Note on a Theorem of Passow*

by

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Consider a system of functions \( u_i \in C^m[a,b] \), \( i = 0, 1, \ldots, m \). We are interested in Hermite interpolation. Thus if 
\[
a \leq x_0 < x_1 < \ldots < x_k \leq b \text{ and } y_j^{(1)}(x_i) = 0, \ldots, y_j^{(m)}(x_i) = 0, j = 0, 1, \ldots, k
\]
are arbitrarily specified with \( \sum_{j=0}^{m} a_j = m + 1 \) and \( \max_j a_j \leq n - 1 \) we are interested in when there exists a unique \( u(x) = \sum_{i=0}^{m} c_i u_i(x) \) such that \( u^{(r)}(x_j) = y_j^{(r)} \), \( j = 0, 1, \ldots, k; r = 0, 1, \ldots, a_j - 1 \). Under such conditions the system \( \{u_i\} \) is called an extended Tchebycheff system (ETS) of order \( n+1 \). In the case \( n = 0 \) the system is referred to as simply a Tchebycheff system (TS) and if \( n = m \) the reference to the order is omitted. Note that an ETS of order \( n+1 \) is a ETS of any lower order. 

It is well known that \( x^k \), \( k = 0, 1, \ldots, m \) where \( t_0 = 0 < t_1 < \ldots < t_m \) is an ETS on any interval \([a,b]\) for \( 0 \leq a < b \). If \( t_0 = 0 \) and for all \( k, t_{2k} \) is even and \( t_{2k+1} \) is odd, then \( t_0, t_1, \ldots, t_m \) is said to have the alternating parity property (APP). Recently E. Passow \[2\] \[3\] proved the following: 

**Theorem 1.** The system \( \{x^k\}_{0}^{m} \) is an ETS of order \( n + 1 \) if and only if \( t_i = i, i = 0, \ldots, n \) and \( \{t_i\} \) has APP.

The purpose of this note is to generalize this result slightly to a larger class of systems \( \{u_i\} \). Let \( w_k, k = 0, 1, \ldots, \) be strictly positive or \((-\infty, \infty)\) and \( r - k \) times differentiable. Then define

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\[ u_0(x) = w_0(x) \]
\[ u_1(x) = w_0(x) \int_0^x w_1(\xi_1) \, d\xi_1 \]
\[ u_2(x) = w_0(x) \int_0^x w_1(\xi_1) \int_0^{\xi_1} w_2(\xi_2) \, d\xi_2 \, d\xi_1 \]
\[ \vdots \]
\[ u_r(x) = w_0(x) \int_0^x w_1(\xi_1) \ldots \int_0^{\xi_{r-1}} w_r(\xi_r) \, d\xi_r \ldots \, d\xi_1 \]

For \( x \) negative the integrals are assumed oriented in the obvious sense so that \( u_k(x) \) is negative iff \( k \) is odd. The system \( (1) \) is a basis for any \( r \)th order differential operator for which successive Wronskians do not vanish; see [1]. The value \( r \) here corresponds to \( t_m \). We will show:

**Theorem 1.** The system \( \{u_k\}_{k=0}^m \) is an ETS of order \( n+1 \) if and only if \( t_i = 1, i = 0, 1, \ldots, n \) and \( \{t_i\} \) has APP provided \( \left| \frac{u_i(x)}{u_j(x)} \right| \to 0 \) for \( |x| \to \infty \) when \( i < j \).

The proof follows [2] and [3] however, some changes are necessary.

The proof is divided into a three lemmas. The first of which is nearly obvious.

**Lemma A.** If \( f(x) \) is continuous on \((-\infty, \infty)\) and has at most \( n \) distinct zeros, then \( g(x) = \int_0^x f(\xi) \, d\xi \)

(a) has at most \( n \) zeros if \( x = 0 \) is a zero of \( f \).

(b) always has at most \( n + 1 \) distinct zeros.

**Lemma B.** The system \( u_{k_1}, \ldots, u_{k_r} \) is a T.S. on \([a,b]\) for \( 0 < a < b \).

**Proof.** The proof is by induction on \( r \). For \( r = 1 \), \( u_{k_1} \) has no zeros on \([a,b]\). Assuming the result true for \( r - 1 \) we consider

\[ \sum_{i=1}^{r} a_i u_{k_i} (x) = U(x). \]
We write \( U(x) \) in the form

\[
U(x) = w_1(x) \int_0^x w_1(\xi) \cdots \int_0^{k_1-1} [v(\xi)] d\xi
\]

where

\[
v(\xi) = a_1 w_{k_1}(\xi) + a_2 w_{k_1}(\xi) \int_0^{k_1+1} w_{k_1+1} \text{ etc.}
\]

Now \( V(\xi) \) can have at most \( r-1 \) zeros, since otherwise

\[
\frac{d}{d\xi}\left( \frac{V(\xi)}{w_{k_1}(\xi)} \right)
\]

would have at least \( r-1 \) zeros violating the induction hypothesis.

