An Exact Comparison of the Waitingtimes under
Three Priority Rules

by

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Mimeograph Series No. 213
December 1969

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*The research of the author was partly supported by the Office of Naval Research Contract NONR 1100 (26) at Purdue University.
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1. Introduction

This paper is a sequel to "A Priority Rule Based on the Ranking of the
Service Times for the $M|G|1$ Queue" which appeared in Operations Research,
Vol. 17, 1969, 466-477. We use the notation and concepts defined there with-
out repeating their formal definitions.

Informally summarized, we are considering the classical $M|G|1$ queue with
Poisson input of rate $\lambda$ and service time distribution $H(\cdot)$ and we distinguish
between successive "generations" of customers as in Kendall [1] and Neuts [3].
The customers present at $t = 0$ form the first generation; the new arrivals
during the total time required to process them form the second generation, the
third generation consists of those arriving during the service of those in the
second generation and so on. This continues until the initial busy period
comes to an end and starts over (regeneratively) with the arrival of the first
customer in the next busy period.

The priority rules discussed in [2] consist of serving within each genera-
tion the customers in the order of shortest (SPT) or longest(LPT) service times
first.

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In [2] a number of comparisons between these two rules and the first come, first served (FCFS) discipline in regards to expected waiting times in the equilibrium state were carried out.

Many questions involving more than the expected values can be asked. In order to answer them an exact comparison of the waiting times as random variables needs to be made. For example, a referee of [2], asked for the (limiting) probability that a customer "does better" under one priority rule than under each of the other two. We attempt to answer such questions in the present paper.

We denote by \( \eta(t) \) the virtual waiting time at time \( t \), i.e. the waiting time of a (virtual) customer joining the queue at time \( t \). By \( \overline{\eta}(t,x) \) we mean the virtual waiting time under the LPT priority rule of a customer, arriving at \( t \) whose service time is \( x \). Likewise \( \underline{\eta}(t,x) \) is the waiting time of this customer under the SPT rule. Clearly under the FCFS rule the waiting time of a customer does not depend on the amount of service he requests, but it does under the other two disciplines.

In this paper we discuss the joint distribution of the random variables \( \eta(t) \), \( \overline{\eta}(t,x) \), \( \underline{\eta}(t,x) \) and their limiting joint distribution as \( t \to \infty \).

We can "visualize" the definition of these three random variables on a common probability space as follows. Imagine that a customer joining the queue at time \( t \) consists of three identical parts 1, 2, 3 each requiring a processing time \( x \geq 0 \). Part 1 waits in front of a server operating under the FCFS rule, part 2 in front of a server operating under the LPT rule and finally part 3 waits in front of a unit governed by the SPT rule. Then \( \eta(t) \), \( \overline{\eta}(t,x) \) and \( \underline{\eta}(t,x) \) are the waiting times of parts 1, 2 and 3 respectively.

2. An Auxiliary Calculation

Consider the time points \( t \) and \( t + t', t' > 0 \). The probability that during the interval \((0, t)\), \( j \) customers arrive whose service times are less than
\( x, j_2 \) whose service times are greater than \( x \) and that during \((t, t + t')\), \( j_3 \) and \( j_4 \) arrive with service times respectively less and greater than \( x \) is given by:

\[
\begin{align*}
\text{e}^{-\lambda t - \lambda t'} & \frac{[\lambda t H(x)]^{j_1}}{j_1!} \frac{[\lambda t' H(x)]^{j_3}}{j_3!} \\
\cdot & \frac{\lambda t [1 - H(x)]^{j_2}}{j_2!} \frac{\lambda t' [1 - H(x)]^{j_4}}{j_4!}
\end{align*}
\]

We assume that \( x \) is a point of continuity of \( H(\cdot) \) so that the probability that one or more customers have service times exactly equal to \( x \) is zero.

The distributions \( \tilde{H}(\cdot) \) and \( \overline{H}(\cdot) \) are defined by:

\[
\begin{align*}
\tilde{H}(y) &= \frac{H(y)}{H(x)}, & 0 &< y \leq x, \\
&= 1, & y &\geq x, \\
\overline{H}(y) &= 0, & y &\leq x, \\
&= \frac{H(y) - H(x)}{1 - H(x)}, & y &> x,
\end{align*}
\]

\( \tilde{H}(\cdot) \) and \( \overline{H}(\cdot) \) are clearly the conditional service time distributions given the information that the required processing time of a customer is respectively less or greater than \( x \).

