Loss Due to Missing Data in Efficiency of a
Locally Optimal Test for Homogeneity
with Respect to Very Rare Events*,**, 

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Mimeograph Series No. 234
August, 1970

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*This investigation was supported in part by research grants GM 10525-07
at University of California, Berkeley, from the National Institutes of
Health, Public Health Service, and NONR 1100(26) from the Office of Na-
val Research at Purdue University.

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**To appear in the proceedings of the National Academy of Science (1970).
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Abstract. With reference to observations of supernovae in galaxies, a
locally optimal test for the hypothesis of homogeneity of the observa-
tional units with respect to occurrence of supernovae (treated as a very
rare event) is obtained for the case where the data are available only
for those galaxies where at least one supernova is observed. It is shown
that the loss in the asymptotic efficiency of the test due to lack of re-
porting of galaxies with no supernovae is very heavy and in fact is in-
finite for the case of very rare events.

1. Introduction. This paper may be regarded as a sequel to an earlier
paper by Bühler at el², where the authors consider the notion of very rare
events, arising in experimental situations such as the observation of super-
novae in galaxies. The specific problem that was dealt with there was to
construct a locally optimal test of the hypothesis of homogeneity of observ-
ational units with respect to occurrence of very rare events. Let N denote
the number of units of observations for each of which the number of occurrences
of an event E is observed during a fixed period of observation. Let M denote
the total number of events E observed. An event was defined² as 'rare' if for
a large N, the quotient M/N is of the order of unity. A 'very rare' event was
defined to be the one where this

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quotient is smaller in order than unity. More specifically $M^2/N$ instead was assumed to be of the order of unity. In particular, the problem was considered there in reference to occurrence of supernovae in galaxies and mathematically it can be described as follows:

Let $U_{ij}$ denote the $j$th observational unit for the $i$th observatory with $j=1,2,\ldots,N_i$ and $i=1,2,\ldots,s$. Each of the units $U_{ij}$, $j=1,2,\ldots,N_i$ is observed for the same period $t_i$ of time. The number of events observed in $U_{ij}$ over time $t_i$ is treated as a random variable and is denoted by $\mu_{ij}$. It is assumed that all these random variables are mutually independent, with $\mu_{ij}$ distributed as a Poisson variable with expectation $\lambda_{ij} t_i$, where $\lambda_{ij}$ is the expected number of events $E$ in $U_{ij}$ per unit time. The hypothesis $H_0$ of homogeneity of the observational units with respect to the very rare event $E$, is then equivalent to the statement that all the $\lambda_{ij}$ have the same unspecified value $\lambda$. The nature of the heterogeneity of $\lambda$'s used as hypothesis alternative to $H_0$, was exhibited by assuming that each observational unit takes a particular value of a random variable $\Lambda$ as the value corresponding to its $\lambda_{ij}$. Thus the hypothesis $H_0$ of homogeneity would correspond to the case where $\Lambda=\lambda$, a constant, with probability one. With a slight
modification of what was assumed before, we assume here that

\[ \uplambda = \lambda_0 \exp[X \xi^{1/2}] \]  

(1)

where \( \lambda_0 > 0 \), \( \xi > 0 \) are two unspecified parameters; \( X \) is treated as a random variable taking values in an interval \([c,d]\) where \( c \) and \( d \) are two arbitrary but fixed finite constants. Let \( F(x) \) be the distribution function of \( X \). About \( F(x) \) all we shall assume that

\[ \int x dF(x) = \int x^2 dF(x) = 0 \; ; \; \int x^2 dF(x) = \int dF(x) = 1 \]  

(2)

Thus, with \( F(x) \) otherwise arbitrary, the random variable \( \mu_{ij} \) observed during controlled time \( t_i \) in a randomly selected \( j \)th galaxy at the \( i \)th observatory would be distributed as a mixture of Poisson variables and its distribution may be described by

\[ \Pr(\mu_{ij} = m_{ij}) = \frac{(\lambda_0 t_i)^{m_{ij}}}{m_{ij}!} \int \exp\{-\lambda_0 t_i \exp[x \xi^{1/2}]\} + \]

\[ + m_{ij} x \xi^{1/2} dF(x), \]

(3)

\( j=1,2,\ldots,N_i ; \; i=1,2,\ldots,s \). While \( \lambda_0 \) is a nuisance parameter, the hypothesis of homogeneity by virtue of (1) is described by \( H_0: \xi = 0 \). Thus given (3), the problem reduces to developing a test of \( H_0: \xi = 0 \) against the alternative \( H_1: \xi > 0 \). Earlier a locally optimal test was constructed for this problem. Also, an asymptotic Poisson
approximation to the distribution of the test statistic was realized for large \( N \) under the assumptions that \( N_i/N \rightarrow \nu_i \) as \( N \rightarrow \infty \) with \( 0 < \nu_i < 1 \), and that \( \lambda_0 = O(N^{-1/2}) \); the latter assumption being due to the very rarity of the events in question.

