ON A MONOTONICITY PROPERTY RELATING
TO THE GAMMA DISTRIBUTIONS

by

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1. **Introduction and Summary.** Let \( \{F_\lambda\}, \lambda \geq 1 \), be the family of gamma distributions with density function \( f_\lambda \) given by

\[
(1.1) \quad f_\lambda(x) = \begin{cases} \\
\frac{\lambda^x e^{-\lambda x}}{\Gamma(\alpha)} & , x \geq 0 \\
\alpha & , \text{elsewhere,}
\end{cases}
\]

where the shape parameter \( \alpha > 0 \) is same for all distributions of the family. We define

\[
(1.2) \quad A(\lambda) = \int_0^\alpha \frac{F^{-1}_\lambda(x)}{b} dF_\lambda(x),
\]

where \( k \geq 2 \) is an integer and \( 0 < b \leq 1 \). Gupta [2] has discussed the subset selection problem for gamma populations with the same degrees of freedom in terms of their shape parameters and his numerical computations indicate the monotonic behavior of \( A(\lambda) \) in \( \lambda \geq 1 \). This monotonic behavior of \( A(\lambda) \) can be easily proved by appealing to the fact that the gamma distributions are convex ordered, i.e., for \( 0 < \lambda < \lambda' \), \( \frac{F^{-1}_{\lambda'}F_\lambda}{\lambda} \), \( (x) \) is

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convex on $[0, \infty)$ (see Barlow and Gupta [1]). However, the proof of this convex ordering property of the gamma family is very tedious (see Van Zwet [5]). McDonald [3] gives a direct proof of the monotonicity of $A(\lambda)$ with the purpose of avoiding recourse to the convex ordering property, but only for the case of $k = 2$ and integer-valued $\lambda$. We first note that a general result of the author [4] is relevant here. This result states that, if \{F_\lambda\}, $\lambda \in \Lambda$, an interval on the real line, is a family of absolutely continuous distributions and $\Psi(x, \lambda)$ is a real valued function differentiable in $x$ and $\lambda$, then, for any positive integer $t$, $B(\lambda) = \int_0^x \Psi'(x, \lambda) dF_\lambda(x)$ is non-decreasing in $\lambda$ provided that $\Psi(x, \lambda) \geq 0$ for all $x$ and $\lambda$ and

\begin{equation}
\frac{d}{d\lambda} \Psi(x, \lambda) - \frac{d}{dx} \frac{\partial}{\partial \lambda} \Psi(x, \lambda) \frac{d}{d\lambda} F_\lambda(x) \geq 0.
\end{equation}

In the case of $A(\lambda)$ defined in (1.2), the condition (1.3) reduces to

\begin{equation}
\frac{d}{d\lambda} F_\lambda(x) \frac{\partial}{\partial \lambda} F_\lambda(x) - \frac{1}{b} \frac{d}{dx} \frac{\partial}{\partial \lambda} F_\lambda(x) \geq 0.
\end{equation}

However, verification of this condition in the case of the gamma family when $\lambda \in [1, \infty)$ presents difficulties. The aim of this note is to obtain sufficient conditions for $B(\lambda)$ to be non-decreasing in $\lambda$ where $\lambda \in \Lambda_d = (\lambda_1 < \lambda_2 < \ldots \ldots )$. This forms the content of the next section. The last section applies this result to establish the monotonicity of $A(\lambda)$ defined in (1.2) for $k \geq 2$ and $\Lambda_d = (1, 2, 3, \ldots )$.

2. **The Main Result.** Let \{F_\lambda\}, $\lambda \in \Lambda_d = (\lambda_1 < \lambda_2 < \ldots \ldots )$, be a family of absolutely continuous distributions on the real line all having the same support and $\Psi(x, \lambda)$ be a non-negative function differentiable in $x$.

Then, for any positive integer $t$, $B(\lambda) = \int_0^x \Psi'(x, \lambda) dF_\lambda(x)$ is nondecreasing
in $\lambda$ over $\Lambda_d$, provided that, for $i = 1, 2, \ldots$,

$$
(2.1) \quad \Delta_{\psi}(x, \lambda_i^{\prime}) \lambda_j^{\prime} (x) - \Delta F_{\lambda_i} (x) \psi' (x, \lambda_j) \geq 0, \quad j = 1, i + 1,
$$

where $\Delta \psi(x, \lambda_i) = \psi(x, \lambda_i + 1) - \psi(x, \lambda_i)$, $\Delta F_{\lambda_i} (x) = F_{\lambda_i + 1} (x) - F_{\lambda_i} (x)$

and the prime over $\psi$ denotes the derivative w.r.t. $x$.

