D'ALEMBERT'S FUNCTIONAL EQUATION ON GROUPS

by

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I. Reduction to a representation theory problem. We consider a functional equation of the form

$$\phi_1(gh) + \phi_2(g^{-1}h) = \sum_{i=1}^{n} \kappa_i(g)\lambda_i(h)$$

(*)

where $\phi_1, \phi_2, \kappa_i, \lambda_i, i = 1, \ldots, n$ are some (measurable) complex functions given on a locally compact group $G$. We can and shall assume that functions $\kappa_i$ and $\lambda_i, i = 1, \ldots, n$ are linearly independent. The equation (*) can be viewed as a generalization of the known D'Alembert's functional equation

$$\phi(gh) + \phi(g^{-1}h) = 2\phi(g)\phi(h),$$

which was studied by many authors (cf 4-6). A particular case of (*) when $\phi_1 = -\phi_2$ arises in some statistical applications and was studied by one of the authors in the situation when $G$ is a compact Lie group [6]. The solution of (*) in the case $G = \mathbb{R}^1$ can be found in the Aczel's book [1] (pp. 171-176, 199).

In this paper we present the solution of (*) in the case when

$$\int |\phi_i(g)|^2 dv(g) < \infty \quad i = 1, 2,$$

where $v$ is the left Haar measure on $G$. Henceforth we denote the space of such functions as $L_2(G)$. We also consider the case when $\phi_i, i = 1, 2$ can be represented as a finite combination of positive definite functions.

Note that if $q(g) = \phi_1(g) + \phi_2(g), \xi(g) = \phi_1(g) - \phi_2(g)$ then

$$q(gh) + q(g^{-1}h) = \sum_{i=1}^{n} [\kappa_i(g) + \kappa_i(g^{-1})]\lambda_i(h) = \sum_{i=1}^{m} a_i(g)\xi_i(h)$$

(**)

and
\[ \xi(gh) - \xi(g^{-1}h) = \sum_{j=1}^{n} \left[ \kappa_{j}(g) - \kappa_{j}(g^{-1}) \right] \lambda_{j}(h) = \sum_{j=1}^{r} \beta_{j}(g) \xi_{j}(h) \quad (***) \]

with linearly independent functions \( \alpha_{i} \), \( \phi_{i} \), \( i = 1, \ldots, m \) and \( \beta_{j} \), \( \xi_{j} \), \( j = 1, \ldots, r \).

Therefore we restrict our attention to the case \( \phi_{1} = \phi_{2} \) or \( \phi_{1} = -\phi_{2} \) in (*)

We reduce the solution of (*) to a certain problem in representation theory in the following way. Let us begin with the equation (**). If \( \eta(g) \), \( g \in G \), of the function \( \eta \) then the left regular representation \( U \) acts in \( H \): \( U(g)\eta(\cdot) = \eta(g^{-1} \cdot) \), \( \eta \in H \) and \( U(g) \) is a unitary operator. The relation (***) implies that

\[ [U(g) + U(g^{-1})] \eta = \sum_{i=1}^{m} \alpha_{i}(g) \phi_{i}, \]

where \( \eta \) denotes the vector of \( L \) corresponding to the function \( \eta(\cdot) \), and \( \phi_{1}, \ldots, \phi_{m} \) are vectors from \( H \). Because of the definition of \( H \), \( U \) is a cyclic representation with a cyclic vector \( \phi \) (i.e. the space spanned by vectors \( U(g) \phi, g \in G \) is dense in \( H \)). The following results is a corollary of these considerations and the theorem 2.1.

THEOREM 1.1. Let \( G \) be a locally compact group of type one such that the elements of the form \( g^{4}, g \in G \), generate a dense subgroup of \( G \). Every solution \( \eta \in L_{2}(G) \) of the equation (**) has the form

\[ \eta(g) = \langle U(g^{-1}) \eta, \eta \rangle \]

where \( U \) is a finite dimensional, unitary, cyclic (with a cyclic vector \( \eta \)) representation of \( G \) such that the space spanned by vectors \([U(g) + U(g^{-1})] \eta, g \in G \) has dimension \( m \), and \( \eta \) is some vector of the representation space \( U \).

Note that a formula for the dimension of the representation \( U \) can be obtained from the theorem 2.1.
It is immediately seen that the functional equation (***) is equivalent to the finite dimensionality of the space spanned by vectors \([U(g) - U(g^{-1})]\xi\), \(g \in G\) where again \(U\) is a unitary representation with a cyclic vector \(\xi\).

