SIMULTANEOUS DIAGONALIZATION OF RECTANGULAR MATRICES(1)
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1. INTRODUCTION

Let A, B be matrices of order \( m \times n \) with elements from a field \( \mathcal{F} \). The vectorspace spanned by such matrices is denoted by \( \mathcal{F}^{m \times n} \). A matrix \( D \in \mathcal{F}^{m \times n} \) is said to be diagonal if \( (D)_{ij} \), the element in the \( (i,j) \)th position of \( D = 0 \) whenever \( i \neq j \). We ask ourselves the following question:

'Given a pair of matrices \( A, B \in \mathcal{F}^{m \times n} \) does there exist nonsingular matrices \( S \in \mathcal{F}^{m \times m} \) and \( T \in \mathcal{F}^{n \times n} \) such that

\[
SAT = D_a, \quad SBT = D_b
\]

(1.1)

where \( D_a \) and \( D_b \) are diagonal matrices in \( \mathcal{F}^{m \times n} \)?'

If \( A \) and \( B \) represent linear transformations from an \( n \) dimensional vectorspace \( V_n(\mathcal{F}) \) to an \( m \)-dimensional vectorspace \( V_m(\mathcal{F}) \) with reference to chosen bases in \( V_m(\mathcal{F}) \) and \( V_n(\mathcal{F}) \) we are thus essentially seeking changes in bases so that the transformations could be described in simpler terms through diagonal matrices.

Theorem 3.1 provides necessary and sufficient conditions for (1.1) to hold. Simultaneous diagonability of a set \( \{A_0\} \) of matrices in \( \mathcal{F}^{m \times n} \) is studied in Theorem 4.1. We note here that since the vectorspace \( \mathcal{F}^{m \times n} \)

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is finitedimensional one may without any loss of generality assume that set \( \{A_{\theta}\} \) so studied consists of only a finite number of such matrices.

Williamson [12] showed that complex matrices \( A \) and \( B \) can be simultaneously diagonalized as in (1.1) through unitary matrices \( S \) and \( T \) iff \( AB^* \) is normal, where \( * \) on a matrix indicates its complex conjugate transpose. Necessary and sufficient conditions for the existence of unitary matrices \( S \) and \( T \) such that

\[
SA_{\theta}T = D_{\theta}
\]

is diagonal for each \( A_{\theta} \) in a set \( \{A_{\theta}\} \) of complex matrices are given by Gibson [3]. The reader is referred to Gibson [3] for a bibliography on other related work in this area.

2. SOME OTHER NOTATIONS AND PRELIMINARY RESULTS

\( \mathfrak{B}^m \) denotes the vector space of \( m \)-tuples with elements in \( \mathfrak{B} \). Lower case letters \( a, b \) indicate column vector representations of such \( m \)-tuples. For a matrix \( A \), \( \mathfrak{m}(A) \) denotes its column span and \( \mathfrak{n}(A) \) its null space. \( A^T \) denotes the transpose of \( A \). \( A^\dagger \), a generalized inverse (g-inverse) of \( A \), is a matrix \( A^\dagger \) satisfying the equation \( AA^\dagger A = A \) [11]. The class of all possible g-inverses of \( A \) is denoted by \( \{A^\dagger\} \). Two subspaces of a vector space are said to be virtually disjoint if they have only the null vector in common.

**Definition 2.1:** Given a matrix \( A \in \mathfrak{B}^{m \times n} \) and subspaces \( \mathfrak{g} \subset \mathfrak{B}^m, \mathfrak{j} \subset \mathfrak{B}^n \), the shorted matrix \( S(A|\mathfrak{g}, \mathfrak{j}) \) is a matrix \( C \in \mathfrak{B}^{m \times n} \) such that

\[
\mathfrak{m}(C) \subset \mathfrak{g}, \quad \mathfrak{n}(C^\dagger) \subset \mathfrak{j}
\]

and if \( E \) is any matrix \( \in \mathfrak{B}^{m \times n} \) that satisfies (2.1) then

\[
\text{Rank} (A - E) \geq \text{Rank} (A - C)
\]
This definition extends the notion of a shorted positive operator studied by Krein [6], Anderson and Trapp [1] and Mitra and Puri [8]. Shorted matrices are studied in greater detail elsewhere [9].

