Asymptotic Expansions for the Distribution of the
Characteristic Roots of $S_1 S_2^{-1}$

When Population Roots Are Not All Distinct*

by

Nung C. Li and K.C.S. Pillai

Southern Colorado State College and Purdue University

Department of Statistics
Division of Mathematical Sciences
Mimeograph Series No. 231
July 1970

*This research was supported by the National Science Foundation, Grant No. GP-11473.
Asymptotic Expansions for the Distribution of the
Characteristic Roots of $\frac{S_1}{\sim_1} \frac{S_2^{-1}}{\sim_2}$

When Population Roots Are Not All Distinct

by

Hung C. Li and K.C.S. Pillai

Southern Colorado State College and Purdue University

1. Introduction and Summary. The asymptotic expansions of the first two terms for the distribution of the characteristic (ch.) roots of $\frac{S_1}{\sim_1} \frac{S_2^{-1}}{\sim_2}$ have been obtained by the authors [4] when all population roots are distinct, both in the real and complex cases, where $S_1$ and $S_2$ are independent sample covariance matrices of degrees $n_1$ and $n_2$ respectively from normal populations. However, if the population roots are not all distinct, the formulae derived break down and the situations become more complicated. In the case of $q$ smallest roots equal, the first term of the asymptotic expansion in the real case has also been obtained by the authors [4]. In the present paper, we extend this result to the second term (Section 2) and derive the corresponding formulae (Section 3) for the complex case, because not all results in the complex case are counterparts of the real. Finally, we show in Section 4 that the results obtained by other authors [1], [2] and [3] are all special cases of the formulae obtained here.

*This research was supported by the National Science Foundation, Grant No. GP-11473.
2. **Two-sample real case.** We follow the notations in Section 2 of [4]. Let $S_j$ ($j = 1, 2$) be independently distributed as Wishart $(n_j, p, \Sigma_j)$, and let the ch. roots of $S_1 S_2^{-1}$ and $(\Sigma_1 \Sigma_2^{-1})^{-1}$ be $b_1$ and $a_1$ ($l = 1, \ldots, p$) respectively such that $b_1 > b_2 > \ldots > b_p > 0$ and $0 < a_1 < \ldots < a_k < a_{k+1} = \ldots = a_p = a$, ($1 \leq k \leq p-1$). Further, let us denote

$$A = \text{diag} (a_1, \ldots, a_k, a, \ldots, a)$$

$$B = \text{diag} (b_1, b_2, \ldots, b_p)$$

and $n = n_1 + n_2$. Then from (3.7) of [4], the joint distribution of $b_1, b_2, \ldots, b_p$ depends on the definite integral

$$J = 2^p \int_{\overset{N(I)}{\sim}} |I + A \sim B \sim Q|^{\frac{1}{2n}} (Q'd Q),$$

where $N(I)$ and $(Q'd Q)$ are defined in [4].

Under the condition $a_j = a$ ($j = k + 1, \ldots, p$) then from the definitions of $r_j, r_{jt}$ and $c_{jt}$ in [4], we have

$$r_j = \frac{a}{1 + ab_j} \quad \text{if} \quad j = k + 1, \ldots, p$$

$$r_{tj} = r_t - r_j = \frac{a^2 b_{jt}}{(1 + ab_j)(1 + ab_t)} \quad \text{if} \quad j, t = k + 1, \ldots, p$$

and

$$(2.1) \quad c_{jt} = r_{tj} b_{jt} - r_j r_t b_{jt}^2 = c_{tj} = 0, \quad \text{if} \quad j, t = k + 1, \ldots, p.$$
From (4.4) of [4], we also have

\begin{equation}
(2.2) \quad s_{jt} = 0 \quad \text{if } j, t = k + 1, \ldots, p.
\end{equation}

Since we wish to compute up to the second term in the asymptotic expansion of \( \sim \), we need to investigate the groups of terms up to the fourth order of \( \sim \). Under conditions (2.1) and (2.2), (3.16) of [4] reduces to

\begin{equation}
(2.3) \quad K = \pi \left( \frac{2 \pi}{n c_{jt}} \right)^{1/2} \pi \frac{p}{n} \frac{1}{j=1} t=k+1 \left( \frac{2 \pi}{n c_{jt}} \right)^{1/2},
\end{equation}

where

\begin{equation}
c_{jt}^0 = r_{tj} b_{jt} - r_{j} r_{t} b_{jt}^2 = c_{tj}^0 \quad j = 1, \ldots, k; \quad t = k + 1, \ldots, p.
\end{equation}

