ON THE RATE OF CONVERGENCE
FOR THE WEAK LAW OF LARGE NUMBERS

by

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1. INTRODUCTION

Let $X, X_1, X_2, \ldots$ be independent random variables with the common distribution function $F(t) = P(X \leq t)$, and let $S_n = X_1 + \ldots + X_n (n \geq 1)$. In studying the rate of convergence in weak laws of large numbers, the convergence of the series

$$\sum_{n=1}^{\infty} P(|S_n| \geq n\epsilon)$$

for some $\epsilon > 0$, was found to be connected with the existence of second moment of $X$ (see Hsu and Robbins [6], Erdős [3] or Révész [9]). In particular, Erdős [2] has shown that the series (1.1) converges for some $\epsilon > 0$, if and only if, $EX^2$ and $|EX|<\epsilon$.

Subsequently, number of authors (notably Heyde and Rohatgi [5], Chow and Lai [2] and Lai and Lan [8]) analysed the convergence of the series

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of the form

\[(1.2) \quad \sum_{n=1}^{\infty} c_n P(|S_n| \geq a_n)\]

for various \(\{c_n\}\) and \(\{a_n\}\), again connecting it with the appropriate moment conditions.

Certain considerations arising in stochastic modeling for the growth of cancer tumors (see [1]), led us to the analysis of convergence of series of type \((1.1)\) with the index of summation restricted to a subsequence.

The problem of this note may be formulated as follows. Let \(\{K_n\}\) be a sequence of integers satisfying

\[(1.3) \quad 1 \leq K_1 \leq K_2 \leq \ldots\]

and

\[(1.4) \quad \lim_{n \to \infty} K_n = \infty.\]

Consider the series

\[(1.5) \quad \sum_{n=1}^{\infty} P(|S_{K_n}| \geq \varepsilon K_n)\]

for some \(\varepsilon > 0\). By grouping the terms corresponding to identical indices \(K_n\), we may write \((1.5)\) as

\[(1.6) \quad \sum_{n=1}^{\infty} c_n P(|S_{b_n}| \geq \varepsilon b_n)\]

where the sequences \(\{b_n\}\) and \(\{c_n\}\) are defined by

\[(1.7) \quad c_0 = 0, \quad c_{n+1} = \min\{r : K_r > K_{c_n+1}\} - 1 - c_0 - \ldots - c_n\]
and

\[(1.8) \quad b_{n+1} = K_{c_0} + c_1 + \ldots + c_{n+1}\]

for \( n = 0,1, \ldots \). 

Note that since \( \lim K_n = \infty \) we have \( 1 \leq c_n < \infty \) for all \( n \geq 1 \), and \( 1 \leq b_1 < b_2 < \ldots \).

We shall now drop the condition that \( c_n \)'s are integers, and consider generally the problem of convergence of series \((1.6)\), where \( \{c_n\} \) is some sequence of positive real numbers and \( \{b_n\} \) is a strictly increasing sequence of positive integers.

Clearly, we have here an interplay of three types of conditions: (i) convergence of series \((1.6)\), (ii) an appropriate moment condition and (iii) a condition imposing constraints on the behaviour of the sequences \( \{c_n\} \) and \( \{b_n\} \). We shall prove three theorems, in each of them two among (i)-(iii) implying the third, with theorems implying the general case where the random variables involved are not necessarily independent and identically distributed (I.I.D.).

\section{The Results}

We start by presenting a lemma due to von Bahr and Esséen, which will be needed below.

\textbf{Lemma 1.} Let \( Y_1, \ldots, Y_n \) be a finite sequence of random variables. Denote \( S_i = Y_1 + \ldots + Y_i \) and assume that \( \mathbb{E}(Y_i | S_{i-1}) = 0 \), \( \mathbb{E}|Y_i|^{1+\lambda} < \infty \), \( i = 1, \ldots, n \), for some \( \lambda \) with \( 0 < \lambda < 1 \). Then there exists a constant \( C(\lambda) > 0 \), such that

\[(2.1) \quad \mathbb{E}|S_n|^{1+\lambda} \leq C(\lambda) \sum_{i=1}^{n} \mathbb{E}|Y_i|^{1+\lambda} .\]
In fact, as pointed out by Rubin [10], we have

\[(2.2) \quad C(\lambda) = \sup_x \left[ \frac{|1+x|^{1+\lambda} - 1 - (1+\lambda)x}{|x|^{1+\lambda}} \right] \]

with \(1 \leq C(\lambda) \leq 2\) for \(0 \leq \lambda \leq 1\).