**THEOREM 1.2.** Let \(G\) be a locally compact group of type one such that the elements of the form \(g^2\), \(g \in G\) generate a dense subgroup of \(G\). Every solution \(\xi \in L_2(G)\) has the same form as indicated in theorem 1.1. with \(\xi\) instead of \(\eta\), \([U(g) - U(g^{-1})]\xi\) instead of \([U(g) + U(g^{-1})]\eta\) and \(r\) instead of \(m\).

The solution of the general equation (*) now follows easily from Theorems 1.1 and 1.2.

The content of these theorems is that if \(G\) is noncompact then D'Alembert's functional equation has few solutions, as non-compact groups, usually, have few finite dimensional unitary representations.

**THEOREM 1.3.** Under assumptions of the theorem 1.1. if there exists a non-zero solution of (***) then \(G\) is compact.

**Proof.** It follows from the theorem 1.1. that every solution of (***) has the form

\[ q(g) = <U(g^{-1})q, n> \]

with a finite dimensional representation \(U\). However, such a matrix element cannot be square integrable unless \(G\) is compact. Indeed let \(K\) be the kernel of \(U\). Clearly \(K\) is closed, normal subgroup of \(G\) and \(q\) is constant on cosets of \(K\) in \(G\). In order for \(q\) to be square integrable \(K\) must have finite volume under Haar measure which implies compactness of \(K\).

To prove that \(G\) is compact it suffices to show that \(G/K\) is compact. To this end we may assume that \(K = \{e\}\) so that \(U\) is injective. But then \(G\) is compactly injectible, and hence \(C\) is the product of a compact group and \(R^p\) for
some $p$ (cf. [2] s. 16.4.2). But $\mathbb{R}^p$ has no injective, finite dimensional unitary representations unless $p = 0$. Thus $G$ is compact.

We give another version of theorem 1.1.

THEOREM 1.4. Theorem 1.1 holds if one assumes that $\varphi$ is a linear combination of positive definite functions instead of $\varphi \in L_2(G)$.

Proof. It follows from Godement [3] that there exists a unitary representation $U$ and vectors $\varphi$ and $\eta$ such that

$$\varphi(g) = \langle U(g^{-1})\varphi, \eta \rangle.$$  

These vectors $\varphi$ and $\eta$ can be assumed to be cyclic for $U$ (the latter since $\varphi(g) = \langle \varphi, U(g)\eta \rangle$). Moreover we can replace $\eta$ by its projection onto the closed subspace spanned by the vectors $U(g^{-1})\varphi$. Because of (**) the space of functions $\langle [U(g) + U(g^{-1})]\varphi, U(\cdot)\eta \rangle$, $g \in G$ if finite dimensional. The cyclicity of $\eta$ implies that the space spanned by the vectors $[U(g) + U(g^{-1})]\varphi$ is finite dimensional and the cyclicity of $\varphi$ and theorem 2.1 imply that $U$ is finite dimensional.

2. Symmetric and anti-symmetric intertwining operators. Let $U$ be a unitary representation of the topological group $G$ in a Hilbert space $H$. Let $H^*$ be the continuous dual of $H$ and let $U^*$ be the contragradient representation to $U$, i.e.

$$\langle \xi, U^*(g)\eta \rangle = \langle U(g^{-1})\xi, \eta \rangle$$

Clearly $H^*$ is a Hilbert space which is conjugate isomorphic with $H$ and $U^*$ is a unitary operator.

Let $I(U,U^*)$ denote the set of all continuous operators $A$ mapping $H$ into $U^*$ such that

$$U^*(g)A = A U(g).$$
If \( A^* \) is the dual operator, \( A^*: H^{**} = H \to H^* \), then \( A \) is said to be symmetric if \( A^* = A \) and anti-symmetric if \( A^* = -A \). The space of symmetric elements of \( I(U, U^*) \) is denoted \( I_s(U, U^*) \) and the space of anti-symmetric elements is denoted \( I(U, U^*) \). Clearly

\[
I(U, U^*) = I_s(U, U^*) + I_a(U, U^*).
\]

Note that \( A^* \) is not the same as the Hilbert space adjoint of \( A \) which is a mapping of \( H^* \) to \( H \).

We prove the following.

