Jensen's Inequality for Conditional Expectations

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Let $f$ be a real-valued Borel function, $X$ and $f(X)$ be integrable random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{G}$ be a sub $\sigma$-field of $\mathcal{F}$. Jensen's inequality states that, if $f$ is convex on an interval $I$ containing the range of $X$, then

$$ (1) \quad E[f(X)] \geq f(EX) $$

(where $E$ denotes expectation); its generalization is

$$ (2) \quad E[f(X) | \mathcal{G}] \geq f(E(X) | \mathcal{G}) $$

with probability one.

One application of the generalized Jensen's inequality is in martingale theory where it is used to show that "convex functions of martingales" and "convex non-decreasing functions of submartingales" are submartingales.

The usual proof of (1), e.g. Loève [6; p. 159], uses the fact that under the hypothesis there must exist a non-decreasing function, $m(\cdot)$, satisfying, for all $x$ and $y$ in $I$:

$$ (3) \quad f(x) - f(y) \geq m(y) (x - y) $$

(e.g. take $m$ to be either the right or left hand derivative of $f$). Then, since $EX$ must lie in $I$, we have

$$ (4) \quad f(X) - f(EX) \geq m(EX) (X - EX). $$
Take the expectations of both sides, giving

\[(5) \quad \mathbb{E}f(X) - f(\mathbb{E}X) \geq m(\mathbb{E}X) (\mathbb{E}X - X) = 0\]

which proves \((1)\).

Inequality \((4)\) can be generalized to

\[(6) \quad f(X) - f[\mathbb{E}(X|\mathcal{Q})] \geq m[\mathbb{E}(X|\mathcal{Q})] [X - \mathbb{E}(X|\mathcal{Q})]\]

and taking the conditional expectations of both sides with respect to \(\mathcal{Q}\), would yield the analogue of \((5)\), thereby proving \((2)\) -- provided that the conditional expectations exist. If I can be bounded above or below then so can \(m[\mathbb{E}(X|\mathcal{Q})]\) and there is no difficulty. But in general, the hypothesis is not sufficient to guarantee the existence of the mean of the right side of \((6)\).

For example, take \(f(x) = \exp|x|\), let \(Y\) be any symmetric random variable with \(\mathbb{E}\exp 2|Y| < \infty\), but \(\mathbb{E}\exp|Y|\) failing to exist. Let \(Z\) be independent of \(Y\) with values 0 and 2 each with probability \(1/2\). Take \(X = YZ\) and \(\mathcal{Q}\) the \(\sigma\)-field generated by \(Y\). Then the right side of \((6)\) becomes

\[(7) \quad |Y|(Z-1)\exp|Y|\]

which has the same distribution as

\[(8) \quad Y\exp|Y|,\]

hence no mean. So while we are strongly tempted to say that the conditional expectation of \((7)\) with respect to \(\mathcal{Q}\) is

\[(9) \quad (|Y|\exp|Y|) E(Z-1) = 0 ,\]

we may not do so.
Loève does not give a proof of (2), but merely asserts (p. 348) that it follows from (1) and the fact that \( P\{X \geq Y\} = 1 \) implies
\[ P\{E(X|\mathcal{F}) \geq E(Y|\mathcal{F})\} = 1 \] for any \( \mathcal{F} \subset \mathcal{G} \). We leave this as an exercise for the reader: one which we have not been able to solve.

Feller [4; p. 214] mentions (2) without proof. Neveu [7; p. 122] mentions only the case \( X \geq 0 \), without proof as an easy generalization of (1), which it is since the interval I can be bounded below. Chung [2; p. 281] has a proof of (2) which is not based on (3) and which is not quite complete.

Doob [3; p. 33] shows that once the existence of regular conditional distributions is established, (2) can be obtained from (1) in an elementary way. Indeed Breiman [1; p. 80] assigns the proof of (2) as an exercise with the above as a hint.

I prefer to build a proof around (6) as follows:

Choose \( a > 0 \) and let
\[ A = A(a) = \{ |E(X|\mathcal{Q})| \leq a \} . \] (10)

Then (6) is true with \( X \) replaced by \( X I_A \) -- unless \( 0 \) is not in \( I \), in which case \( X \) should be replaced by \( X I_A + bI_A \) for some \( b \) in \( I \), and \( f(0) \) should be replaced by \( f(b) \) below. Now \( m[E(XI_A|\mathcal{Q})] \) is bounded, so we are justified in concluding that
\[ E[f(XI_A)|\mathcal{Q}] \geq f(E(XI_A|\mathcal{Q})) . \] (11)

Because \( A \in \mathcal{Q} \), the left side of (11) is
\[ E[f(X)I_A + f(0)I_A c|\mathcal{Q}] \]
\[ = E[f(X)|\mathcal{Q}]I_A + f(0)I_A c , \] (12)

while the right side is
\[ f[E(X|\mathcal{Q})I_A] = f[E(X|\mathcal{Q})]I_A + f(0)I_A c . \] (13)
Comparing (12) and (13) we see that, in effect, on A, the $I_A$'s can be deleted from (11). Since $P(A) \to 1$ as $a \to \infty$, this completes the proof.

Recently, I noticed (6) in Hunt [5; p. 48] with the remark that (2) then follows immediately if $X$ is bounded and "in general by a passage to the limit". So the preceding proof fills in the details omitted by Hunt. But note that the proof will not go through if (10) is replaced by $A = \{|X| \leq a\}$ since this set is not in $\Psi$. 
REFERENCES