**Proof.** For $\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_t} + 1 \in \Lambda_d$, define

$$
(2.2) \quad A_{r}(\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_t} + 1) = \sum_{j=1}^{t+1} \psi_{j}(x) dF_{j}(x), \quad r = 1, 2, \ldots, t + 1,
$$

and

$$
(2.3) \quad A(\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_t} + 1) = \sum_{r=1}^{t+1} A_{r}(\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_t} + 1),
$$

where $F_{j} = F_{\lambda_{j}}$ and $\psi_{j} = \psi(x, \lambda_{j})$. For any positive integer $s$,

$$
(2.4) \quad A(\lambda_{s+1}, \lambda_{s+1}, \ldots, \lambda_{s+1}) = A(\lambda_{s}, \lambda_{s}, \ldots, \lambda_{s})
$$

$$
= (A_{0} - A_{1}) + (A_{1} - A_{2}) + \ldots + (A_{t} - A_{t+1}),
$$

where $A_{m}$ stands for $A$ with the first $m$ arguments equal to $\lambda_{s+1}$ and the remaining $t+1-m$ equal to $\lambda_{s}$. Let us consider a typical term, namely, $A_{m} - A_{m+1}$, in (2.4). We can see that

$$
(2.5) \quad A_{m} - A_{m+1} = m \int s^{m-1}^{t+1-m} dF_{s}(x) + (t+1-m) \int s^{m} s^{t-m} dF_{s+1}(x)
$$

$$
- (m+1) \int s^{m} s^{t-m} dF_{s}(x) - (t-m) \int s^{m+1} s^{t-m-1} dF_{s+1}(x).
$$

Using integration by parts, we obtain
\[ (2.6) \quad \int \psi^m \psi^{t-m} \, dF_{s+1}(x) - \int \psi^m \psi^{t-m} \, dF_s(x) \]
\[ = \int (F_s - F_{s+1}) \psi^{m-l} \psi^{t-m-1} \left[ m \psi^l_s \psi^l_{s+1} + (t-m) \psi_s \psi^l_{s+1} \right] \, dx. \]

Using (2.6) in (2.5) and regrouping the terms, we can write (2.5) as

\[ (2.7) \quad A_m - A_{m+1} = m \int \psi^{m-l} \psi^{t-m} \left[ \psi_s \Delta \psi_{s+1} - \psi^l_s \Delta F_s \right] \, dx \]
\[ + (t-m) \int \psi^{m-l} \psi^{t-m-1} \left[ \psi_{s+1} \Delta \psi_{s+1} - \psi^l_{s+1} \Delta F_s \right] \, dx, \]

which is non-negative if (2.1) is satisfied. This completes the proof of the main result.

**Remark.** The non-negativity of \( \psi(x,\lambda) \) is essentially needed when \( t > 1 \).

3. **Application to the Gamma family.** Let \( F_\lambda \) be the distribution function with density \( f_\lambda \) given by (1.1) and let \( \lambda \in [1,2,3,\ldots] \). In order to establish the monotonicity of \( A(\lambda) \), we need to show that for any positive integer \( i \),

\[ (3.1) \quad f_i(x) \Delta F_j \left( \frac{x}{b} \right) - \frac{1}{b} \quad f_j \left( \frac{x}{b} \right) \Delta F_i(x) \geq 0 \quad \text{for} \quad j = i, i+1. \]

It is well-known that \( \Delta F_i(x) = \frac{1}{\alpha} f_{i+1}(x) \). Hence, the left hand side of (3.1)

\[ = - \frac{1}{\alpha} f_{i+1} \left( \frac{x}{b} \right) \quad f_j(x) + \frac{1}{\alpha} f_{i+1} (x) f_j \left( \frac{x}{b} \right) \]
\[ = e^{-\alpha(\frac{1}{b} + 1)x} \quad e^\alpha x^{i+j} (\frac{x}{b})^{i+j} - \frac{1}{b} \]
\[ \geq 0 \quad \text{for} \quad j = i, i+1, \text{since} \quad b \leq 1. \]
REFERENCES


