Buffon in the Round

by

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1. Introduction. Suppose that a needle of length 2d is thrown at ran-
dom onto a circle of radius R so that its mid-point M falls within or on the
circumference. We can define the term "at random" in a number of ways. In
this paper we consider the following alternative models.

A. The needle is tossed so that the position of M is uniformly dis-
tributed along the radius vector passing through it. Let U be the distance
from the center O of the circle to M. Then U is a random variable with dens-
ity function f given by

\[ f(u) = \begin{cases} \frac{1}{R} & 0 \leq u \leq R \\ 0 & \text{otherwise} \end{cases} \]

B. The needle is tossed so that the position of M is uniformly dis-
tributed over the whole circle. In this case U has density function g,

where

\[ g(u) = \begin{cases} \frac{2u}{R^2} & 0 \leq u \leq R \\ 0 & \text{otherwise} \end{cases} \]

In both cases we assume that the angle the needle makes with a fixed vector
in the plane of the circle is uniformly distributed on [0,2\pi]

We shall find the probability distribution of the number of intersections
of the needle with the circumference of the circle. In case A these probab-

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ities are expressed in terms of elliptic integrals of the 1st and 2nd kinds. On the other hand the results for case B are expressed in terms of elementary functions only. In appendix 1 a table of these probabilities is given for certain $d|R$ ratios.

2. **Derivation of the Probabilities.**

We define the random variable $Z$ as the number of intersections of the needle with the circumference of the circle. It is clear, that for $d > 2R$, $P[Z=2]=1$, so we need only consider the case $0 < d \leq 2R$. There are 2 main sub-cases to consider (a) $R \leq d \leq 2R$ and (b) $0 < d \leq R$. We shall deal with the case (a) first.

(a) $R \leq d \leq 2R$. We shall break this case down into two further sub-cases depending on the distance from 0 to M. Let $p_i(u)$ be the conditional probability $P[Z=i|U = u]$, $i = 0, 1, 2$ then:

(i) $0 < u \leq d-R$. Under these assumptions we always have exactly two intersections, so that:

$$p_0(u) = p_1(u) = 0$$

$$p_2(u) = 1$$

(ii) $d-R < u \leq R$. Referring to fig. 1 for notation we have:

\[
\begin{align*}
    p_0(u) &= 0 \\
    p_1(u) &= 2 - 2\Psi/\pi \\
    p_2(u) &= 2\Psi/\pi - 1 \\
    R^2 &= u^2 + d^2 - 2ud \cos(\pi-\Psi)
\end{align*}
\]

On expressing $\Psi$ in terms of $u, d$ and $R$ and substituting we get

\[
\begin{align*}
    p_0(u) &= 0 \\
    p_1(u) &= 2 - 2/\pi \arccos \left( \frac{R^2-d^2-u^2}{2ud} \right) \\
    p_2(u) &= 2/\pi \arccos \left( \frac{R^2-d^2-u^2}{2ud} \right) - 1
\end{align*}
\]
\( P_A[Z = i] \) denotes the probability of \( i \) intersections in case A. Similarly for \( P_B[Z = i] \). It follows that for \( R \leq d \leq 2R \) we have:

\[
\begin{align*}
P_A[Z=0] &= 0, \\
P_A[Z=1] &= \int_0^R \left[ \frac{2 - \frac{2}{\pi} \arccos \left( \frac{R^2 - d^2 - u^2}{2ud} \right) }{R} \right] \, du, \\
P_A[Z=2] &= \int_0^R \left[ \frac{2}{\pi} \arccos \left( \frac{R^2 - d^2 - u^2}{2ud} \right) - 1 \right] \, du.
\end{align*}
\]

In case B we have

\[
\begin{align*}
P_B[Z=0] &= 0, \\
P_B[Z=1] &= \int_0^R \left[ \frac{2 - \frac{2}{\pi} \arccos \left( \frac{R^2 - d^2 - u^2}{2ud} \right) }{R^2} \right] \, 2udu, \\
P_B[Z=2] &= \int_0^R \left[ \frac{2}{\pi} \arccos \left( \frac{R^2 - d^2 - u^2}{2ud} \right) - 1 \right] \, 2udu.
\end{align*}
\]