**Theorem 2.1.** Let \( G \) be a locally compact group and \( U \) a type one unitary representation of \( G \) which possesses a cyclic vector \( \varphi \). Let \( L_+ \) and \( L_- \) be the subspaces of \( H \) defined as

\[
L_+ = \text{span}\{[U(g) \pm U(g^{-1})]\varphi, \ g \in G\}.
\]

Then

(a) If \( L_- \) is finite dimensional and the elements of the form \( g^2, \ g \in G \), generate a dense subgroup of \( G \), then \( U \) is finite dimensional and

\[
dim U = \dim L_- + \dim I_s(U, U^*).
\]

(b) If \( L_+ \) is finite dimensional and the elements of the form \( g^4, \ g \in G \), generate a dense subgroup of \( G \) then \( U \) is finite dimensional and

\[
dim U = \dim L_+ + \dim I_a(U, U^*).
\]

**Proof.** Since the proofs of (a) and (b) are similar we prove only (b). Let \( L' \) be the annihilator of \( L_+ \) in \( H^* \). We shall establish a one to one correspondence between \( L' \) and \( I_a(U, U^*) \). Specifically, the correspondence will be obtained as follows. Let \( \lambda \in L' \). For each vector of the form
\[ \xi = \sum_i c_i U(g_i) \varphi \quad \text{define} \quad B_\lambda(\xi) = \sum_i c_i U^*(g_i) \lambda. \]

We shall show that this correspondence is well defined. Granting this \( B_\lambda \) becomes a densely defined intertwining operator from \( H \) to \( H^* \). The main problem in proving our theorem is to demonstrate that \( B_\lambda \) is in fact a bounded operator. This will be achieved by expanding the domain of \( B \) as much as possible.

Now, let \( V = U \oplus U^* \), and let \( \mathcal{A} \) be the von-Neumann algebra on \( H \oplus H^* \) generated by the set of operators \( \{V(g), g \in G\} \). Each element of \( \mathcal{A} \) is an operator of the form \( A \oplus A' \) where

\[ A = \lim_{\alpha} \sum_{g \in G} c^\alpha(g) U(g), \quad A' = \lim_{\alpha} \sum_{g \in G} c^\alpha(g) U^*(g), \]

and \( c^\alpha \) is a net of functions on \( G \) which are supported on finite sets on \( G \). The limits are taken in the strong operator topology.

**Lemma 2.1.** Suppose \( A \oplus A' \in \mathcal{A} \) and \( \lambda \in L' \). Then \( A\varphi = 0 \) implies that \( A'\lambda = 0 \).

**Proof.** Let \( A \) and \( A' \) be represented as described above and consider the function

\[ f(h) = \langle U(h) \varphi, A' \lambda \rangle. \]

By definition

\[ f(h) = \lim_{\alpha} \sum_{g \in G} \langle U(h) \varphi, c^\alpha(g) U^*(g) \lambda \rangle = \lim_{\alpha} \sum_{g \in G} \langle U(g^{-1} h) \varphi, c^\alpha(g) \rangle \]

Since \( \lambda \in L' \)

\[ \langle U(g) \varphi, \lambda \rangle = -\langle U(g^{-1}) \varphi, \lambda \rangle. \]
Thus
\[ f(h) = \lim_{\alpha} \sum_{g \in G} <U(h^{-1})c_{\alpha}(g)U(g)\psi, \lambda> = <U(h^{-1})A\psi, \lambda> = 0 \]
and \( A'\lambda = 0 \) because of the cyclicity of \( \psi \). The lemma is proven.

Let \( C \) be the subspace of \( H \) defined by
\[ C = \{ A\psi, A, A' \in \mathcal{A} \text{ for some } A' \}. \]

For each \( \lambda \in L' \) let \( B_\lambda \) be the mapping of \( C \) into \( H^* \) defined as \( B_\lambda(A\psi) = A'\lambda \) where \( A \oplus A' \in \mathcal{A} \). By the Lemma 2.1 this definition makes sense.

Lemma 2.2. For all \( v, w \in C \)
\[(i) \quad <v, B_\lambda w> = -<B_\lambda v, w> \]
\[(ii) \quad B_\lambda U(g)v = U^*(g)B_\lambda v \]

Conversely, any linear operator \( B: C \rightarrow H^* \) which satisfies (i) and (ii) has the form \( B = B_\lambda \) where \( \lambda = B\psi \in L' \).