Let \( X \in \mathcal{A}^{m \times p}, Y \in \mathcal{A}^{q \times n} \) be such that
\[
\mathcal{S} = \mathbb{M}(X), \quad \mathcal{J} = \mathbb{M}(Y)
\]
and 0 be the null matrix in \( \mathcal{A}^{q \times p} \). We consider the bordered matrix
\[
F = \begin{pmatrix}
A & X \\
Y & 0
\end{pmatrix}
\]  
(2.3)

and let
\[
G = \begin{pmatrix}
C_1 & C_2 \\
C_3 & -C_4
\end{pmatrix} \in \mathcal{F}^{-}
\]  
(2.4)

where \( C_1 \in \mathcal{A}^{n \times m}, C_2 \in \mathcal{A}^{n \times q}, C_3 \in \mathcal{A}^{p \times m}, \) and \( C_4 \in \mathcal{A}^{p \times q} \).

Theorem 2.1 gives a set of necessary and sufficient conditions for the existence of an unique shorted matrix \( S(A|\mathcal{S},\mathcal{J}) \) and provides an explicit expression for the same.

**Theorem 2.1.** (a) The shorted matrix \( S(A|\mathcal{S},\mathcal{J}) \) exists and is unique iff the matrix \( F \) satisfies the rank additivity conditions
\[
\text{Rank } F = \text{Rank } (A:X) + \text{Rank } Y = \text{Rank } \begin{pmatrix} A \\ Y \end{pmatrix} + \text{Rank } X.
\]  
(2.5)

(b) When (2.5) is satisfied,
(i) \( C_2 \in \{Y^-\}, C_3 \in \{X^-\}, \)

(ii) \( AC_2 Y, XC_3 A \) and \( XC_4 Y \) are invariant under the choice of \( G \) in (2.4) and further
\[
AC_2 Y = XC_3 A = XC_4 Y = A - AC_1 A \quad \text{(say)}.
\]  
(2.6)

(iii) The matrix \( C \) in (2.6) is the unique shorted matrix \( S(A|\mathcal{S},\mathcal{J}) \).
Proof. The 'if' part of (a) and the (b) part of Theorem 2.1 are proved for complex matrices in [7]. (See Theorems 1 and 2 and Remark 1 following Theorem 2 in [7].) The transition from the complex field to arbitrary field \( \mathcal{F} \) presents no special difficulties. Theorem 1 in [7] is a generalization of similar theorems due to Khatri [5] and Rao [10].) To prove the 'only if' part of (a) assume now that \( A_0 = S(A|\mathcal{S}, \mathcal{F}) \) is the unique shorted matrix. Write \( A = A_0 + A_1 \) and observe that the uniqueness of the shorted matrix \( S(A|\mathcal{S}, \mathcal{F}) \) implies that \( \mathcal{M}(A_1) \) is virtually disjoint with \( \mathcal{S} \) and \( \mathcal{M}(A_1^\prime) \) with \( \mathcal{F} \). If \( \mathcal{M}(A_1) \) is not virtually disjoint with \( \mathcal{S} \), let \( \ell_1 \) be a nonnull \( m \)-tuple in \( \mathcal{M}(A_1) \cap \mathcal{S} \). Let \( A_1 \) be of rank \( s \). Consider a rank factorization of \( A_1 \)

\[
A_1 = LR
\]

where \( L = (\ell_1; \ell_2; \ldots; \ell_s) \), \( R^\prime = (r_1; r_2; r_s) \). For any nonnull \( n \)-tuple \( t_1 \) in \( \mathcal{F} \), the matrix \( E = A_0 + \ell_1 t_1 \) satisfies condition (2.1) and further \( A - E \) has the same rank as \( A - A_0 = A_1 \). This contradicts the uniqueness of the shorted matrix \( S(A|\mathcal{S}, \mathcal{F}) \). A similar argument shows that \( \mathcal{M}(A_1^\prime) \) is virtually disjoint with \( \mathcal{F} \). If \( \mathcal{M}(A_1^\prime) \) is not virtually disjoint with \( \mathcal{M}(A_1^\prime) \), let vectors \( a \in \mathcal{F}^n \), \( b \in \mathcal{F}^m \) be such that

\[
\begin{align*}
Aa &= Xb \neq 0, \\
Ya &= 0.
\end{align*}
\]

(2.7) \( A_1 a = Xb \neq 0 \) which contradicts the assumption that \( \mathcal{M}(A_1) \) is virtually disjoint with \( \mathcal{S} \). The other part of (2.5) is similarly established.