Simplifying these expressions we get

\[
\begin{align*}
P_A[Z=0] &= 0, \\
P_A[Z=1] &= 4 - \frac{2d}{R} - \frac{2}{\pi R} \int_{d-R}^R \arccos \left( \frac{R^2 - d^2 - u^2}{2ud} \right) \, du, \\
P_A[Z=2] &= \frac{2d}{R} - 3 + \frac{2}{\pi R} \int_{d-R}^R \arccos \left( \frac{R^2 - d^2 - u^2}{2ud} \right) \, du, \\
P_B[Z=0] &= 0, \\
P_B[Z=1] &= \frac{2d}{R^2} (2R-d) - \frac{4}{\pi R^2} \int_{d-R}^R u \arccos \left( \frac{R^2 - d^2 - u^2}{2ud} \right) \, du, \\
P_B[Z=2] &= 1 - \frac{2d}{R^2} (2R-d) + \frac{4}{\pi R^2} \int_{d-R}^R u \arccos \left( \frac{R^2 - d^2 - u^2}{2ud} \right) \, du.
\end{align*}
\]

We will leave the evaluation of the above integrals to Section 3.
(b) $0 < d < R$. We break this case down into three sub-cases depending on the distance from 0 to M.

(i) $0 < u < R - d$. Under these conditions no intersections are possible so we have

$$p_0(u) = 1$$
$$p_1(u) = p_2(u) = 0.$$ 

(ii) $R - d < u < \sqrt{R^2 - d^2}$. Now only zero or one intersections are possible.

Referring to fig. 2 we see that:

$$p_0(u) = 1 - 2\psi/\pi$$
$$p_1(u) = 2\psi/\pi$$
$$p_2(u) = 0$$
$$\psi = \arccos\left(\frac{R^2 - d^2 - u^2}{2ud}\right)$$

(iii) $\sqrt{R^2 - d^2} < u < R$. Only one or two intersections are possible and, referring to fig. 3, we see that:

$$p_0(u) = 0$$
$$p_1(u) = 2 - 2\psi/\pi$$
$$p_2(u) = 2\psi/\pi - 1$$
As for case (a) we obtain

\[
P_A[Z=0] = \frac{\sqrt{R^2-d^2}}{R} - 2 \pi R \int_{R-d}^{\sqrt{R^2-d^2}} \arccos \left( \frac{R^2-d^2-u^2}{2ud} \right) \, du,
\]

\[
P_A[Z=1] = 2 - 2 \frac{\sqrt{R^2-d^2}}{R} + \frac{2}{\pi R} \int_{R-d}^{\sqrt{R^2-d^2}} \arccos \left( \frac{R^2-d^2-u^2}{2ud} \right) \, du
\]

\[- \frac{2}{\pi R} \int_{\sqrt{R^2-d^2}}^{R} \arccos \left( \frac{R^2-d^2-u^2}{2ud} \right) \, du,
\]

\[
P_A[Z=2] = \frac{\sqrt{R^2-d^2}}{R} - 1 + \frac{2}{\pi R} \int_{R-d}^{\sqrt{R^2-d^2}} \arccos \left( \frac{R^2-d^2-u^2}{2ud} \right) \, du.
\]

\[
P_B[Z=0] = \frac{R^2-d^2}{R^2} - 4 \frac{\sqrt{R^2-d^2}}{\pi R^2} \int_{R-d}^{\sqrt{R^2-d^2}} u \arccos \left( \frac{R^2-d^2-u^2}{2ud} \right) \, du,
\]

\[
P_B[Z=1] = 2 - \frac{2}{R^2} (R^2-d^2) + \frac{4}{\pi R^2} \int_{R-d}^{\sqrt{R^2-d^2}} u \arccos \left( \frac{R^2-d^2-u^2}{2ud} \right) \, du
\]

\[- \frac{4}{\pi R^2} \int_{\sqrt{R^2-d^2}}^{R} u \arccos \left( \frac{R^2-d^2-u^2}{2ud} \right) \, du,
\]

\[
P_B[Z=2] = \frac{1}{R^2} (R^2-d^2) - 1 + \frac{4}{\pi R^2} \int_{R-d}^{\sqrt{R^2-d^2}} u \arccos \left( \frac{R^2-d^2-u^2}{2ud} \right) \, du.
\]