We wish to point out here that it is a common practice among the astronomers that while looking for occurrences of supernovae, they tend to report only those cases where they succeed in observing at least one supernova; the galaxies showing no supernova are left unreported. This being the case, our random variables \( \mu_{ij} \) are observed as truncated only to the positive integer values, so that one needs to construct tests appropriate for the situation, given that \( \mu_{ij} \geq 1 \). Thus, subject to this we have modified (3) as given by

\[
\text{Pr}(\mu_{ij} = m_{ij} \mid \mu_{ij} \geq 1) = \frac{(\lambda_0 t_i)^{m_{ij}} \int \exp(-\lambda_0 t_i \exp[x \xi^{1/2}] + m_{ij} x \xi^{1/2})dF(x)}{m_{ij}! \left[ 1 - \int \exp(-\lambda_0 t_i \exp(x \xi^{1/2}))dF(x) \right]} \tag{4}
\]

for \( m_{ij} = 1, 2, \ldots ; \; j=1, 2, \ldots, n_i ; \; i=1, 2, \ldots, s \). Here we have used letter \( n_i \) instead of \( N_i \) to differentiate between the observations truncated as in (4) and the nontruncated ones as in (3). Let \( n = \Sigma_{i=1}^{s} n_i \). Essentially for each \( i \), \( n_i \) is the random variable representing the number of those galaxies
observed with at least one supernova out of a total of \( N_1 \) galaxies. As such, it follows from the law of large numbers that

\[
\frac{n_1}{N_1} \xrightarrow{P} 1 - \int \exp\left(-\lambda_0 t_1 \exp[x \frac{1}{2}]\right) dF(x),
\]

as \( N_1 \to \infty \). We shall need this result later.

In section 2, we apply the theory of optimal \( C(\alpha) \)-tests developed by Neyman\(^3\), and as generalized later by Bartoo and Puri\(^1\), for constructing a locally optimal test of the hypothesis \( H_0 \) based on the observations subject to (4), treating \( \lambda_0 \) as a fixed nuisance parameter. Then, keeping with the spirit of very rare events, the distribution of the relevant test statistic is approximated for large \( n \) to a Poisson distribution with \( \lambda_0 \) tending to zero in an appropriate manner as \( n \to \infty \).

A similar optimal \( C(\alpha) \)-test of \( H_0 \) based on the non-truncated observations subject to (3), is already given elsewhere\(^1\).

In section 3, performances of the two tests, one based on truncated data subject to (4) and the other based on complete data subject to (3), are compared. It is shown that the loss in asymptotic efficiency of the test, because of being unable to use the part of data of galaxies with no supernova, is very heavy and in fact is infinite for the case of very rare events.

2. An Optimal \( C(\alpha) \)-test based on truncated Data. In this section, even though for a given \( N_1 \), \( n_1 \) is a random variable,
nevertheless we shall consider \( n_i \) as nonrandom and fixed. This is reasonable in view of (5) since we shall concentrate ourselves to the asymptotic case with \( N_i \to \infty \), for each \( i \). The theory for constructing an optimal test of the hypothesis \( H_0: \xi = 0 \), based on the observations \( \mu_{ij} \) which are independently but not identically distributed, is dealt with by Bartoo and Puri. Following their notation, the construction of the test involves first with finding the partial derivatives \( \phi_{ij}(\xi) \) and \( \phi_{ij}(\lambda_0) \) of the log-likelihood for the observation \( \mu_{ij} \). Here both the derivatives are evaluated at \( H_0: \xi = 0 \). Next, one obtains a regression constant \( a \) such that the variance of

\[
\psi(\mu) = \sum_{i=1}^{S} \sum_{j=1}^{n_i} [\phi_{ij}(\xi)(\mu_{ij};\lambda_0) - a \phi_{ij}(\lambda_0)(\mu_{ij};\lambda_0)] , \tag{6}
\]

obtained under \( H_0 \), is minimized. Using this constant, let

\[
f_{ij}^* = \phi_{ij}(\xi)(\mu_{ij};\lambda_0) - a \phi_{ij}(\lambda_0)(\mu_{ij};\lambda_0). \tag{7}
\]