We shall first prove

**THEOREM 1.** Let \(Y_1, Y_2, \ldots\) be a sequence of random variables with \(E(Y_i | S_{i-1}) = 0, i=1,2,\ldots\), where \(S_0 = 0, S_i = Y_1 + \ldots + Y_i, i=1,2,\ldots\).

Assume that for some sequence \(\{\lambda_n\}\) with \(0 < \lambda_n \leq 1\) we have \(E|Y_i|^{1+\lambda_n} < \infty, i=1,2,\ldots\), where \(\lambda = \sup_{n} \lambda_n\), and the sequences \(\{c_n\}\) and \(\{b_n\}\) satisfy the condition

\[(2.3) \quad \sum_{n=1}^{\infty} c_n \bar{b}_n^{-\lambda_n} b_n < \infty \]

where

\[(2.4) \quad \bar{b}_n = \frac{1}{b_n} \sum_{j=1}^{b_n} E|Y_i|^{1+\lambda_n} \cdot \]

Then for every \(\varepsilon > 0\) we have

\[(2.5) \quad \sum_{n=1}^{\infty} c_n P\{|Y_1 + \ldots + Y_{b_n}| \geq \varepsilon b_n\} < \infty \cdot \]

**Proof.** We may estimate the terms of the series in (2.5), using Markov inequality and Lemma 1, as follows.
(2.6) \[ c_n P\left( |Y_1 + \ldots + Y_n| \geq \epsilon b_n \right) = c_n P\left( S_{b_n}^{1+\lambda} \geq (\epsilon b_n)^{1+\lambda} \right) \]

\[ \leq c_n \left( \frac{E[S_{b_n}^{1+\lambda}]}{\epsilon b_n^{1+\lambda}} \right) \]

\[ \leq c_n C(\lambda_n) \theta_n b_n^{-\lambda} \epsilon^{-(1+\lambda_n)} \]

The theorem now follows from (2.3), since \( \sup_n C(\lambda_n) \leq 2 \), and \( \epsilon^{-(1+\lambda_n)} \leq \epsilon^{-1} \) or \( \epsilon^{-2} \), depending on whether \( \epsilon < 1 \) or \( \epsilon \geq 1 \).

In particular, in the case of I.I.D. random variables \( X, X_1, X_2, \ldots \) we obtain

**COROLLARY 1.** Assume that \( E|X|^{1+\lambda} < \infty \) for some \( \lambda \) with \( 0 < \lambda \leq 1 \).

Moreover, let \( EX = 0 \), and assume that the sequences \( \{b_n\} \) and \( \{c_n\} \) satisfy the condition

(2.7) \[ \sum_{n=1}^{\infty} c_n b_n^{-\lambda} < \infty. \]

Then the series (1.6) converges for every \( \epsilon > 0 \).

Observe that for \( \tau < 0 \), if we put \( c_n = n^{\tau} \), \( b_n = n \) and \( \lambda \rightarrow 1 + \tau \), we obtain the sufficiency part of Theorem 1 of Katz [7].

We shall now prove
THEOREM 2. Assume that \( \lim \inf_{n \to \infty} c_n > 0 \). If for some \( \lambda > 0 \) we have

\[
2.8 \quad \lim_{n \to \infty} \sup \frac{b_{n+1}^\lambda (b_{n+1} - b_n)}{c_n b_n} < \infty
\]

and the series (1.6) converges for some \( \varepsilon > 0 \), then \( E|X|^{1+\lambda} < \infty \) and \( |EX| < \varepsilon \).

Proof. Using the inequality (see Feller [4], p.149)

\[
2.9 \quad P\{|X_1 + \ldots + X_n| \geq t\} \geq \frac{1}{n} \left(1 - e^{-n[1-F(t)+F(-t)]}\right)
\]

we infer from the convergence of series (1.6) that

\[
2.10 \quad \sum_{n=1}^{\infty} c_n \left(1 - e^{-n[1-F(\varepsilon b_n)+F(-\varepsilon b_n)]}\right) < \infty.
\]

Since \( b_n \to \infty \) and \( c_n \)'s are bounded away from 0 for \( n \) large enough, we must have

\[
2.11 \quad \lim_{n \to \infty} b_n [1-F(\varepsilon b_n)+F(-\varepsilon b_n)] = 0
\]

and hence

\[
2.12 \quad \sum_{n=1}^{\infty} c_n b_n [1-F(\varepsilon b_n)+F(-\varepsilon b_n)] < \infty.
\]