3. Evaluation of the integrals.

Let \( I = \int_{a}^{b} \arccos \left( \frac{R^2-d^2-u^2}{2ud} \right) \, du. \)
On integrating by parts we get

\[ I = u \arccos \left( \frac{R^2 - d^2 - u^2}{2ud} \right) \bigg|_a^b - \int_a^b \frac{(R^2 - d^2 - u^2) \, du}{\sqrt{u^2 - (R-d)^2} \left[ (R+d)^2 - u^2 \right]} \]

Let \( I_1 \) denote the remaining integral. By means of the substitution \( u = (R+d) \cos \phi \), we obtain

\[ I_1 = -\int_{a_1}^{b_1} \frac{[(R^2 - d^2) + (R+d)^2 \cos^2 \phi] \, d\phi}{\sqrt{4Rd - (R+d)^2 \sin^2 \phi}} \]

For the moment we do not bother to express \( a_1 \) and \( b_1 \) in terms of \( a \) and \( b \). Now, let \( (R+d) \sin \phi = 2 \sqrt{Rd} \sin \theta \), we obtain

\[ I_1 = -\frac{2R}{R+d} \int_{a_2}^{b_2} \frac{[(R+d) - 2d \sin^2 \theta] \, d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} \]

where \( k^2 = \frac{4Rd}{(R+d)^2} \)

Upon rewriting this expression we get

\[ I_1 = -2R \int_{a_2}^{b_2} \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}} + \frac{4R}{R+d} \int_{a_2}^{b_2} \frac{\sin^2 \theta d\theta}{\sqrt{1-k^2 \sin^2 \theta}}. \]

Setting:

\[ F(n|k) = \int_0^n \frac{d\theta}{\sqrt{1-k^2 \sin^2 \theta}} \]

\[ E(n|k) = \int_0^n \sqrt{1-k^2 \sin^2 \theta} \, d\theta \]
where $F(\eta|k)$ is the elliptic integral of the 1st kind and $E(\eta|k)$ is the
elliptic integral of the 2nd kind and observing that

$$
\int_{a_2}^{b_2} \frac{\sin^2 \theta d\theta}{\sqrt{1-k^2 \sin^2 \theta}} = \frac{1}{k^2} \int_{a_2}^{b_2} \frac{[1-(1-k^2 \sin^2 \theta)] d\theta}{\sqrt{1-k^2 \sin^2 \theta}}
$$

$$
= \frac{1}{k^2} [F(b_2|k) - F(a_2|k) - E(b_2|k) + E(a_2|k)]
$$

we obtain,

$$
I = u \arccos \left( \frac{R^2-d^2-u^2}{2ud} \right) \bigg|_a^b - \left\{ (d-R)[F(b_2|k) - F(a_2|k)] \\
- (d+R)[E(b_2|k) - E(a_2|k)] \right\}
$$

To express $a_2$ and $b_2$ in terms of $a$ and $b$ we must express $\theta$ in terms of $u$.

Following through the substitutions we get

$$
\theta = \arcsin \sqrt{\frac{(R+d)^2-u^2}{4Rd}}.
$$

So we can easily find $a_2$ and $b_2$ in terms of $a$ and $b$. We leave off doing
this until later.

Let

$$
I' = \int_a^b u \arccos \left( \frac{R^2-d^2-u^2}{2ud} \right) du.
$$

On integrating by parts we obtain

$$
I' = \frac{u^2}{2} \arccos \left( \frac{R^2-d^2-u^2}{2ud} \right) \bigg|_a^b - \frac{1}{2} I_1'
$$
where\[I_1' = \int_{a}^{b} \frac{u(R^2-d^2-u^2)}{\sqrt{u^2 - (R-d)^2} \cdot [(R+d)^2-u^2]}\, du\]

By the substitutions, \(u = (R+d) \cos \phi\), followed by \((R+d) \sin \phi = 2\sqrt{Rd} \sin \theta\), we obtain

\[I_1' = -2R^2(b_2-a_2) - 2Rd \sin \theta \cos \theta \bigg|_{\theta=a_2}^{b_2}\]

Hence we have

\[I' = \frac{u^2}{2} \arccos \left( \frac{R^2-d^2-u^2}{2ud} \right) \bigg|_{u=a}^{b} + R^2(b_2-a_2) + Rd \sin \theta \cos \theta \bigg|_{\theta=a_2}^{b_2}\]

We gather together in table 1 the values of \((a,b)\) and \((a_2,b_2)\) we need to evaluate the integrals involved.