Next, let \( U_1^t \) and \( U_2^t \) be the total service time of all customers in \((0, t)\) with service time respectively less and greater than \( x \). Similarly \( U_3^t \) and \( U_4^t \) are the corresponding quantities for the customers arriving in \((t, t + t')\).

The following auxiliary probability mass function is of importance in the sequel. We define \( W(t, t'; x_1, x_2, x_3, x_4) \) as the probability that for given \( t > 0 \) and \( t' > 0 \), the random variables \( U_1^t, U_2^t, U_3^t, \) and \( U_4^t \) satisfy:

\[
U_1^t \leq x_1, \quad U_2^t \leq x_2, \quad U_3^t \leq x_3, \quad U_4^t \leq x_4,
\]
It follows readily, using (1), that:

\[
W(t, t'; x_1, x_2, x_3, x_4) = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} \sum_{j_4=0}^{\infty} e^{-\lambda t - \lambda t'} \frac{[\lambda t H(x)]^1}{j_1!} \frac{[\lambda t' H(x)]^3}{j_3!} \cdot \frac{[\lambda t' [1 - H(x)]]^2}{j_2!} \cdot \frac{[\lambda t [1 - H(x)]]^4}{j_4!} \cdot \tilde{H}(x_1)\tilde{H}(x_3),
\]

where \(\tilde{H}(x_1)\) is the \(j_1\)-th convolution power of \(\tilde{H}(\cdot)\) evaluated at \(x_1\). Similar interpretations are given to the other factors.

This expression does not simplify directly. As many formulae in applied probability it involves series in the convolution powers of distribution functions. Upon taking Laplace-Stieltjes transforms:

\[
\tilde{W}^*(t, t'; s_1, s_2, s_3, s_4) = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} \sum_{j_4=0}^{\infty} e^{-s_1 x_1 - s_2 x_2 - s_3 x_3 - s_4 x_4} W(t, t'; x_1, x_2, x_3, x_4),
\]

we obtain a more familiar series.

\[
\tilde{W}^*(t, t'; s_1, s_2, s_3, s_4) = \sum_{j_1=0}^{\infty} \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} \sum_{j_4=0}^{\infty} e^{-\lambda t - \lambda t'} \frac{[\lambda t H(x)]^1}{j_1!} \frac{[\lambda t' H(x)]^3}{j_3!} \cdot \frac{[\lambda t' [1 - H(x)]]^2}{j_2!} \cdot \frac{[\lambda t [1 - H(x)]]^4}{j_4!}.
\]
\[
\{ \lambda t [ 1 - H (x) ] \}^{j_2} \quad \{ \lambda t' [ 1 - H (x) ] \}^{j_4} \quad \tilde{h} (s_1) \quad \tilde{h} (s_3) \quad \tilde{h} (s_2) \\
\sim^{j_3} \quad \sim^{j_4} \quad \sim^{j_2} \quad \sim^{j_1} \quad \sim^{j_2} \quad \sim^{j_4} \quad \sim^{j_1} \quad \sim^{j_3} \\
\tilde{h} (s_4),
\]

where \( \tilde{h} (s) \) and \( \tilde{h} (s) \) are the L. S. transforms of \( \tilde{H} (\cdot) \) and \( \tilde{H} (\cdot) \).

Summing we obtain:

\[(7) \quad \tilde{W}^* (t, t'; s_1, s_2, s_3, s_4) =
\]

\[
\exp \left\{ - \lambda t - \lambda t' + \lambda t H (x) \tilde{h} (s_1) + \lambda t [ 1 - H (x) ] \tilde{h} (s_2) 
\right. \\
+ \lambda t' H (x) \tilde{h} (s_3) + \lambda t' [ 1 - H (x) ] \tilde{h} (s_4) \right\},
\]

We now return to the \( M|G|1 \) queue, which we consider at time \( t \). We define the following five random variables. \( U_0 \) is the length of time beyond \( t \) until the generation of customers in service at time \( t \) completes its service. \( U_1 \) and \( U_2 \) are respectively the total service times of the customers with processing times less than and greater than \( x \), who joined the queue between the beginning of the service of the current generation but before \( t \). \( U_3 \) and \( U_4 \) are respectively the total service times of the customers with processing times less than and greater than \( x \), who join the queue during the time interval \( (t, t + U_0) \).

