \cap n_k} \leq 1/2.
\]

By (2.4),

\[
E_{\tau_{m_k \cap n_k}} = (m_k/n_k)E_{\tau_c \cap n_k},
\]

so

\[
\mathbb{P}(v_{m_k \cap n_k} \geq 1) \leq 1/2,
\]

and, using the left inequality in (2.5), we have

\[
\mathbb{P}(\tau_{n_k} \cap m_k) \geq \varphi_{n_k,m_k}
\]

\[
\geq m_k \mathbb{P}(\tau_c \geq n_k) \mathbb{P}(v_{m_k \cap n_k} \leq 1)
\]

\[
\geq m_k \mathbb{P}(\tau_c \geq n_k)/2
\]

\[
\geq m_k \mathbb{E}_{\tau_c \cap n_k}/2n_k
\]

\[
\geq \alpha/6.
\]
3. Proof of (1.6). The arguments involving $B_C$ are very similar to those
of the last section and proofs will just be sketched. Let
$\eta_a = \inf\{t \geq 0 : W_t < a\sqrt{\xi}\}$. Precise information on the moments of $\eta_a$ has
been supplied by Novikov in [6]. Here we need only the following analog
of Lemma 2.2.

**Lemma 3.1.** If $r < 1$ there is a $q = q(r) < 1$ such that $E_{\eta_a}^q = \infty$.
Furthermore $E_{1,2\eta_1} = \infty$.

**PROOF.** That $E_{0,1\eta_1} = \infty$ follows from $E_{0,1\eta_1} = E_{0,1}(W_{\eta_1} - 1)^2$ if
$E_{0,1\eta_1} < \infty$, because $P_{0,1}(W_{\eta_1}^2 = \eta_1) = 1$. Since $P_{1,2}(\eta_1 - 1 > \lambda) > P_{0,1}(\eta_1 > \lambda)$,
$\lambda > 0$, we get $E_{1,2\eta_1} \geq E_{0,1\eta_1} = \infty$.

The proof of the first assertion of Lemma 3.1 can be patterned on the
proof of the first assertion of Lemma 2.2. The analogs of the times $\gamma_i$
here are

$$\tilde{\gamma}_1 = \inf\{t \geq 1 : W_t = 2\sqrt{\xi} \text{ or } W_t \leq r\sqrt{\xi}\},$$

and in general

$$\tilde{\gamma}_{2k} = \inf\{t \geq \tilde{\gamma}_{2k-1} : W_t \leq \sqrt{\xi}\}, \text{ and}$$

$$\tilde{\gamma}_{2k+1} = \inf\{t \geq \tilde{\gamma}_{2k} : W_t = 2\sqrt{\xi} \text{ or } W_t \leq r\sqrt{\xi}\}.$$

Now let $M^* = \min_{0 \leq t \leq 1} W_t$, and let $U_r = \inf\{t \geq 1 : W_t = r\sqrt{\xi} - 1 + M^*\}$

**Lemma 3.2.** If $c > 1$, $EU_c < \infty$.

**PROOF.** For each $t > 1$, (2.2) gives
\[ \text{EU}_c \cap t = E(W_{U_c} \cap t)^2 \geq E(c \sqrt{U_c} - t + M)^2, \]

and the rest of the proof resembles the proof of Lemma 2.3.

**Theorem 3.1.** If \( c > 1 \), \( P(B_c) = 0 \) and, if \( c < 1 \), \( P(B_c) \neq 0 \).

**PROOF.** Note that if \( [a,b] = I \) is a subinterval of \([0,1]\), and if \( M_I = \min_{a \leq t \leq b} W_t - W_a \), then

\[
\{a \leq t \leq b: W_{t+h} - W_t > c \sqrt{h} \quad \forall \, h \in (0,1]\}
\subseteq \{W_{t+h} - W_t > c \sqrt{h} - (b-a) + M_I, \, b - a \leq h \leq 1\},
\]

and the rest of the proof that \( P(B_c) = 0, \, c > 1 \), follows from Lemma 3.2 just like the proof that \( P(A_c) = 0, \, c < 1 \), followed from Lemma 3.3. Furthermore, the proof that \( P(B_c) > 0, \, c < 1 \), is almost the same as the proof that \( P(A_c) > 0, \, c > 1 \).