In the present case, using (4) we have

\[
\begin{align*}
\phi_{ij}(\xi) &= \frac{1}{2}(\mu_{ij}(\mu_{ij}-1) - \frac{(\lambda_0 t_i)^2}{1-\exp(-\lambda_0 t_i)}) + \frac{1}{2} - \lambda_0 t_i \lambda_0 \phi_{ij}(\lambda_0) \\
\phi_{ij}(\lambda_0) &= \frac{1}{\lambda_0}(\mu_{ij} - \frac{\lambda_0 t_i}{1-\exp(-\lambda_0 t_i)}) . \tag{8}
\end{align*}
\]
The desired constant \( \theta \) is given by

\[
\theta = \lambda_0 (1 + A) / 2 ,
\]

where

\[
A = \frac{s \sum_{i=1}^{s} n_i [(\lambda_0 t_i)^3 \exp(-\lambda_0 t_i)]/[1 - \exp(-\lambda_0 t_i)]^2}{s \sum_{i=1}^{s} n_i [\lambda_0 t_i - \lambda_0 t_i (1 + \lambda_0 t_i) \exp(-\lambda_0 t_i)]/[1 - \exp(-\lambda_0 t_i)]^2} .
\]

Also, the variance of (6) under \( H_0 \) is given by

\[
\sum_{i=1}^{s} \sum_{j=1}^{s} \text{Var}_{ij}^* = \frac{1}{4} \left\{ \sum_{i=1}^{s} \sum_{j=1}^{s} \frac{2(\lambda_0 t_i)^2 - (\lambda_0 t_i)^2 [2 + (\lambda_0 t_i)^2] \exp(-\lambda_0 t_i)]}{(1 - \exp(-\lambda_0 t_i))^2} \right\}
\]

\[
- \left\{ \frac{s \sum_{i=1}^{s} n_i [(\lambda_0 t_i)^3 \exp(-\lambda_0 t_i)]}{s \sum_{i=1}^{s} n_i [\lambda_0 t_i - \lambda_0 t_i (1 + \lambda_0 t_i) \exp(-\lambda_0 t_i)]} \frac{1}{(1 - \exp(-\lambda_0 t_i))^2} \right\} .
\]

Following the standard lines (see an earlier paper\(^1\) for details), in view of the condition (2) satisfied by \( F \), it can be easily demonstrated that the sequence \( \{ f_{ij}^* \} \), as defined in (7), is a Cramér sequence. Finally the test criterion for the optimal \( C(\alpha) \)-test for the hypothesis \( H_0 \) is to reject \( H_0 \) if \( Z^*(\lambda_0) > v(\alpha) \), where \( v(\alpha) \) satisfies \( \Phi(v(\alpha)) = 1 - \alpha \), \( \Phi \) being the cumulative standard normal distribution function, and where
and where

\[ Z^*(\lambda_0) = [\sum_{i=1}^{S} \sum_{j=1}^{n_i} f_{ij}^*] / [\sum_{i=1}^{S} \sum_{j=1}^{n_i} \text{Var} f_{ij}^*]^{1/2} \]  \quad (12) 

The test statistic \( Z^* \) is asymptotically \( \mathcal{N}(0,1) \) under \( H_0 \) as \( n \to \infty \), and \( n_i \to \infty \) for each \( i \) such that \( n_i/n \to \delta_i \) with \( 0 < \delta_i < 1 \). \( \lambda_0 \) in (12) may be replaced by some suitable root \( n \) consistent estimator. The asymptotic power of the above test for alternatives closely to \( H_0 : \xi = 0 \), is given by

\[ 1 - \Phi(\sqrt{n} \cdot \xi \sum_{i=1}^{S} \sum_{j=1}^{n_i} \text{Var} f_{ij}^* )^{1/2} \]  \quad (13) 