Now let \( S' = \sum_{j<t}^{p} c_{jt}^{-1} \) and

\begin{equation}
S'' = \sum_{j<s<t}^{p} \left( \frac{c_{st}}{c_{js} c_{jt}} + \frac{c_{jt}}{c_{js} c_{st}} + \frac{c_{js}}{c_{jt} c_{st}} \right).
\end{equation}

Under conditions (2.1) and (2.2), \( S' \) turns out to be

\begin{equation}
\sum_{j<t}^{k} c_{jt}^{-1} + \sum_{j=1}^{k} \sum_{t=k+1}^{p} c_{jt}^{-1}.
\end{equation}
In $S''$, if both $j$ and $s$, or $j$ and $t$, or $s$ and $t$ are greater than $k$, the corresponding term containing $c_{js}$ or $c_{jt}$ or $c_{st}$ as a factor vanishes.

Now $\sum_{j>t}^{p} \frac{c_{jt}}{c_{jt}}$ which originally contributed $\frac{3}{4m} \binom{p}{2}$, now under conditions (2.1) and (2.2), gives

\[\frac{3}{4n} \left\{ \binom{k}{2} + kq \right\}, \quad \text{and} \quad \frac{1}{2n} \sum_{j<s<t}^{p} \left( \frac{c_{js} c_{jt}}{c_{js} c_{jt}} + \frac{c_{js} c_{st}}{c_{js} c_{st}} + \frac{c_{jt} c_{st}}{c_{jt} c_{st}} \right)\]

which originally gave $\frac{3}{2n} \binom{p}{3}$, now becomes

\[\frac{1}{2n} \left\{ \binom{k}{3} + \binom{k}{2} q + k \frac{q(q-1)}{2} \right\} + \frac{1}{2n} \left\{ \binom{k+1}{3} + \binom{k+1}{2} q \right\} + \frac{1}{2n} \left\{ \binom{k}{3} + \binom{k}{2} q \right\}\]

i.e. $\frac{3}{2n} \left\{ \binom{k}{3} + \binom{k}{2} q \right\} + \frac{k}{4n} \left( q(q-1) \right)$,

where $q = p - k$. After simplification, (3.17) of [4] turns out to be

\[(2.4) \quad K \left\{ \frac{1}{2n} s' + \frac{3}{4n} \left[ \binom{k}{2} + kq \right] + \frac{p-2}{3n} s' - \frac{1}{8n} s'' \right. \]

\[+ \frac{3}{2n} \left[ \binom{k}{3} + \binom{k}{2} q \right] + \frac{k}{4n} \left( q(q-1) \right) \}\]

Similarly, (3.18) of [4] becomes
(2.5) \[ K \left\{ \frac{1}{6n} s'' - \frac{p-2}{2n} s' - \frac{1}{2n} \left[ \binom{k}{3} + \binom{k}{2} q \right] \right\} , \]

Finally, since

\[ \text{tr} \tilde{s}^2 = -2 \left\{ \sum_{j<t} s_{jt}^2 + \sum_{j=l}^{k} \sum_{t=k+1}^{p} s_{jt}^2 \right\}, \]

it is easy to see that \((p-2)\text{tr} \tilde{s}^2 / 4!\) gives

\[ \frac{p - 2}{12} n K s' . \]

Adding (2.3) - (2.6) and factoring \(K\) out, we have the following theorem:

**Theorem 2.1.** Let \(A\) and \(B\) be diagonal matrices with \(0 < a_1 < \ldots < a_k < a_{k+1} = \ldots = a_p = a, (1 \leq k \leq p-1)\) and \(b_1 > b_2 > \ldots > b_p > 0,\) then for large degrees of freedom \(n = n_1 + n_2,\) the first two terms in the expansion for \(J\) are given by

\[ (2.7) \quad J = 2^p \frac{\pi^2 q^2}{\Gamma q (\frac{1}{2} q)} \frac{1}{\pi} \left( \sum_{j=1}^{k} (1 + a_j b_j) \right) \frac{p}{\pi} \left( 1 + ab_j \right) \frac{n}{2} \frac{k}{\pi} \left( \frac{2\pi}{n c_{jt}} \right)^{\frac{1}{2}} \]