By Lemma A, \( U(x) \) has at most \( r \) zeros one of which is zero so that \( U(x) \) has at most \( r-1 \) zeros on \([a,b] \).

**Lemma C.**

(a) If \( \{t_i\}_{i=0}^m \) has APP then \( \{u_i\}_{i=0}^m \) is a TS.

(b) If \( \left| \frac{u_i(x)}{u_j(x)} \right| \to 0 \) as \( |x| \to \infty \) for \( i < j \) then \( \{u_i\}_{i=0}^m \) is a TS implies that \( \{t_i\}_{i=0}^m \) has APP.

**Proof.** We first show (a) that APP implies TS. The proof is by induction.

For \( m = 0 \) we have \( t_0 = 0 \) and \( u_0(x) \) is assumed to be positive on \((-\infty, \infty)\). Assuming the result for \( m-1 \) we consider

\[
U(x) = \sum_{i=0}^m a_i u_{t_i}(x).
\]

If \( U(x) \) has at least \( m+1 \) distinct zeros then we consider

\[
D_0 U(x) = \frac{d}{dx} U(x) = \sum_{i=1}^m a_i V_{t_i}(x)
\]
where
\[ v_{t_1}(x) = D_0 u_{t_1}(x) \]

As in the proof of Lemma B we write \( D_0 U(x) \) in the form
\[
w_1(x) \int_0^x w_2(\xi_2) \cdots \int_0^{\xi_{t_1-2}} w_{t_1-1}(\xi_{t_1-1}) \int_0^{\xi_{t_1-1}} V(\xi) \, d\xi \tag{2}
\]

where
\[
V(\xi) = a_1 \, w_{t_1}(\xi) + a_2 \, w_{t_1}(\xi) \int_0^{\xi} \cdots
\]

is a linear combination of \( m \) functions again satisfying the APP.

Therefore, by the induction \( V(\xi) \) has at most \( m-1 \) zeros. As in [2]; if \( a_1 = 0 \) then \( D_0 U(x) \) has at most \( m-1 \) zeros by Lemma A part (a). If \( a_1 \neq 0 \), then since the number of integrals \( t_1 - 1 \) in (2) is even
\( x = 0 \) is not a separating zero of \( D_0 U(x) \) and \( U(x) \) has at most \( m \) zeros.

We turn now to the converse (b), i.e. TS implies APP. The case \( m = 0 \) and \( m = 1 \) are easily checked. We then assume the result for \( m-1 \) and suppose \( t_0, t_1, \ldots, t_m \) does not have APP. The two cases. (i)
\( t_0 \cdots t_{n-1} \) has APP and (ii) \( t_0 \cdots t_{n-1} \) does not have APP can be handled as in [2]. For case (i) we assume that \( t_{n-1} \) and \( t_n \) are both odd and consider
\[
U_{n-1} = \sum_{i=0}^{n-1} a_i u_{t_i} \text{ with } n-1 \text{ distinct simple zeros on } [a,b]
\]
with \( a_{n-1} > 0 \). Then \( U_{n-1} \in u_{t_n} \) has \( n+1 \) zeros for \( \xi \) sufficiently small using the assumptions in (b), i.e. \( U_{m-1} \in u_{t_n} \) will have a zero near every simple zero of \( U_{n-1} \) and will gain two more zeros for large \( x \). Case (ii) is again handled as in [2].

**Proof of Theorem 1.** If \( \{t_i\} \) does not have APP then by Lemma C, \( \{u_{t_i}\} \) is not a TS so it is also not an ETS.

Suppose that \( t_j > j \) for some \( j \leq n \) and consider the minimal such \( j \).
We then take \( x_0 = 0, y_0 \neq 0, y_i = 0, i = 0, \ldots, j - 1 \) (see introduction) and consider any 
\[ U(x) = \sum_{i=0}^{j} a_i u_{t_i}^j(x) \]  
A little reflection shows that any function of this form has the \( j \)th derivative at \( x = 0 \) equal to zero.

This follows from the fact that if we define 
\[ D_i f(x) = \frac{d}{dx} w_i(x) \quad i = 0, 1, \ldots \]

then \( f(0) = 0 \) and \( D_i D_{i+1} \ldots D_j f(0) = 0, i = 0, \ldots, j - 1 \) if and only if \( f^{(i)}(0) = 0, 1, \ldots, j \). The proof of the converse also follows [3] using the operators \( D_0, \ldots, D_{n-1} \) instead of the ordinary derivatives.


3. Passow, Eli. Extended Tchebycheff Systems on \((-\infty, \infty)\)