<table>
<thead>
<tr>
<th>(u)</th>
<th>(d-R)</th>
<th>(R-d)</th>
<th>(\sqrt{R^2-d^2})</th>
<th>(R)</th>
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</thead>
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<td>(\pi/2)</td>
<td>(\arcsin \sqrt{\frac{1+\gamma}{2}})</td>
<td>(\arcsin \sqrt{\frac{2+\gamma}{4}})</td>
</tr>
</tbody>
</table>

where \(\gamma = \frac{d}{R}\).

Let \(\theta_1 = \pi/2\), \(\theta_2 = \arcsin \sqrt{\frac{1+\gamma}{2}}\) and \(\theta_3 = \arcsin \sqrt{\frac{2+\gamma}{4}}\)
Case A

(i) \[ \int_{d-R}^{R} \arccos \left( \frac{R^2 - d^2 - u^2}{2ud} \right) \, du \]

\[ = 2\pi R - \pi d - R \arccos \left( \frac{\gamma}{2} \right) - \left\{ (d-R)[F(\theta_3 | k) - F(\theta_1 | k)] - (d+R)[E(\theta_3 | k) - E(\theta_1 | k)] \right\} \]

(ii) \[ \int_{R-d}^{\sqrt{R^2 - d^2}} \arccos \left( \frac{R^2 - d^2 - u^2}{2ud} \right) \, du. \]

\[ = \frac{\pi}{2} \sqrt{R^2 - d^2} - \left\{ (d-R)[F(\theta_2 | k) - F(\theta_1 | k)] - (d+R)[E(\theta_2 | k) - E(\theta_1 | k)] \right\} \]

(iii) \[ \int_{R}^{\sqrt{R^2 - d^2}} \arccos \left( \frac{R^2 - d^2 - u^2}{2ud} \right) \, du \]

\[ = \pi R - R \arccos \left( \frac{\gamma}{2} \right) - \frac{\pi}{2} \sqrt{R^2 - d^2} - \left\{ (d-R)[F(\theta_3 | k) - F(\theta_2 | k)] - (d+R)[E(\theta_3 | k) - E(\theta_2 | k)] \right\} \]

Case B

(i) \[ \int_{d-R}^{R_u} \arccos \left( \frac{R^2 - d^2 - u^2}{2ud} \right) \, du \]

\[ = \frac{\gamma}{2} \left[ 2dR - d^2 - R^2 \right] - \frac{R^2}{2} \arccos \left( \frac{\gamma}{2} \right) + R^2 \arcsin \sqrt{\frac{2+\gamma}{4}} + \frac{Rd}{4} \sqrt{4-\gamma^2} \]
(ii) \[ \int_{R-d}^{\sqrt{R^2-d^2}} u \arccos \left( \frac{R^2-d^2-u^2}{2ud} \right) du. \]

\[ = R^2 \arcsin \sqrt{\frac{1+\gamma}{2}} + \frac{Rd}{2} \sqrt{1-\gamma^2} - \frac{\pi}{4} (R^2+d^2) \]

(iii) \[ \int_{\sqrt{R^2-d^2}}^{R} u \arccos \frac{R^2-d^2-u^2}{2ud} du \]

\[ = \frac{\pi}{4} (R^2+d^2) + \frac{Rd}{2} \left( \frac{1}{2} \sqrt{4-\gamma^2} - \sqrt{1-\gamma^2} \right) - \frac{R^2}{2} \arccos \left( \frac{\gamma}{2} \right) \]

\[ + R^2 (\arcsin \sqrt{\frac{2+\gamma}{4}} - \arcsin \sqrt{\frac{1+\gamma}{2}}). \]

4. Main Results. Using the results of section 3 we can write expressions for \( P_A[Z=1], P_B[Z=1] \)

**Case A**

(i) \[ 0 < \gamma \leq 1 \]

\[ P_A[Z=0] = \frac{2}{\pi} \left\{ (1-\gamma) \left[ F(\theta_1 | k) - F(\theta_2 | k) \right] + (1+\gamma) \left[ E(\theta_1 | k) - E(\theta_2 | k) \right] \right\} \]

\[ P_A[Z=1] = \frac{2}{\pi} \arccos \frac{\gamma}{2} \]