If at time \( t \) the server is idle all five variables are zero.

We express the joint distribution of the waiting times \( \bar{\eta} (t, x) \), \( \eta (t) \) and \( \tilde{\eta} (t, x) \) in terms of the joint distribution of the random variables \( U_j \), \( j = 0, \ldots, 4 \).
3. The Joint Distribution of the $U_j$, $j = 0, \ldots, 4$.

$R_1^0(t, x_0, x_1, x_2, x_3, x_4)$ is the probability that in $(0, t)$ the queue has never become empty and the variables $U_j$, $j = 0, \ldots, 4$ associated with the timepoint $t$ satisfy $U_j \leq x_j$, $j = 0, \ldots, 4$, given that at $t = 0$ at least 1 customers were in the queue, one beginning service at that time.

This probability is given by:

$$R_1^0(t, x_0, x_1, x_2, x_3, x_4) =$$

$$\sum_{n=0}^{\infty} \sum_{\nu=1}^{\infty} \int_0^x \int_0^t \int_0^\nu dQ_{iv}^{(n)}(\tau) dH_{iv}^{(v)}(t' - \tau)$$

$$W(t' - \tau, t'; x_1, x_2, x_3, x_4),$$

by the following argument. At some time $\tau$ prior to $t$, the generation in service at $t$ enters service. There are some number $\nu \geq 1$ customers in it, so that the duration of the total service time distribution of these $\nu$ customers is the $\nu$-fold convolution $H^{(\nu)}(\cdot)$ of $H(\cdot)$. If $U_0 \leq x_0$ must hold, the total service time of these $\nu$ customers cannot exceed $t + x_0$. The other requirements $U_1 \leq x_1, U_2 \leq x_2, U_3 \leq x_3, U_4 \leq x_4$ account for the factor $W(t - \tau, t' ; x_1, x_2, x_3, x_4)$.

The probabilities $Q_{iv}^{(n)}(\cdot)$ were defined in [2].

Taking Laplace-Stieltjes transforms in (8), i.e. evaluating:

$$\tilde{R}_1^0(\xi, s_0, s_1, s_2, s_3, s_4) =$$

$$\int_0^{\infty} e^{-\xi t} dt \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} \int_0^{\infty} e^{-s_0 x_0 - s_1 x_1 - s_2 x_2 - s_3 x_3 - s_4 x_4}$$

$$d_{x_0, x_1, x_2, x_3, x_4} R_1^0(t, x_0, x_1, x_2, x_3, x_4),$$

and recalling (7) we obtain:
(10) \( R_i^o (\xi, s_o, s_1, s_2, s_3, s_4) = \)

\[
\sum_{n=0}^{\infty} \sum_{\nu=1}^{\infty} \int_{0}^{\infty} e^{-\xi t} dt \int_{0}^{\infty} e^{-s_0 x_o} \int_{0}^{\infty} e^{-s_1 x_1} \int_{0}^{\infty} e^{-s_2 x_2} \int_{0}^{\infty} e^{-s_3 x_3} \int_{0}^{\infty} e^{-s_4 x_4} d\xi d\nu
\]

\[
\int_{0}^{t} \int_{0}^{(\nu)} \int_{0}^{(\nu)} d\xi d\nu \int_{0}^{t - x_o - \tau} \int_{0}^{(\nu)} d\tau \int_{0}^{(\nu)} d\tau \int_{0}^{(\nu)} d\tau \int_{0}^{(\nu)} d\tau \int_{0}^{(\nu)} d\tau
\]

\[
= \sum_{n=0}^{\infty} \sum_{\nu=1}^{\infty} a_i^{(n)} (\xi) \int_{0}^{\infty} \int_{0}^{\infty} \exp \left\{ -\xi t_1 - \lambda t_1 - \lambda x_o - s_o x_o + \lambda x_o \left[ 1 - H(x) \right] \h (s_2) + \lambda x_o H(x) \h (s_3) + \lambda x_o \left[ 1 - H(x) \right] \h (s_4) \right\}
\]