4. Independent Wiener processes. The arguments in this section are similar to those of Section 2, but we make use of Shepp's results in [2]. Fix \( r > 0 \), and for \( 0 < t < 2 \) and \( |s| < r \sqrt{t} \) let \( f_{t,s} \) be the continuous version of the density of \( W_2 I(\tau_r > 2) \) under \( P_{t,s} \). Of course \( f \) vanishes off \((-r \sqrt{r}, \, r \sqrt{r})\). Then if \( \alpha(r) = \alpha = 2/P_{1,0}(\tau_r < 2) \), we have

\[ (4.1) \quad f_{1,y}(s)/f_{1,0}(s) \leq \alpha, -r \sqrt{r} < s < r \sqrt{r}. \]

To see this let I be a closed subinterval of \((-r \sqrt{2}, \, r \sqrt{2})\) and define the set \( F \subseteq \{(t,s): 1 \leq t \leq 2, |s| < r \sqrt{t} \} \) by \((t,s) \in F\) if \( g(t,s) \geq g(1,y) \), where

\[ g(a,b) = P_{a,b}(W_2 \in I \text{ and } \tau_r > 2). \]
Then $F$ is a closed set containing a curve joining $(1, y)$ and the midpoint of $I$. Let $v$ be the first time $(t, W_t) \in F$. Using the Strong Markov Property, we get $g(1, 0) \geq g(1, y) P_{1, 0}(v < \tau_r^2 - 2)$. For $y > 0$ we have $P_{1, 0}(v < \tau_r^2 - 2) \geq P_{1, 0}(\tau_r < 2, W_{\tau_r} > 0)$ with a similar formula for $y < 0$, so $g(1, 0) \geq \alpha g(1, y)$, implying (4.1).

Similarly, we can prove that for each $y \in (-r, r)$ there is a $K(r, y) = K > 0$ such that

$$(4.2) \quad f_{1, r}(s)/f_{1, 0}(s) \geq K, \quad -\sqrt{2r} < s < \sqrt{2r}.$$ 

Shepp shows in [8] that if $z$ is as in Section 1 and $r > z$, there exists $\gamma(r) = \gamma \in (0, 2)\setminus \frac{1}{2}$ such that $E_{1, 0}^Y \tau_r^Y = \infty$, and so, using (4.2) and conditioning on $W_{\tau_r^2}$, we have $E_{1, 0}^{Y, \tau_r^Y} = \infty$ for each $y \in (-r, r)$, implying

$$(4.3) \quad \lim_{\lambda \to \infty} P(\tau_r > \lambda) \lambda^p = \infty \text{ for each } p > \gamma.$$ 

Now let $X_t$ and $Y_t$ be independent Wiener processes. Put $\theta_r = \tau_r(X) \tau_r(Y)$. Then $P(\theta_r > \lambda) = P(\tau_r > \lambda)^2$, so, by (4.3),

$$\lim_{\lambda \to \infty} P(\theta_r > \lambda) \lambda^{2p} = \infty \text{ if } p > \gamma.$$ 

In particular, there is an $\alpha = \alpha(r) < 1$ such that $E\theta_r^\alpha = \infty$. Now, methods similar to those employed in Section 2 show that, for $r > z$, $P(D_r^X \cap D_r^Y \neq \emptyset) = 1$. Note the set corresponding to $A_c$ is

$$\{ a \in [0, 1]: |X_{t+h} - X_t| + |Y_{t+h} - Y_t| < r\sqrt{n}, \forall \ h \in (0, 1]\},$$

and that $\theta_r = \inf\{ t \geq 1: |X_{t+h} - X_t| + |Y_{t+h} - Y_t| > r\sqrt{n}\}$. The sets $A_r$ and $D_r$ are defined in Section 1.