Proof. The proof of (i) is analogous to that of the lemma 2.1 and (ii) follows from the definition of \( B_\lambda \). The last statement of lemma 2.2 is true since
\[ <U(g)\psi, B\psi> = -<\psi BU(g)\psi> = -<\psi, U^*(g)B\psi> = \]
\[ <-U(g^{-1})\psi, B\psi> \]

Now let \( \pi: H \rightarrow H \) be a central projection for \( U \), i.e. \( \pi \) commutes with \( U \) and with the commuting algebra of \( U \). We call \( \pi \) balanced if \( \pi \quad \pi^* \) is a central projection for \( V \). More specifically, the general "matrix" form of an interwining operator for \( V \) is
\[
\begin{pmatrix}
\alpha & \beta \\
\gamma & \delta
\end{pmatrix}
\]
where \( \alpha \in I(U,U) \), \( \delta \in I(U^*,U^*) \), \( \beta \in I(U^*,U) \) and \( \gamma \in I(U,U^*) \). Hence \( \pi \) is balanced if \( \pi \) is central for \( U \) and \( \delta \pi = \pi \gamma \) for all \( \gamma \in I(U,U^*) \). (The \( \beta \)-identity follows from \( \gamma \)-identity by transposition). Intuitively, "balanced central projection" means that if an irreducible representation \( U_0 \) "occurs" in \( \pi \), then \( U_0 \) also "occurs" in \( \pi \) given that \( U_0 \) "occurs" in \( U \). Of course, \( U \) might have no discrete spectrum, so this is only formal.

Lemma 2.3. If \( \pi \) is balanced, then \( \pi L_+ \subset L_+ \). If there is a non-zero balanced projection \( \pi \) such that \( \pi L_+ = 0 \) then the image of \( \pi \) is one-dimensional and \( U \) is trivial on the image of \( \pi \).

Proof. It suffices to show that \( \pi^* \) maps \( L' \) into \( L' \). Let \( \lambda \in L' \). Then
\[
\lambda = B_\lambda \varphi \text{ and } \pi^* \lambda = \pi^* B_\lambda \varphi.
\]
But then
\[
\langle U(g) \varphi, \pi^* \lambda \rangle = \langle U(g) \varphi, \pi^* B_\lambda \varphi \rangle = -\langle \varphi, B_\lambda \pi U(g) \varphi \rangle = -\langle \varphi, \pi^* U^* (g) \lambda \rangle = -\langle U(g^{-1}) \varphi, \pi^* \rangle
\]
since \( \pi U(g) \Theta \pi U^*(g) \in U \). Thus, as claimed, \( \pi^* \lambda \in L' \).

Now, if \( \pi (U(g) + U(g^{-1})) \varphi = 0 \) then \( U(g) \pi \varphi = -U(g^{-1}) \pi \varphi \), so that \( U(g^4) \pi \varphi = \pi \varphi \). Since the elements \( g^4 \) generate a dense subgroup, \( U \) is trivial on the image of \( \pi \). Since a cyclic representation can contain the identity representation at most once, our lemma follows.

Corollary. There are only a finite number of disjoint balanced projections for \( U \). Also \( \sum_{i=1}^{q} \pi_i = 1 \) if \( \pi_1, \ldots, \pi_q \) is a maximal family of disjoint balanced projections.

Proof. Let \( \pi_1 \) be the unique balanced projection such that \( \pi_1 L_+ = 0 \) (it is possible that \( \pi_1 = 0 \)). If \( \{ \pi_i \} \) is any family of disjoint balanced projections, then \( \bigoplus \pi_1 L_+ \subset L_+ \). By finite dimensionality there can be at most a finite number of such \( \pi_i \). The equality \( \sum_{i=1}^{q} \pi_i = I \) for a maximal family follows from the fact that \( \pi \) is balanced iff \( I - \pi \) is balanced.
This corollary allows us to assume that in the proof of finite dimensionality of \( U \), the identity is the only non-zero balanced projection. In this case \( U \) is "almost" a primary representation as the next lemma shows.

**Lemma 2.4.** \( U \) has at most two disjoint central projections.

**Proof.** Let \( \pi \) be a central projection for \( U \). Let \( H_{\pi} \) be the image of \( \pi \) in \( H \) and let \( H_{\pi}^{\perp} \) be the closure of the image of \( H_{\pi} \) under \( I(U, U^*) \). Let \( \pi \tilde{\pi} \) be the projection in \( H^* \) onto \( H_{\pi}^{\perp} \). It is clear that \( H_{\pi}^{\perp} \) is invariant under \( U^* \) so that \( \pi \oplus \pi \tilde{\pi} \in U' \). We claim also that \( \pi \oplus \pi \tilde{\pi} \in U \). To see this it is sufficient to prove that \( \pi \oplus \pi \tilde{\pi} \) commutes with \( U^* \). It is obvious that \( H_{\pi}^{\perp} \) is invariant under any operator which commutes with \( U^* \), so it suffices to show that \( \pi A = A \pi \) for all \( A \in I(U, U^*) \) and \( \pi A' = A' \pi \) for all \( A' \in I(U^*, U) \).