Again, (see Feller [4], p.151), we have \( E|X|^{1+\lambda} < \infty \) iff

\[
2.13 \quad \int_0^\infty x^{1+\lambda} [1-F(x)+F(-x)]dx < \infty.
\]

Also, from (2.8) it follows that for some constant \( M \) we have

\[
2.14 \quad b_{n+1}^\lambda (b_{n+1} - b_n) \leq M c_n b_n, \quad n=1,2,\ldots.
\]

Since the sequence \( \{b_n\} \) is strictly increasing, while \( 1-F(t)+F(-t) \) is nonincreasing, we bound the integral in (2.13) as follows:
Theorem 2. Suppose that $\lambda_{n+1}/\lambda_n \to 1$ and $a_n \to \infty$ as $n \to \infty$. If $U$ is a monotone function such that

$$\lim_{n \to \infty} [\lambda_n U(a_n x)] = x(x) \leq \infty$$

exists on a dense set, and $x$ is finite and positive in some interval, then $U$ varies regularly and $x(x) = c|x|^\rho$ for some $-\infty < \rho < \infty$.

We shall now prove

**Theorem 3.** Let $b_n/b_{n+1} \to 1$. Assume that for some $\lambda > 0$

$$\lim_{x \to \infty} x^{1+\lambda}[1-F(x)+F(-x)]$$

exists.
exists and is positive, say equal to c. Then the convergence of series (1.6) for some \( \epsilon > 0 \) implies (2.7).

**Proof.** As in the proof of Theorem 2, convergence of (1.6) implies (2.12). Let us write the series in (2.12) as

\[
\sum c_n b_n [1-F(\epsilon b_n)+F(-\epsilon b_n)] = \sum (c_n b_n^{-\lambda}) \left\{ b_n^{1+\lambda} [1-F(\epsilon b_n)+F(-\epsilon b_n)] \right\}.
\]

We now apply Lemma 2 with \( \lambda_n = b_n^{1+\lambda} \), \( a_n = b_n \), \( U(t) = 1 - F(t) + F(-t) \) and \( x = \epsilon \). As a result, \( \lim_{n \to \infty} \lambda_n U(a_n \epsilon) \) becomes \( \lim_{n \to \infty} b_n^{1+\lambda} [1-F(\epsilon b_n)+F(-\epsilon b_n)] \), which exists and is positive in view of the assumption of the theorem. Consequently, the latter limit equals \( ce^\rho \) for some \( \rho \). In fact, replacing \( x \) by \( \epsilon x \) in (2.18) we infer that \( \rho = -(1+\lambda) \). From the convergence of (2.19) it follows now that \( \sum c_n b_n^{-\lambda} < \infty \), as asserted. \( \square \)

As an example, consider the case when \( X \) has the central t-distribution with 2 degrees of freedom, so that \( E X^2 = \infty \) and \( E|X| < \infty \). Here the limit (2.18) exists with \( \lambda = 1 \) and \( c = 1/2 \), so that Theorem 3 applies.

Note that since the sequence \( \{b_n\} \) is strictly increasing, the condition (2.8) may be written as

\[
\lim_{n \to \infty} \inf \frac{c_n b_n^{-\lambda}}{(b_n+1/b_n)^\lambda \left(\frac{b_n+1}{b_n} - 1\right)} > 0.
\]

Now, if (2.7) holds, then \( c_n b_n^{-\lambda} \to 0 \), so that condition (2.20) (and hence (2.8)) may hold only if \( b_{n+1}/b_n \to 1 \).

Let us also note that the existence of the positive limit (2.18) implies \( E|X|^{1+\lambda} = \infty \), although \( E|X|^{1+\sigma} < \infty \), for all \( 0 < \sigma < \lambda \). Conversely, if
(2.21) \[ \sigma_0 = \sup \left\{ \sigma : \int_0^\infty x^\sigma [1-F(x) + F(-x)] dx < \infty \right\} \]

and

(2.22) \[ \int_0^\infty x^{\sigma_0} [1-F(x) + F(-x)] dx = \infty , \]

then

(2.23) \[ \lim_{x \to \infty} x^{1+\sigma} [1-F(x) + F(-x)] = 0 \]

for all \( \sigma < \sigma_0 \). Here we cannot say that the limit (2.23) is positive or 0 in the case with \( \sigma = \sigma_0 \).
References