Shepp also proves that, if $s < z$, there exists a $\delta = \delta(s) > \frac{1}{2}$ such that $E_{1, 0}^{\delta} < \infty$. Conditioning on $X_{\tau_s^2}$ and using (4.1), this gives
\[
\lim_{\lambda \to \infty} \sup_{y \in (-s, s)} P_{1, y} (\tau_s > \lambda)^{\delta} < \infty
\]

Now fix \( r \in (0, z) \) and let \( s = (r + z)/2 \). Put
\[
\Gamma = \Gamma_{k, n} = \{ \exists t \in [(k/n, (k+1)/n)] : |X_{t+h} - X_t| + |Y_{t+h} - Y_t| < r \sqrt{n} \quad \forall h \in (0, 1) \}.
\]

Let \( M \) be the smallest integer such \( (s-r)M \geq r \). Define the events \( C_{j, k, n} = C_j, -M \leq j \leq M \), and \( G_{i, k, n} = G_i, -M \leq i \leq M \), by
\[
C_j = \{ (t, X_t) \in (t, x) : |x - \alpha_j| \leq s \sqrt{t - (k/n)}, (k+1)/n \leq t \leq (k/n) + 1 \},
\]
where \( \alpha_j = (k+1)/n + (s-r)j/\sqrt{n} \), and
\[
G_i = \{ (t, Y_t) \in (t, x) : |x - \beta_i| \leq s \sqrt{t - (k/n)}, (k+1)/n \leq t \leq (k/n) + 1 \},
\]
where \( \beta_i = Y_{(k+1)/n} + (s-r)j/\sqrt{n} \).

Conditioning on \( X_{(k+1)/n} \), and using Brownian scaling, we see both \( P(C_j) \) and \( P(G_i) \) are maximized by \( \sup_{y \in (-r, r)} P_{1, y} (\tau_s > n) \) so that

\[
P(C_j \cap G_i) = o(n^{-2\delta}) = o(n^{-1}) \quad \text{by} (4.4). \]

A geometrical argument gives

\[
r \subset \bigcup_{i,j} C_j \cap G_i, \quad \text{so that} \quad P(r) = o(n^{-1}), \quad \text{yielding}
\]

\[
P_{k=0}^{n-1} (\bigcup_{k=0}^{n-1} \Gamma_{k, n}) = o(1),
\]

which implies

\[
P(D_r^X \cap D_r^Y) = 0.
\]

These arguments easily generalize to \( n \) independent Wiener processes, with the aid of the results in [8]. Let \( z_n \) be the smallest positive zero of \( M(-1/n, 1/2, \sqrt{n}) \), where \( M \) is the confluent hypergeometric function.

Then we have

**Theorem 4.1** \( \bigcap_{i=1}^n D_i \) is a.s. empty if \( r < z_n \) and not empty if \( r > z_n \).
A proof of (1.4) can be made which is very similar to that of Theorem 4.1. Here it is convenient to work with a Brownian motion $Z_t$, $t \in (-\infty, \infty)$. We note that $S_t = Z((k+1)/n + t) - Z(k+1)/n$, $t \geq 0$, and $R_t = Z(k/n - t) - Z(k/n)$, $t \geq 0$, are independent Wiener processes. Furthermore

$$\{\forall \ t \in [\frac{k}{n}, \frac{k+1}{n}]: \ |Z_{t+h} - Z_t| < r \sqrt{h} \ \forall \ h \in (0,1] \cup [-1,0)\}$$

can be shown to be contained in a set defined in terms of $S_t$ and $R_t$ in a manner similar to the way $UC_i \cap G_j$ was defined in terms of $X_t$ and $Y_t$ earlier, and thereby shown to have probability equal to $o(1/n)$ if $r < z$, from which we get

$$P(\forall \ t \in [0,1]: \ |Z_{t+h} - Z_t| < r \sqrt{h} \ \forall \ h \in (0,1] \cup [-1,0)) = 0,$$

which is equivalent to (1.4).
References