Q.E.D.

We also need an explicit representation of a \( g \)-inverse of \( F \), given in Theorem 2.2. The proof is by direct computation. The complex version of Theorem 2.2 appears as Theorem 3 in [7]. This generalizes a theorem of Hall and Meyer [4].
Theorem 2.2. For any choice of the g-inverses of $X$, $Y$ and $E_XA^F_Y$,
\[
\begin{pmatrix}
0 & Y^- \\
X^- & -X^\top AY^-
\end{pmatrix}
+ \begin{pmatrix}
I & (I - AY^-) \\
-X^-A & 0
\end{pmatrix}
\tag{2.8}
\]
is a g-inverse of $F$, where $Q = F_Y(E_XA^F_Y)^\top E_X E_X = I - XX^-$ and $F_Y = I - Y^- Y$.

3. SIMULTANEOUS DIAGONALIZATION OF A PAIR OF MATRICES

Theorem 3.1. Let $A, B \in \mathbb{C}^{m \times n}$. There exists a pair of nonsingular matrices satisfying (1.1) iff.

(a) $\text{Rank} \begin{pmatrix} A & B \\ B & 0 \end{pmatrix} = \text{Rank} (A; B) + \text{Rank} B = \text{Rank} \begin{pmatrix} A \\ B \end{pmatrix} + \text{Rank} B \tag{3.1}$

and

(b) $AC_2BC_2$ is semisimple (or equivalently $C_3BC_3A$ is semisimple) \(3.2\)

where \(\begin{pmatrix} C_1 & C_2 \\ C_3 & -C_4 \end{pmatrix}\) is any g-inverse of $F = \begin{pmatrix} A & B \\ B & 0 \end{pmatrix}$.

Proof. ('only if' part) We assume here that nonsingular $S$ and $T$ exist such that

\[SAT = D_a, SBT = D_b\]

where $D_a$ and $D_b$ are diagonal matrices. It is easily seen that

\[\text{Rank} \begin{pmatrix} D_a & D_b \\ D_b & 0 \end{pmatrix} = \text{Rank} (D_a; D_b) + \text{Rank} D_b = \text{Rank} \begin{pmatrix} D_a \\ D_b \end{pmatrix} + \text{Rank} D_b\]

Hence (3.1) follows.

Further the matrix \(\begin{pmatrix} C_1 & C_2 \\ C_3 & -C_4 \end{pmatrix}\) is a g-inverse of $F$ iff $C_1 = T \tilde{C}_1 S$, 


\[ C_2 = T\overline{C}_2S, \ C_3 = T\overline{C}_3S, \ C_4 = T\overline{C}_4S \] where \[ \begin{pmatrix} \overline{c}_1 & \overline{c}_2 \\ \overline{c}_3 & -\overline{c}_4 \end{pmatrix} \] is a g-inverse of \[ \begin{pmatrix} D_a & D_b \\ D_b & 0 \end{pmatrix}. \]

We now show that there exists a choice of a g-inverse of \[ \begin{pmatrix} D_a & D_b \\ D_b & 0 \end{pmatrix} \] such that \( \overline{c}_2 \) and \( \overline{c}_3 \) are both diagonal. For this we use formula (2.8) and substitute for \( D_b^- \) and \( Q \) the matrices defined as follows:

\[ (D_b^-)^{-1}_{ii} = 1/(D_b)_{ii} \text{ if } (D_b)_{ii} \neq 0, \ (D_b^-)^{-1}_{ij} = 0, \text{ otherwise,} \] \hspace{1cm} (3.3)

\[ (Q)^{-1}_{ii} = 1/(D_a)_{ii} \text{ if } (D_a)_{ii} \neq 0 \text{ and } (D_b)_{ii} = 0 \] \hspace{1cm} (3.4)

Since \( D_b^- \) and \( Q \) are diagonal matrices

\[ \overline{c}_2 = D_b^- - QD_a D_b^- \]

is diagonal and

\[ AC_2 BC_2 = S^{-1} D_a T^{-1} T\overline{C}_2 SS^{-1} D_b T^{-1} T\overline{C}_2 S = S^{-1} D_1 S, \]

where \( D_1 = D_a \overline{c}_2 D_b \overline{c}_2 \in \mathcal{M}_{m \times m} \), and is diagonal. This establishes the fact that \( AC_2 BC_2 \) is semisimple. We now show that if (3.1) holds the semisimplicity of \( AC_2 BC_2 \) is equivalent to semisimplicity of \( AC_2 BB^- \) for any choice of \( B^- \).