\[ \cdot \left[ \sum_{j=1}^{k} c_{jt}^{-1} + \sum_{j=1}^{p} \sum_{t=k+1}^{p} c_{jt}^{0-l} + \alpha(p,k) \right] + \ldots , \]

where

\[ (2.8) \quad \alpha(p,k) = \frac{k}{12} \left( (k-1) (hk+1) + 6 (p^2 - k^2) \right) \]

and \(q = p - k.\)
Theorem 2.2. The asymptotic distribution of the ch. roots, $b_1 > b_2 > \ldots > b_p > 0$ of $S_1 S_2^{-1}$, for large degrees of freedom $n = n_1 + n_2$, when ch. roots of $(\Sigma_1^{-1} \Sigma_2^{-1})^{-1}$ are $0 < a_1 < \ldots < a_k < a_{k+1} = \ldots = a_p = a$ $(1 \leq k \leq p-1)$ is given by

$$c_0 = \frac{1}{2} \frac{n_1}{\pi} \frac{1}{\pi} \frac{1}{\pi} \left( \frac{1+a_j b_j}{1+a_j b_j} \right)_{j=1}^{k} \frac{1}{\pi} \frac{1}{\pi} \left( \frac{1+a_j b_j}{1+a_j b_j} \right)_{j=k+1}^{p}$$

$$\cdot \frac{b_j - b_i}{\pi} \frac{b_j - b_i}{\pi} \left( \frac{2c_j}{n c_j} \right)_{j<i}^{k} \frac{b_j - b_i}{\pi} \frac{b_j - b_i}{\pi} \left( \frac{2c_j}{n c_j} \right)_{j=i=1}^{k} \frac{b_j - b_i}{\pi} \frac{b_j - b_i}{\pi} \left( \frac{2c_j}{n c_j} \right)_{j=k+1}^{p} \frac{b_j - b_i}{\pi} \frac{b_j - b_i}{\pi} \left( \frac{2c_j}{n c_j} \right)_{j=k+1}^{p}$$

$$\left\{ 1 + \frac{1}{2n} \left[ \sum_{j<k} c_j^{-1} \sum_{j=1}^{k} \sum_{t=k+1}^{p} c_j^{-1} + \alpha(p,k) \right] + \ldots \right\},$$

where

$$c_0 = \frac{1}{2} q^2 \Gamma p \left( \frac{1}{2} n \right) \left\{ \Gamma q \left( \frac{1}{2} q \right) \Gamma p \left( \frac{1}{2} n_1 \right) \Gamma p \left( \frac{1}{2} n_2 \right) \right\}^{-1}$$

and $\alpha(p,k)$ is defined by (2.8).

3. Two-sample complex case. Let $S_j$ $(j = 1, 2)$ be independently distributed as complex Wishart $(n_j, p, \Sigma_j)$ and let $0 < a_1 < \ldots < a_k < a_{k+1} = \ldots = a_p = a$ $(1 \leq k \leq p-1)$ and $b_1 > b_2 > \ldots > b_p > 0$ be the ch. roots of $(\Sigma_1 \Sigma_2^{-1})^{-1}$ and $S_1 S_2^{-1}$ respectively. We still denote

$$A = \operatorname{diag} (a_1, \ldots, a_k, a, \ldots, a)$$

and

$$B = \operatorname{diag} (b_1, b_2, \ldots, b_p)$$"
then from (5.6) of [4], we have

\[(3.1) \quad J_1 = \int_{N(I)} |I + A \sim B \sim U^*|^{-n} (U^* \, d \, U)\]

where \( n = n_1 + n_2 \), and \( N(I) \) and \( (U^* \, d \, U) \) defined as in [4].

Partition the unitary matrix \( U \) into the submatrices \( U_1 \) consisting of its first \( k \), and \( U_2 \), the remaining \( q \) rows. If the integrand of \( (3.1) \) does not depend on \( U_2 \) then we can integrate over \( U_2 \) for fixed \( U_1 \) by the formula

\[(3.2) \quad \int_{U_2} c_1 (d \, U) = c_2 (d \, U_1),\]

where \( c_1 = \pi^{p(p-1)} \left\{ \tilde{f}_p (p) \right\}^{-1} \), \( c_2 = \pi^{k(p-1)} \left\{ \tilde{f}_k (p) \right\}^{-1} \)

and \( \tilde{f}_x(y) \) and \( (d \, U_1) \) defined similarly as in [4].

