MELLIN TRANSFORMS FROM FOURIER TRANSFORMS

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ABSTRACT

Let $F$ be of bounded variation, $\hat{F}$ its Fourier-Stieltjes transform. Then if $0 < \Re(\alpha) < 1$, we obtain an explicit formula for $\int_{-\infty}^{\infty} x^{\alpha} dF$ if that integral exists, and if $\Re(\alpha) = 0, \alpha \neq 0$, we give an explicit limit for the integral with no restrictions.
Let $F$ be of bounded variation, $\hat{F}$ its Fourier-Stieltjes transform.

Then

**Theorem 1:** If $0 < \lambda = \Re(\alpha) < 1$ and $|x|^{-\lambda}$ is $F$-integrable,

\[
\begin{align*}
&\left(\frac{1}{\pi} \right)^{\frac{1}{2}} e^{\frac{1}{2} \pi \alpha} \int_{0}^{\infty} y^{-\alpha} dF(y) + \left(\frac{1}{\pi} \right)^{\frac{1}{2}} e^{\frac{1}{2} \pi \alpha} \int_{-\infty}^{0} |y|^{-\alpha} dF(y) = \frac{1}{\Re(\alpha)} \int_{0}^{\infty} t^{\alpha-1} \hat{F}(t) dt.
\end{align*}
\]

**Theorem 2:** If $\Re(\alpha) = 0$, $\alpha \neq 0$, and $(\delta, \sigma)$ approaches $(0,0)$ through positive pairs such that $\delta^{-\sigma}$ is bounded, then

\[
\begin{align*}
&\left(\frac{1}{\pi} \right)^{\frac{1}{2}} e^{\frac{1}{2} \pi \alpha} \int_{0}^{\infty} y^{-\alpha} dF(y) + \left(\frac{1}{\pi} \right)^{\frac{1}{2}} e^{\frac{1}{2} \pi \alpha} \int_{-\infty}^{0} |y|^{-\alpha} dF(y) \\
&= \frac{1}{\Re(\alpha)} \lim_{t \to 0} \int_{0}^{\infty} t^{\sigma+\alpha-1} e^{-\delta t} \hat{F}(t) dt - \delta^{-\sigma-\alpha} F(0).
\end{align*}
\]

The results of the abstract follow easily, since $e^{\frac{1}{2} \pi i \alpha} \neq e^{-\frac{1}{2} \pi i \alpha}$, and since the argument of $\delta^{-\alpha}$ can be made arbitrary.

We recall the following results from classical analysis: Let $\Re(\alpha) > 0, \Re(\beta) > 0, \Re(t) > 0, \mu$ not zero or a negative real number. Then without using analytic function theory, we can show that

\[
(A) \quad r(\alpha) t^{-\alpha} = \int_{0}^{\infty} x^{\alpha-1} e^{-tx} dx
\]

and

\[
(B) \quad \frac{r(\alpha)r(\beta)}{r(\alpha+\beta)\mu^\beta} = \int_{0}^{\infty} \frac{x^{\alpha-1}}{(x+\mu)^{\alpha+\beta}} dx,
\]

where the powers are all principal, i.e., if $w = e^v, v$ of imaginary part less than $\pi$ in magnitude, then $w^y = e^{yv}$. 
Let us now demonstrate the theorems. For the first, from (A)

(1) \[ t^{\alpha-1} = \frac{1}{\Gamma(1-\alpha)} \int_0^\infty x^{-\alpha}e^{-tx}dx, \]

and hence

(2) \[ \int_\varepsilon^M t^{\alpha-1} \int_{-\infty}^\infty e^{ity}dF(y)dt \]
\[ = \int_\varepsilon^M \int_0^\infty \frac{1}{\Gamma(1-\alpha)} x^{-\alpha} e^{-tx} \int_{-\infty}^\infty e^{ity}dF(y)dxdt \]
\[ = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^\infty \int_0^\infty \frac{x^{-\alpha}}{x-i\gamma} (e^{-\varepsilon(x-i\gamma)} - e^{-M(x-i\gamma)})dx dF(y), \]

since the right side is absolutely integrable in all variables. Now if \( y \neq 0 \) and \( \alpha = \lambda \), the inner integral is bounded by \( C_\lambda |y|^{-\lambda} \). Thus we can use the bounded convergence theorem and (B) to obtain

(3) \[ \int_0^\infty t^{\alpha-1} \int_{-\infty}^\infty e^{ity}dF(y)dt \]
\[ = \Gamma(\alpha) \int_{-\infty}^\infty (-iy)^{-\alpha}dF(y) \]
\[ = \Gamma(\alpha)(e^{\frac{1}{2}i\alpha} \int_0^\infty y^{-\alpha}dF(y) + e^{-\frac{1}{2}i\alpha} \int_{-\infty}^0 |y|^{-\alpha}dF(y)). \]

If \( \alpha = 0 \), it can be shown that the inner integral in the last expression of (2) is bounded, but no convergence is possible as \( \varepsilon \to 0 \).

However

(4) \[ \int_0^\infty t^{\sigma+\alpha-1}e^{-\delta t} \int_{-\infty}^\infty e^{ity}dF(y)dt \]
\[ = \frac{1}{\Gamma(1-\sigma-\alpha)} \int_0^\infty \int_0^\infty x^{-\sigma-\alpha} e^{-t(x+\delta)} \int_{-\infty}^\infty e^{ity}dF(y)dxdt \]
\[ = \frac{1}{\Gamma(1-\sigma-\alpha)} \int_{-\infty}^\infty \int_0^\infty x^{-\sigma-\alpha} \frac{1}{x+\delta-i\gamma} dxdF(y), \]
and the inner integral is equal to $\Gamma(\sigma+\alpha)\Gamma(1-\sigma-\alpha)(\delta-iy)^{-\sigma-\alpha}$. Thus

$$
(5) \quad \frac{1}{\Gamma(\sigma+\alpha)} \left[ \int t^{\sigma+\alpha-1} e^{-\delta t} \int e^{iyt} dF(y) dt - \delta^{-\sigma-\alpha} F(0) \right] \\
= \int_{y \neq 0} (\delta-iy)^{-\sigma-\alpha} dF(y).
$$

The result then follows from the continuity of the gamma function and the application of the bounded convergence theorem to the right side of (5).