This follows from the fact that if \( x \) is an eigenvector of \( AC_2 BC_2 \) for a nonnull eigenvalue \( \lambda \),

\[ AC_2 BC_2 x = \lambda x \Rightarrow AC_2 x = \lambda x, \] \hspace{1cm} (3.5)

since \( x \in \mathcal{M}(AC_2 B) = \mathcal{M}(BC_3 A) \subset \mathcal{M}(B) \) and \( C_2 \in \{ B^- \} \). For the same reason

\[ AC_2 BB^- x = AC_2 x = \lambda x. \] \hspace{1cm} (3.6)

This shows that \( x \) is an eigenvector of \( AC_2 BB^- \) for the same eigenvalue \( \lambda \)
and vice versa. Since \( \text{Rank}(AC_2BC_2) = \text{Rank}(AC_2BB^-) = \text{Rank}(AC_2B) \), the equivalence of the two statements follows.

Since \( AC_2B \) is invariant under choice of a \( g \)-inverse of \( F \), if \( AC_2BC_2 \) is semisimple for one choice of this \( g \)-inverse it is so for every other choice.

('if' part): Let \( B \) be of rank \( r \). Consider a rank factorization of \( B \),

\[
B = UV,
\]

where \( U \in \mathcal{S}^{m \times r} \), \( V \in \mathcal{S}^{r \times n} \).

Since \( \mathcal{M}(AC_2B) \subseteq \mathcal{M}(B) \), \( \mathcal{M}(B^*C_2A^*) \subseteq \mathcal{M}(B^*) \)

\[
AC_2B = UKV
\]

for some \( K \in \mathcal{S}^{r \times r} \). Choose and fix a \( g \)-inverse of \( B \), \( B^- = V^{-1}U^{-1}_L \) where \( U^{-1}_L \) and \( V^{-1}_R \) are respectively left and right inverses of \( U \) and \( V \). Semi-

simplicity of \( AC_2BC_2 \) implies semisimplicity of \( AC_2BB^- = UKU_L^{-1} \) which in turn implies semisimplicity of \( K \). Put \( K = WDW^{-1} \) where \( W, D \in \mathcal{S}^{r \times r} \) and \( D \) is diagonal. Then

\[
AC_2B = UKV = UWDW^{-1}V = S_1DT_1
\]

where \( S_1 = UW \), \( T_1 = W^{-1}V \). Check that \( B = S_1T_1 \). Also, let \( S_2T_2 \) be a rank factorization of \( A-AC_2B \).

\[ \mathcal{M}(A-AC_2B) \cap \mathcal{M}(B) = \{0\} \] and

\[ \mathcal{M}(A^*-B^*C_2A^*) \cap \mathcal{M}(B^*) = \{0\} \] follows from (3.1) and the proof of Theorem 2 of [7].

Hence \( \mathcal{M}(S_2) \) is virtually disjoint with \( \mathcal{M}(S_1) \) and \( \mathcal{M}(T_2^*) \) with \( \mathcal{M}(T_1^*) \). Let \( S_3 \) and \( T_3 \) be so chosen that \( (S_1:S_2:S_3) \) and \( (T_1^*:T_2^*:T_3^*) \) are nonsingular. Put \( S^{-1} = (S_1:S_2:S_3) \), \( (T^*)^{-1} = (T_1^*:T_2^*:T_3^*) \)

and check that

\[
\text{SAT} = D_a \text{ and } \text{SBT} = D_b
\]

where \( D_a = \text{diag}(D,I,0) \), \( D_b = \text{diag}(I,0,0) \) are clearly diagonal matrices.

This completes the proof of the 'if' part and of Theorem 3.1. Q.E.D.
4. SIMULTANEOUS DIAGONALIZATION OF SEVERAL MATRICES

Without any loss of generality let us assume here that $m \leq n$. We shall further assume here that the field $\mathcal{F}$ contains more than $m$ distinct nonnull elements.

We need the following result.