Apply the transformation \( \sim = \exp \left( i \, H \right) \) (See (5.7) of [4]) the parametrization of \( U \) may be obtained by writing

\[(3.3) \quad U = \left( \begin{array}{c} U_1 \\ U_2 \end{array} \right) = \exp \left\{ i \left( \begin{array}{cc} H_{11} & H_{12} \\ H_{12}^* & 0 \end{array} \right) \right\},\]

where \( H_{11} \) is a \( k \times k \) Hermitian matrix and \( H_{12} \) is a \( k \times q \) rectangular complex matrix.
From (5.8) of [4], it can be shown that

\[(3.4) \quad \frac{k(k-1)}{\tilde{\Gamma}_k(p)} (d \frac{U_t}{\tilde{\Gamma}_1}) \sim (d \frac{H_{11}}{\tilde{\Gamma}_2}) (d \frac{H_{12}}{\tilde{\Gamma}_2}) \{ 1 + O (\text{squares of } |h_{jt}|')s \}. \]

where the symbols \((d \frac{H_{11}}{\tilde{\Gamma}_1})\) and \((d \frac{H_{12}}{\tilde{\Gamma}_2})\) stand for \(\prod_{j=t}^{k} dh_{jt} R \prod_{j=t}^{k} dh_{jt} I\) and \(\prod_{j=1}^{k-p} dh_{jt} R \prod_{j=1}^{k-p} dh_{jt} I\).

Note that

\[(3.5) \quad \begin{cases} c_{jt} = r_{jt} b_{jt} - r_j r_t b_{jt}^2 = c_t j = 0 \\ h_{jtR} = h_{jtI} = 0 \end{cases} \]

for \(j, t = k + 1, \ldots , p,\)

where \(r_{jt} = r_j - r_t = \frac{a_j}{1 + a_j b_j} - \frac{a_t}{1 + a_t b_t}\)

and \(b_{jt} = b_j - b_t.\)

Since we wish to compute up to the second term in the asymptotic expansion of \(J_1\) we need to investigate the groups of terms up to the fourth order of \(H.\) Under conditions (3.5), then (5.19), (5.20) and (5.21) of [4] become
\begin{align}
\int \ldots \int \exp \left( -n \sum_{j=t}^{k} c_{jt} h_{jt} \bar{h}_{jt} - n \sum_{j=1}^{k} \sum_{t=k+1}^{p} c_{jt} h_{jt} \bar{h}_{jt} \right) \\
\cdot \left( \pi \right)^{k} dh_{jt R} \cdot dh_{jt I} \cdot \left( \pi \right)^{k} dh_{jt R} \cdot dh_{jt I} \\
= \left( \pi \right)^{k} \left( \pi \right)^{k} \left( \pi \right)^{p} \left( \pi \right)^{p} \left( \pi \right)^{n} c_{jt} \\
= c_0,
\end{align}

\begin{align}
\int \ldots \int \exp \left( -n \sum_{j=t}^{k} c_{jt} h_{jt} \bar{h}_{jt} - n \sum_{j=1}^{k} \sum_{t=k+1}^{p} c_{jt} h_{jt} \bar{h}_{jt} \right) \\
\cdot h_{uv c}^{2m} \left( \pi \right)^{k} dh_{jt R} \cdot dh_{jt I} \cdot \left( \pi \right)^{k} dh_{jt R} \cdot dh_{jt I} \\
= \begin{cases} 
(2m-1)(2n \ c_{uv})^{-m} & \text{if } v = 1, \ldots, k \\
(2m-1)(2n \ c_{uv}^0)^{-m} & \text{if } v = k+1, \ldots, p
\end{cases}
\end{align}

and

\begin{align}
\int \ldots \int \exp \left( -n \sum_{j=t}^{k} c_{jt} h_{jt} \bar{h}_{jt} - n \sum_{j=1}^{k} \sum_{t=k+1}^{p} c_{jt} h_{jt} \bar{h}_{jt} \right) \\
\cdot (h_{uv} \bar{h}_{uv})^{m} \left( \pi \right)^{k} dh_{jt R} \cdot dh_{jt I} \cdot \left( \pi \right)^{k} dh_{jt R} \cdot dh_{jt I} \\
= \begin{cases} 
\frac{C (m \ i)}{(n \ c_{uv}^m)} & \text{if } v = 1, \ldots, k \\
\frac{C (m \ i)}{(n \ c_{uv}^0)^m} & \text{if } v = k+1, \ldots, p
\end{cases}
\end{align}
respectively, where $h_{uv}$ denotes either $h_{uvR}$ or $h_{uvI}$ and

$$c^0_{jt} = r_{jt} b_{jt} - r_{jt} r_{jt} b_{jt}^2 = c^0_{jt} \quad j = 1, \ldots, k; \quad t = k + 1, \ldots, p.$$