The first identity is proven as follows

\[
\pi A = \pi A \pi + \pi A (I - \pi) = A \pi + \pi A (I - \pi)
\]

Now, since \( \pi \) is central \( U | \pi \) is disjoint from \( U | (I - \pi) \) (i.e. they contain no equivalent sub-representations) and \( \pi A (I - \pi) = 0 \).

The second identity follows similarly using the inclusion \( A' H_{\pi}^{\perp} \subset H_{\pi} \), which holds since \( A' A \) maps \( H_{\pi} \) into \( H_{\pi} \) for all \( A \in I(U, U^*) \).

Therefore \( \pi \oplus \pi \tilde{\pi} \) and \( (\pi) \ast \oplus (\pi) \ast \tilde{\pi} \ast \) are central projections. Hence their product \( \pi (\pi) \ast \oplus \pi \ast (\pi) \ast \) is central, so \( \pi (\pi) \ast \) is balanced. Thus \( \pi (\pi) \ast = I \) or \( 0 \). If \( \pi (\pi) \ast = I \), then \( \pi = I \). If \( \pi (\pi) \ast = 0 \), then \( \pi + (\pi) \ast \) is a balanced projection and \( \pi + (\pi) \ast = I \). But if \( \sigma \leq \pi \) (i.e. \( \sigma \pi = \sigma \)), then \( (\sigma) \ast \leq (\pi) \ast \), so that \( \sigma + (\sigma) \ast = I \). The latter is impossible unless \( \sigma = \pi \). Thus if \( \pi \neq I \), then \( \pi \) has no smaller central projections. Since \( \pi \) was arbitrary this shows that \( \pi \) and \( I - \pi \) are the only central projections for \( U \). Q.E.D.

Now, let \( \pi \) be a minimal central projection for \( U \). By the above lemma \( I - \pi \) is also such a projection. By restricting \( U \) to the image of \( \pi \) we may
assume that the only central projection for $U$ is $I$. Hence $U$ is a primary representation so $U$ must be of the form $nU_0$ where $U_0$ is an irreducible representation and $n \in \{1, 2, \ldots, \infty\}$. If one shows that $U_0$ is finite dimensional then it will follow that $U$ is finite dimensional since a cyclic representation cannot contain any finite dimensional representation with infinite multiplicity (see [2], 15.5.3). Thus, we can take $U = U_0$.

We prove that in the irreducible case the set $\mathcal{C}$ (the domain of $B_\lambda$) equals $H$. To see this, we use the fact that a topologically irreducible representation of a C*-algebra is algebraically irreducible (cf [2], 2.8.4). Let $C^*(G)$ be the group C*-algebra of $G$. Let $U$ (resp. $U^*$) denote the representation of $C^*(G)$ corresponding to $U$ (resp. $U^*$). If $f \in C^*(G)$, then $\tilde{U}(f) \otimes \tilde{U}^*(f) \in \mathcal{C}$. It follows that $\tilde{U}(f)\varphi \in \mathcal{C}$ for all $f \in C^*(G)$. But the set of $\tilde{U}(f)\varphi$ is invariant under $U$, and $U$ is irreducible. Hence $\mathcal{C} = H$ as claimed. Since $B_\lambda$ has an adjoint operator, it follows from the closed graph theorem that $B_\lambda$ is continuous. As $U$ is irreducible, the space $I(U, U^*)$ is at most one dimensional. The mapping $\lambda \mapsto B_\lambda$ is injective from $L'$ to $I(U, U^*)$, so $L'$ is at most one dimensional. This shows that $L_+^*$ has finite co-dimension in $H$, what implies finite dimensionality of $H$. Thus the proof of the finite dimensionality of $U$ in general is finished.

Since $U$ is finite dimensional, $B_\lambda$ is a continuous operator from $H^*$ to $H^*$. The mapping $\lambda \mapsto B_\lambda$ is injective. It is also surjective for if $B \in I(U, U^*)$ then $\lambda = B\varphi \in L'$. It is easily seen that $B = B_\lambda$. Thus
\[ \dim U = \dim L_+ + \dim L' = \dim L_+ + \dim I_a(U, U^*) \]
and the theorem 2.1 is proven.
References