Lemma 4.1. If matrices $A$ and $B$ satisfy condition (3.1) there exists a nonnull scalar $k$ such that

$$\mathcal{M}(A) \subseteq \mathcal{M}(A+kB), \quad \mathcal{M}(A^-) \subseteq \mathcal{M}(A^-+kB^-)$$

or equivalently

$$\mathcal{M}(A) \subseteq \mathcal{M}(A+kB), \quad \mathcal{M}(A^-) \subseteq \mathcal{M}(A^-+kB^-)$$

and

$$\text{Rank}(B(A+kB)^{-1}B) = \text{Rank} B.$$  \hfill (4.1c)

Conversely (4.1a) or (4.1b) and (4.1c) imply (3.1).

Proof. Assume now that (3.1) holds and let

$$
\begin{pmatrix}
    C_1 & C_3 \\
    C_2 & -C_4
\end{pmatrix} \in \begin{bmatrix}
    (A & B) \\
    (B & 0)
\end{bmatrix}
$$

Let $k$ be so chosen that $k \neq 0$ and

$$\det(BC_4+kI) \neq 0.$$  

Clearly

$$\mathcal{M}(BC_4B) \subseteq \mathcal{M}(B) = \mathcal{M}(BC_4B+kB) = \mathcal{M}(B^-C_4^-B^-) \subseteq \mathcal{M}(B^-) = \mathcal{M}(B^-C_4^-B^-+kB^-).$$

Since $\mathcal{M}(A-BC_4B) \cap \mathcal{M}(B) = \{0\}$ and $\mathcal{M}(A^-B^-C_4^-B^-) \cap \mathcal{M}(B^-) = \{0\}$ follows from (3.1) and the proof of Theorem 2 of [7]. Hence

$$\mathcal{M}(A) = \mathcal{M}(A-BC_4B+BC_4B) = \mathcal{M}(A-BC_4B) + \mathcal{M}(BC_4B)$$

$$\subseteq \mathcal{M}(A-BC_4B)+\mathcal{M}(BC_4B+kB)$$

$$= \mathcal{M}(A-BC_4B+BC_4B+kB) = \mathcal{M}(A+kB).$$
and similarly \( \mathcal{M}(A^r) \subseteq \mathcal{M}(A^r + kB^r) \). This establishes (4.1a). (4.1b) is trivial.

If (4.1a) holds, the matrix \( \begin{pmatrix} A + kB & B \\ 0 & 0 \end{pmatrix} \) can be reduced to \( \begin{pmatrix} A + kB & 0 \\ 0 & B(A + kB)^{-1}B \end{pmatrix} \) through sweep out operations on its rows and columns. Hence

\[
\text{Rank}\begin{pmatrix} A & B \\ B & 0 \end{pmatrix} = \text{Rank}\begin{pmatrix} A + kB & B \\ B & 0 \end{pmatrix} = \text{Rank}(A + kB) + \text{Rank} B(A + kB)^{-1}B
\]

\[
= \text{Rank}\begin{pmatrix} A \\ B \end{pmatrix} + \text{Rank} B(A + kB)^{-1}B = \text{Rank}(A:B) + \text{Rank} B(A + kB)^{-1}B,
\]

and (3.1) implies (4.1c). Conversely the same argument shows that (4.1c) imply (3.1).

Q.E.D.

Theorem 4.1. Let \( A_1, A_2, \ldots, A_p \in \mathfrak{M}^{m \times n} \). The following two statements are equivalent.

(a) There exists nonsingular matrices \( S \in \mathfrak{M}^{m \times m}, T \in \mathfrak{A}^{n \times n} \) such that

\[
SA_i T = D_i, \quad i = 1, 2, \ldots, p
\]

(4.3)

where each \( D_i \) is a diagonal matrix in \( \mathfrak{M}^{m \times n} \).

(b) There exists nonnull scalars \( k_2, \ldots, k_p \) in \( \mathfrak{M} \) such that if

\[
A_0 = A_1 + k_2 A_2 + \ldots + k_p A_p,
\]

then for \( i = 1, 2, \ldots, p; j = 1, 2, \ldots, p, \)

(i) \( \mathcal{M}(A_i) \subseteq \mathcal{M}(A_0) \), \( \mathcal{M}(A_i^2) \subseteq \mathcal{M}(A_0^2) \),

(ii) \( A_i A_0^{-1} \) is semisimple,

(iii) \( A_i A_0^{-1} A_j = A_j A_0^{-1} A_i \).