In the above, the asymptotic normality approximation for the test criterion (6) or equivalently (12) holds only if \( \lambda_0 \) is fixed and is not too small. Unfortunately, in the case of very rare events this is not the case. As such, the normal distribution approximation is very poor even for large \( n_i \) and needs a further approximation when \( \lambda_0 \) is assumed to be extremely small. At this point, we need to add few words concerning the order of values for \( \lambda_0 \) that may be considered appropriate for this approximation. Earlier an optimal test was constructed based on complete data including even those galaxies which show no supernova. There, the approximation for the distribution of the test statistic, owing to the very rarety of the event, was achieved by assumign that \( \lambda_0 = \delta N^{1/2} \). In
the present case, it is easily seen that as \( N \to \infty \) and 
\( N_i \to \infty \) such that \( N_i/N \to v_i \) with \( 0 < v_i < 1 \), for 
\( i = 1, 2, \ldots, s \), the law of large number yields approximately

\[
n_i \sim N v_i \left( 1 - \int \exp[-\lambda_0 t_i \exp(x^{1/2})]dF(x) \right)
\]

and hence

\[
n/N \sim \sum_{i=1}^{s} v_i \left( 1 - \int \exp[-\lambda_0 t_i \exp(x^{1/2})]dF(x) \right).
\]

Here \( \sim \) implies that the ratio of the two sides tends to one-in
probability as \( N \to \infty \). The fact that \( \lambda_0 = O(N^{-1/2}) \) and (15),
immediately yield that in terms of \( n \), \( \lambda_0 = O(n^{-1}) \). Un-
fortunately, by taking \( \lambda_0 = O(n^{-1}) = \Delta/n \) for some \( \Delta > 0 \), the
distribution of \( \sum_{i=1}^{s} \sum_{j=1}^{n_i} f_{ij} \) tends to be degenerate as \( n_i \to \infty \)
and \( n \to \infty \) such that \( n_i/n \to \delta_i \). This is not unexpected how-
ever, for the simple reason that if we are going to base our
decision only on the data truncated at zero, then in order to
make any headway in our problem of testing the homogeneity of
occurrence of supernovae, the events, although are very rare,
better not be considered as rare as to make \( \lambda_0 = O(n^{-1}) = O(N^{-1/2}) \),
remembering that this order of \( \lambda_0 \) was considered very rare
earlier only for the nontruncated data. Thus, in order that
we obtain a reasonable approximation for the asymptotic dis-
tribution of our test statistic (6), in the sense that it be
nondegenerate and that the variance (11) tend to a nonzero but a
finite quantity as $n \to \infty$, an appropriate order of $\lambda_0$ turns out to be $\lambda_0 = o(n^{-1/2})$. Following this analysis in terms of $N$, it can be easily seen that this amounts to taking $\lambda_0 = o(N^{-1/3})$ instead of $\lambda_0 = o(N^{-1/2})$ as was the case before. Our next step then is to approximate the distribution of (6) or equivalently of $\Sigma \Sigma f_{ij}^*$ with $\lambda_0 = \Delta n^{-1/2}$, and $\Delta > 0$. Rewriting (6) we have

$$\psi(\mu) = T_n(\mu) - \xi_n(\mu) + B_n$$  \hspace{1cm} (16)

where

$$T_n(\mu) = \frac{1}{2} \sum_{i=1}^{s} \sum_{j=1}^{n_i} (\mu_{ij} - 1)(\mu_{ij} - 2),$$  \hspace{1cm} (17a)

$$\xi_n(\mu) = \sum_{i=1}^{s} \left( A/2 + \lambda_0 t_i - 1 \right) \left\{ \mu_i - \frac{n_i \lambda_0 t_i}{1 - \exp(-\lambda_0 t_i)} \right\},$$ \hspace{1cm} (17b)

$$B_n = \sum_{i=1}^{s} \left\{ \frac{n_i \lambda_0 t_i}{1 - \exp(-\lambda_0 t_i)} - n_i - \frac{n_i (\lambda_0 t_i)^2}{2 \left[ 1 - \exp(-\lambda_0 t_i) \right]} \right\},$$ \hspace{1cm} (17c)

$$\mu_i = \sum_{j} \mu_{ij},$$ \hspace{1cm} (17d)

and $A$ is as defined in (10). We need now the following lemma, the proof of which, being elementary, is omitted.