\[ - \frac{2}{\pi} \left\{ (1-\gamma) \left[ F(\theta_1 | k) - F(\theta_2 | k) \right] + (1+\gamma) \left[ E(\theta_1 | k) - E(\theta_2 | k) \right] \right\} \]

\[ + \frac{2}{\pi} \left\{ (1-\gamma) \left[ F(\theta_2 | k) - F(\theta_3 | k) \right] + (1+\gamma) \left[ E(\theta_2 | k) - E(\theta_3 | k) \right] \right\} \]
\[ P_A[Z=2] = 1 - \frac{2}{\pi} \arccos \left( \frac{\gamma}{2} \right) \]
\[ - \frac{2}{\pi} \left\{ (1-\gamma) [F(\theta_2|k) - F(\theta_3|k)] - (1+\gamma) [E(\theta_2|k) - E(\theta_3|k)] \right\} \]

(ii) \[ 1 \leq \gamma \leq 2 \]
\[ P_A[Z=0] = 0 \]
\[ P_A[Z=1] = \frac{2}{\pi} \arccos \left( \frac{\gamma}{2} \right) \]
\[ - \frac{2}{\pi} \left\{ (\gamma-1) [F(\theta_1|k) - F(\theta_3|k)] - (\gamma+1) [E(\theta_1|k) - E(\theta_3|k)] \right\} \]

\[ P_A[Z=2] = 1 - \frac{2}{\pi} \arccos \left( \frac{\gamma}{2} \right) \]
\[ + \frac{2}{\pi} \left\{ (\gamma-1) [F(\theta_1|k) - F(\theta_3|k)] - (\gamma+1) [E(\theta_1|k) - E(\theta_3|k)] \right\} \]

Since we can write \( k \) in terms of \( \gamma \), viz \( k^2 = 4\gamma(1+\gamma)^2 \), each of these expressions can be computed as functions of the single variable \( \gamma \).

Case B

(i) \[ 0 < \gamma \leq 1 \]
\[ P_B[Z=0] = 2 - \frac{2\gamma}{\pi} \sqrt{1-\gamma^2} - \frac{4}{\pi} \arcsin \sqrt{\frac{1+\gamma}{2}} \]
\[ P_B[Z=1] = \frac{4}{\pi} \sqrt{1-\gamma^2} - \frac{\gamma}{4} \sqrt{4-\gamma^2} - 2 + \frac{2}{\pi} \arccos \left( \frac{\gamma}{2} \right) \]
\[ + \frac{8}{\pi} \arcsin \sqrt{\frac{1+\gamma}{2}} - \frac{4}{\pi} \arcsin \sqrt{\frac{2+\gamma}{4}} \]
\[ P_B[Z=2] = 1 + \frac{2\gamma}{\pi} \left( \frac{1}{2} \sqrt{4-\gamma^2} - \sqrt{1-\gamma^2} \right) - \frac{2}{\pi} \arccos \left( \frac{\gamma}{2} \right) \]
\[ + \frac{4}{\pi} \arcsin \sqrt{\frac{2+\gamma}{4}} - \frac{4}{\pi} \arcsin \sqrt{\frac{1+\gamma}{2}} \]

(ii) \quad 1 \leq \gamma \leq 2

\[ P_B[Z=0] = 0 \]
\[ P_B[Z=1] = 2 - \frac{\gamma}{\pi} \sqrt{4-\gamma^2} + \frac{2}{\pi} \arccos \left( \frac{\gamma}{2} \right) - \frac{4}{\pi} \arcsin \sqrt{\frac{2+\gamma}{4}} \]
\[ P_B[Z=2] = \frac{\gamma}{\pi} \sqrt{4-\gamma^2} - 1 + \frac{4}{\pi} \arcsin \sqrt{\frac{2+\gamma}{4}} - \frac{2}{\pi} \arccos \left( \frac{\gamma}{2} \right) \]

The form of the above results was chosen to make their evaluation by computer easy. In appendix 1 we give numerical results for values of \( \gamma \) between 0 and 2.
5. Appendix 1.

**Case A**

<table>
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The authors thank Mrs. P. Cohen and Mr. C. Henry for writing the computer programs.
GRAPH I
Buffon in the Round


Neuts, Marcel F. and Purdue, Peter

July, 1970

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The distribution of the number of intersections of a needle and the circumference of a circle is studied under two distinct assumptions on the random manner in which the former is tossed on the circle.

Numerical results and graphs are also provided.