(4.4, 4.5, 4.6, 4.7)

(2) Lemma 4.1 is false if the field contains only \( m \) distinct nonnull elements or less.
Proof. (a) ⇒ (b): Since \(A_1\) and \(A_2\) are simultaneously reducible to diagonal matrices using Theorem 3.1 and then Lemma 4.1, a nonnull scalar \(k_2\) can be determined so that if
\[
A(2) = A_1 + k_2 A_2
\]
then
\[
\mathbb{m}(A_1) \subseteq \mathbb{m}(A(2)), \mathbb{m}(A_1) \subseteq \mathbb{m}(A(2))
\]
\[
\mathbb{m}(A_2) \subseteq \mathbb{m}(A(2)), \mathbb{m}(A_2) \subseteq \mathbb{m}(A(2)).
\]

Since \(A(2)\) and \(A_3\) are simultaneously reducible to diagonal matrices, the same argument can be repeated and the nonnull scalars \(k_2, k_3, \ldots, k_p\) can be recursively determined so as to satisfy (4.5).

Let \(D_0 = D_1 + k_2 D_2 + \ldots + k_p D_p\). Then \(D_0\) is diagonal and
\[
S A_0^T = D_0.
\]

As in the proof of Theorem 3.1 it is seen that if \(A_i A_0^-\) is semisimple for some choice of \(A_0^-\) it is so for every other choice. Choose for \(D_0^-\) the following diagonal matrix in \(\mathbb{F}^{n \times m}\)
\[
(D_0^-)_{ii} = 1/(D_0)_{ii} \text{ if } (D_0)_{ii} \neq 0
\]
\[
(D_0^-)_{ij} = 0, \text{ otherwise.}
\]

It is seen that \(T D_0^- S \in (A_0^-)\) and with this choice of \(A_0^-\) the truth of (4.6) and (4.7) are easily verified. We note that on account of (4.5), \(A_i A_0^- A_j\) is invariant under choice of \(A_0^-\).

(b) ⇒ (a): Consider a rank factorization of \(A_0\),
\[
A_0 = U V,
\]
where \(U \in \mathbb{F}^{m \times r}, V \in \mathbb{F}^{r \times n}\) and \(r = \text{Rank } A_0\). Choose and fix a g-inverse \(A_0^-\) where
\[ A_0^- = V^{-1} U^{-1}_L \]

and \( U^{-1}_L \) and \( V^{-1}_R \) are respectively left and right inverses of \( U \) and \( V \) (4.5)

\[ A_i = U B_i V \]

for some matrix \( B_i \in \mathbb{F}^{r \times r} \)

\[ A_i A_0^- = U B_i U^{-1}_L. \]

Since on account of (4.6) and (4.7) the matrices \( A_i A_0^- \) commute and are semisimple, it follows that the matrices \( B_i \) commute and are semisimple. Hence there exists a nonsingular matrix \( W \in \mathbb{F}^{r \times r} \) such that

\[ W^{-1} B W = D_i, \quad i = 1, 2, \ldots, p \]

where \( D_1, D_2, \ldots, D_p \) are diagonal matrices. The rest of the proof of Theorem 4.1 can be completed on the same lines as in the proof of the 'if' part of Theorem 3.1. Q.E.D.

Theorem 4.2 is an extension of Theorem 6 of Bhimasankaram [2].

Theorem 4.2. Let \( A_1, A_2, \ldots, A_p \) be complex hermitian matrices of order \( n \times n \). Then there exists a nonsingular matrix \( T \) such that \( T A_i T \) is diagonal for each \( i \) iff there exists nonnull real scalars \( k_2, k_3, \ldots, k_p \) such that if

\[ A_0 = A_1 + k_2 A_2 + \ldots + k_p A_p \]

then for \( i = 1, 2, \ldots, p; j = 1, 2, \ldots, p \)

(a) \( \mathcal{R}(A_i) \subseteq \mathcal{R}(A_0) \)

(b) \( A_i A_0^- \) is semisimple with real eigenvalues for some \( g \)-inverse \( A_0^- \) of \( A_0 \).

(c) \( A_i A_0^- A_j = A_j A_0^- A_i \).
Proof: The 'only if' part follows from the corresponding part of Theorem 4.1 since here without any loss of generality one can restrict the scalar \( k_i \) to be real. The 'if' part follows from Theorem 6 of Bhimasankaram [2].

Q.E.D.
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