**Lemma.** Subject to (4), and with $\lambda_0 = \Delta n^{-1/2} = \Delta \delta_i n_i^{-1/2}$, for every $i=1,2,\ldots,s$, as $n_i \to \infty$,

$$\mu_{i.}/n_i \overset{p}{\to} 1,$$ \hspace{1cm} (18)
and
\[ n_i^{-1/2} (\mu_i - \frac{n_i \lambda_0 t_i}{1 - \exp(-\lambda_0 t_i)}) \cdot \frac{\Delta^{1/2}_i t_i}{2} \cdot \frac{E[\exp(2\xi^{1/2}_i)] - 1}{E[\exp(\xi^{1/2}_i)]} \]

the last limit being zero under the hypothesis \( H_0 : \xi = 0 \).

We may recall here that by assumption \( X \) takes values in a finite interval \([c,d]\) with probability one, so that it does have a moment generating function.

Now it can be shown that as \( n_i \to \infty \),
\[ A/2 + \lambda_0 t_i - 1 = \Delta^{1/2}_i n_i^{-1/2} [t_i - \frac{\Sigma \delta_i t_i^2}{3\Sigma \delta_i}] + o(n_i^{-1/2}) \]
and that
\[ \lim_{n \to \infty} B_n = -\frac{\Delta^2}{6} \sum_{i=1}^{s} \delta_i t_i^2 \]

By virtue of (19) and (20), it then follows that as \( n \to \infty \),
\[ \mathcal{V}_n(\mu) \to \frac{\Delta^2}{2} \cdot \frac{E[\exp(2\xi^{1/2}_i)] - 1}{E[\exp(\xi^{1/2}_i)]} \cdot \sum_{i=1}^{s} \delta_i t_i^2 (t_i - \frac{\Sigma \delta_i t_i^2}{3\Sigma \delta_i}) \]
which is zero when \( \xi = 0 \). Thus rejection of \( H_0 \) for large values of \( \Sigma \Sigma f_{ij}^* \) is equivalent asymptotically to its rejection for large values of the statistic \( T_n(\mu) \). As such the optimal \( G(\alpha) \)-test for very rare events in the present case reduces to the rule of rejecting \( H_0 \) whenever
\[ T_n(\mu) > \nu^* \]
where \( \nu^* \) is a constant determined from the asymptotic
distribution of $T_n(\mu)$ for large $n$, such that under $H_0$, the probability of $T_n(\mu) > v^*$ is acceptable as the level of significance. Note that this test remains an optimal $C(\alpha)$-test no matter what the mixing distribution $F$ of (4) is, except that it satisfied condition (2). What remains to be done now is to find the limiting distribution of $T_n(\mu)$ as $n \to \infty$. This is achieved in the following theorem.

**Theorem.** Subject to (4) and $\lambda_0 = \Delta n^{-1/2} = \Delta \delta_i n_i^{-1/2}$, whatever be the mixing distribution $F$ satisfying (2) and that it be concentrated on an arbitrary finite interval $[c, d]$, the limiting distribution of $T_n(\mu)$ as $n \to \infty$ is Poisson with expectation

$$
\tau(\xi) = \frac{E[\exp(3\xi^2/2)]}{E[\exp(\xi^2/2)]} \cdot \frac{\Delta^2}{6} \left[ \sum_{i=1}^{s} \delta_i \tilde{t}_i^2 \right].
$$

(24)

The proof of this theorem is based on taking the limit of the characteristic function of $T_n(\mu)$ in a straightforward manner. Thus under $H_0$, the constant $v^*$ is obtained by finding the maximum $k$ satisfying $\Pr(T_n(\mu) > k) \leq \alpha$, where the distribution of $T_n(\mu)$ is taken to be Poisson with expectation $\tau(0) = (\Delta^2/6) \left( \sum_{i=1}^{s} \delta_i \tilde{t}_i^2 \right)$, and $\alpha$ is the prefixed level of significance. Here since $\delta_i = n_i/n$,

$$
\tau(0) = \frac{\lambda_0^2}{6} \sum_{i=1}^{s} n_i \tilde{t}_i^2.
$$
Also, $\lambda_0$ should be replaced by a suitable root $n$ consistent estimator such as

$$\hat{\lambda} = \left[ \sum_{i=1}^{S} \sum_{j=1}^{n_i} \mu_{ij} (\mu_{ij} - 1) \right] / \left[ \sum_{i=1}^{S} \sum_{j=1}^{n_i} n_{ij} t_i \right].$$

(26)

Using a table of Poisson distribution one can easily find out the constant $v^*$. For asymptotic power calculations again one can proceed as in Buhler et al. For this one needs to calculate the power for different values of the constant $E[\exp(3X_{\xi}^{1/2})]/E[\exp(X_{\xi}^{1/2})]$ in (24) for $\xi > 0$, while using the Poisson approximation for the distribution of $T_n(\mu)$ as suggested by the above theorem.