Therefore we have the following theorem:

**Theorem 3.1.** Let $A$ and $B$ be diagonal matrices with $0 < a_1 < \ldots < a_k < a_{k+1} = \ldots = a_p = a$ ($1 \leq k \leq p-1$) and $b_1 > b_2 > \ldots > b_p > 0$. Then for large $n$, the first two terms in the expansion for $J_1$ are given by

$$(3.9) \quad J_1 = \frac{\pi q(q-1)}{\pi q(q)} \sum_{j=1}^{k} (1 + a_j b_j)^{-n} \sum_{j=k+1}^{p} (1 + a_j b_j)^{-n} \sum_{j=t}^{k} \frac{\pi}{nc_{jt}}$$

$$\times \frac{\pi}{3n} \{ 1 + \frac{1}{3n} \left[ \sum_{j< t} c_{jt}^{-1} + \sum_{j=1}^{k} \sum_{t=k+1}^{t} c_{jt}^0 + \beta(p,k) \right] \ldots \},$$

where

$$(3.10) \quad \beta(p,k) = \frac{k}{2^2} \left[ (k-1)(2k-1) + 3(p-k)(p+k-1) \right]$$

and $q = p - k$.

The proof is analogous to that of Theorem 2.1.

**Theorem 3.2.** The asymptotic distribution of the ch. roots, $b_1 > b_2 > \ldots > b_p > 0$, of $S^{-1}_{12} S_{22}^{-1}$ for large degrees of freedom $n = n_1 + n_2$ when the roots of $(S^{-1}_{12} S_{22}^{-1})^{-1}$ are $0 < a_1 < \ldots < a_k < a_{k+1} = \ldots = a_p = a$ ($1 \leq k \leq p-1$), is given by
\[(3.11) \quad c_3 a_{2n}^{q_1} \prod_{j=1}^k \frac{a_j}{n_j} \frac{n_1}{\pi} \frac{k}{\pi} \frac{(1 + a_j b_j)^{-n}}{j=k+1} \frac{p}{\pi} \frac{p}{\pi} (1 + ab_j)^{-n} \]

\[\frac{p}{\pi} (b_j - b_t)^2 \prod_{j<t} \frac{k}{n_j} \prod_{j=1}^k \frac{p}{\pi} \prod_{t=k+1} \frac{p}{\pi} \prod_{j=1}^p b_j^{n_1-p} \cdot \frac{d}{dj} \]

\[\{ 1 + \frac{1}{3n} \left[ \sum_{j<t} c_{jt}^{-1} + \sum_{j=1}^k \sum_{t=k+1}^p c_{jt}^{0^{-1}} + \beta(p,k) \right] + \ldots \} , \]

where

\[c_3 = \pi^{q(q-1)} \frac{n}{\Gamma_p(n)} \left\{ \frac{\Gamma_q(n)}{\Gamma_p(n_1)} \frac{\Gamma_q(n_2)}{\Gamma_p(n_2)} \right\}^{-1} \]

and \(\beta(p,k)\) is defined by (3.10).

\[4. \quad \text{Remark: Replace } b_j \text{ by } \frac{n_1}{n_2} b_j / n_2 \quad (j = 1, \ldots, p), \text{ and let } n_2 \text{ tend to infinity and rewrite } n_1 \text{ as } n, \text{ then } (2.9) \text{ and } (3.11) \text{ become the one-sample cases. i.e.} \]

\[(4.1) \quad \left( \frac{n}{2} \right)^{q_2} p^n \pi^{q_2} \frac{q_2}{2} \left\{ \frac{\Gamma_q(n)}{\Gamma_p(n_1)} \frac{\Gamma_q(n_2)}{\Gamma_p(n_2)} \right\}^{-1} a_1 \frac{a_2}{n_j} \frac{a_3}{n_j} \frac{a_4}{n_j} \frac{a_5}{n_j} \frac{a_6}{n_j} \frac{a_7}{n_j} \frac{a_8}{n_j} \frac{a_9}{n_j} \]

\[\frac{p}{\pi} (b_j - b_t) \exp \left( -\frac{1}{2n} \sum_{j=1}^k a_j b_j - \frac{1}{2n} \sum_{j=k+1}^p b_j \right) \frac{n}{\pi} \frac{a_j}{n} \left( \frac{2m}{n} \frac{1}{\gamma_{jt}} \right) \]

\[\frac{k}{\pi} \frac{p}{\pi} \left( \frac{2m}{n} \right)^{\frac{1}{2}} \frac{p}{\pi} \frac{b_j^{1/2(n-p-1)}}{b_j} \cdot \frac{d}{dj} \{ 1 + \frac{1}{2n} \left[ \sum_{j<t} \gamma_{jt}^{-1} + \sum_{j=1}^k \sum_{t=k+1}^p \right] \}
\]

\[\gamma_{jt}^{0^{-1}} \} + \ldots \} \]
and

\[(4.2) \quad n^{p} \pi^{(q-1)} \left\{ \frac{\Gamma_{q}(q) \Gamma_{p}(n)}{\Gamma_{p}(n_{1}) \Gamma_{p}(n_{2})} \right\}^{-1} a^{qn} \prod_{j=1}^{k} a_{j}^{n} \prod_{j<k}^{p} (b_{j} - b_{t})^{2} \]

\[\exp \left( -\frac{1}{n} \sum_{j=1}^{k} a_{j} b_{j} - a \sum_{j=k+1}^{p} b_{j} \right) \prod_{j=k+1}^{p} \gamma_{jt}^{n} \prod_{j=1}^{k} \gamma_{jt}^{n} \prod_{j=1}^{n} \gamma_{jt}^{0} \]

\[\prod_{j=1}^{p} b_{j}^{n-p} d_{b_{j}} \left\{ 1 + \frac{1}{3n} \left[ \sum_{j<t}^{k} \gamma_{jt}^{-1} + \sum_{j=1}^{k} \sum_{t=k+1}^{p} \gamma_{jt}^{0} \right] + \ldots \right\} \]

respectively, where

\[\gamma_{jt} = (a_{t} - a_{j})(b_{j} - b_{t}) \quad \text{for } j, t = 1, \ldots, k \]

\[\gamma_{jt}^{0} = (a - a_{j})(b_{j} - b_{t}) \quad \text{for } j = 1, \ldots, k; \quad t = k+1, \ldots, p.\]

The first term in the asymptotic expansion of \((4.1)\) gives the result in (3.11) of \([3]\).

Using the same convention as in Section 4 of \([4]\), then the restriction \(1 \leq k \leq p-1\) can be written as \(0 \leq k \leq p\).

(I) If \(k = 0\), i.e. \(q = p\), then \(a_{1} = a_{2} = \ldots = a_{p} = a, \alpha(p,k) = 0, \beta(p,k) = 0\), (1) (2.9) reduces to (4.9) of \([4]\), and (ii) (3.11) becomes

\[n^{p(p-1)} \Gamma_{p}^{(n)} \left\{ \frac{\Gamma_{p}(p) \Gamma_{p}(n_{1}) \Gamma_{p}(n_{2})}{\Gamma_{p}(n)} \right\}^{-1} a^{pn} \prod_{j=t}^{p} \frac{1}{(b_{j} - b_{t})^{2}} \]

\[\prod_{j=1}^{p} b_{j}^{n_{1} - p} (1 + ab_{j})^{-n} d_{b_{j}} , \]
both of which are the joint distribution of \( b_1, b_2, \ldots, b_p \) under null hypothesis \( \Sigma_1 = a^{-1} \Sigma_2 \) (for the real case see [5]) and are exact forms where we assume no asymptotic condition.

Similarly, we can obtain the corresponding formulae from (4.1) and (4.2).

(II) If \( k = p \), i.e. \( q = 0 \), then \( 0 < a_1 < a_2 < \ldots < a_p \), \( \alpha(p,k) = \alpha(p) \), \( \beta(p,k) = \beta(p) \), and (2.7), (2.8), (2.9), (3.9), (3.10) and (3.11) reduce to (3.14), (3.15), (3.20), (5.22), (5.23) and (5.27) of [4] respectively. Similarly, (4.1) reduces to (1.8) of [1]. If we take the first term in the asymptotic expansion of (3.20) in [4], it is Chang's result [2].
References