3. **Performance Comparison of the Two Tests.** Let us now consider the situation where all the observations including the ones with no supernova are reported. Let $T_{ij}$ denote for this case the number of supernovae reported in the jth galaxy observed for a period of time $t_i$ at the ith observatory with $j=1, 2, \ldots, N_i$; $i=1, 2, \ldots, s$. Let $N_i/N = v_i$ where $N = \Sigma N_i$. Here $T_{ij}$ follows the distribution as given in (3) and they are assumed to be mutually independent. For this case, the optimal $C(\alpha)$-test for the hypothesis of homogeneity $H_0: \xi = 0$ has been given elsewhere. This involves calculation of an expression analogous to (6) given by

$$\tilde{Y}(T) = \frac{1}{2} \sum_{i=1}^{S} \sum_{j=1}^{n_i} n_{ij} (T_{ij} - 1) - \lambda_0 \sum_{i=1}^{S} t_i t_i - \frac{1}{2} \lambda_0 \sum_{i=1}^{S} n_{ij} t_i^2$$

(27)
where \( T_{ij} = \sum_{j=1}^{s} T_{ij} \). Variance of (27) under \( H_0 \) is given by

\[
\text{Var}\ \tilde{\psi} = \frac{\lambda_0^2}{2} \sum_{i=1}^{s} N_i t_i^2.
\]

(28)

The optimal \( C(\alpha) \)-test is the rule of rejecting \( H_0 \) whenever

\[
\tilde{Z}(\lambda_0) = \frac{\tilde{\psi}(\lambda_0)}{[\text{Var}\ \tilde{\psi}]^{1/2}} > z(\alpha),
\]

(29)

where \( z(\alpha) \) is same as defined before. Here the asymptotic distribution of \( \tilde{Z}(\lambda_0) \) is \( \mathcal{N}(0,1) \) for large \( N \). For fixed \( \lambda_0 \), the asymptotic local efficiency of an optimal \( C(\alpha) \)-test based on complete data as against the one based on truncated data, is given by the ratio of the variance (28) to the variance (11), with \( n_i \) in (11) replaced by \( N_i [1 - \exp(-\lambda_0 t_i)] \), which is approximately valid for large \( N \). This efficiency is considerably large in general. In fact in the present case of very rare events with \( \lambda_0 = O(N^{-1/3}) \), this efficiency tends to \( \infty \). This can be easily verified by noticing that the variance (11), in view of the theorem of section 2, tends under \( H_0 \) to \( \tau_0(0) \), while the variance (28) tends to \( \infty \). Thus the practice of not reporting of galaxies with no supernova results in a considerable loss in efficiency of the appropriate test used for detecting any heterogeneity in the occurrence of supernovae over various galaxies.

Finally, we close with the remark that for the nontruncated case treated earlier, the approximation of the test statistic
resulted into a Poisson distribution, where we had taken essentially \( \lambda_0 = O(N^{-1/2}) \). By contrast, under the present conditions on \( \lambda_0 \), the corresponding test statistic such as \( \tilde{Z} \) of (29) is asymptotically \( \mathcal{N}(0,1) \). This is because of the fact that we have taken \( \lambda_0 = O(N^{-1/3}) \) here, which implies that the events in question, although are very rare, but are not as rare as were treated earlier\(^2\). If, on the other hand, in reality the rarity is really to the extent that \( \lambda_0 = O(N^{-1/2}) \), then, as mentioned earlier, the corresponding optimal \( C(\alpha) \)-test statistic based on the truncated data and given by (6), degenerates in the limit, making thereby the complete reporting even more essential.


With reference to observations of supernovae in galaxies, a locally optimal test for the hypothesis of homogeneity of the observational units with respect to occurrence of supernovae (treated as a very rare event) is obtained for the case where the data are available only for those galaxies where at least one supernova is observed. It is shown that the loss in the asymptotic efficiency of the test due to lack of reporting of galaxies with no supernovae is very heavy and in fact is infinite for the case of very rare events.