\[
+ \lambda t_1 H(x) \h (s_1) + \lambda t_1 \left[ 1 - H(x) \right] \h (s_2) + \lambda x_o H(x) \h (s_3)
\]

\[
+ \lambda x_o \left[ 1 - H(x) \right] \h (s_4) \int_{0}^{(\nu)} d\xi d\nu \int_{0}^{(\nu)} d\xi d\nu \int_{0}^{(\nu)} d\xi d\nu \int_{0}^{(\nu)} d\xi d\nu \int_{0}^{(\nu)} d\xi d\nu
\]

\[
= \sum_{n=0}^{\infty} \sum_{\nu=1}^{\infty} a_i^{(n)} (\xi) \left\{ h \left[ \lambda + s_o - \lambda H(x) \h (s_3) - \lambda \left[ 1 - H(x) \right] \h (s_4) \right] \h (s_1) \right\}
\]

\[
- h \left[ \xi + \lambda - \lambda H(x) \h (s_1) - \lambda \left[ 1 - H(x) \right] \h (s_2) \right] \left\{ \xi - s_o - \lambda H(x) \left[ \h (s_1) - \h (s_3) \right] - \lambda \left[ 1 - H(x) \right] \left[ \h (s_2) - \h (s_4) \right] \right\}^{-1}
\]

\[
= \left\{ \xi - s_o - \lambda H(x) \left[ \h (s_1) - \h (s_3) \right] - \lambda \left[ 1 - H(x) \right] \left[ \h (s_2) - \h (s_4) \right] \right\}^{-1}
\]

\[
. \sum_{n=0}^{\infty} \left\{ a_i^{(n)} \left\{ h \left[ \lambda + s_o - \lambda H(x) \h (s_3) - \lambda \left[ 1 - H(x) \right] \h (s_4) \right] \h (s_1) \right\} \right\}
\]

\[
- a_i^{(n)} \left\{ h \left[ \xi + \lambda - \lambda H(x) \h (s_1) - \lambda \left[ 1 - H(x) \right] \h (s_2) \right] \right\}
\]

in terms of the functions \( a_i^{(n)} (\xi, z) \) defined in [3].
Next, let \( R_i (t, x_o, x_1, x_2, x_3, x_4) \) be the probability that at time \( t \), the random variables \( U_j \) associated with \( t \) satisfy \( U_j \leq x_j, j = 0, 1, 2, 3, 4 \) given that at time \( t = 0 \) there were \( i \) customers in the queue.

The standard regeneration argument relates the function \( R_i (t, x_o, x_1, x_2, x_3, x_4) \) to the functions \( R^0_i (t, x_o, x_1, x_2, x_3, x_4) \) as follows:

\[
(11) \quad R_i (t, x_o, x_1, x_2, x_3, x_4) = \]

\[
R^0_i (t, x_o, x_1, x_2, x_3, x_4) + \mathbb{P} \left\{ \xi (t) = 0 \mid \xi (0) = i \right\} U (x_o, x_1, x_2, x_3, x_4)
+ \int_0^t R^0_i (t-u, x_o, x_1, x_2, x_3, x_4) \, dM_1 (u),
\]

where \( \mathbb{P} \{ \xi (t) = 0 \mid \xi (0) = i \} \) is the (conditional) probability that the queue length \( \xi (t) = 0 \), i.e. that the server is idle at time \( t \). \( M_1 (\cdot) \) is the renewal function of the (general) renewal process of beginnings of busy periods. \( U (x_o, x_1, x_2, x_3, x_4) \) is the distribution degenerate at zero in five variables.

We recall that:

\[
(12) \quad m_1 (\xi) = \int_0^\infty e^{-\xi t} \, dM_1 (t) = \frac{1}{\lambda} \gamma (\xi). \left[ \frac{\xi + \lambda - \lambda \gamma (\xi)}{1 - \lambda} \right],
\]

where \( \gamma (\xi) \) is the L.S. transform of the distribution of the busy period of the \( M|G|1 \) queue.

Also:

\[
(13) \quad \int_0^\infty e^{-\xi t} \mathbb{P} \{ \xi (t) = 0 \mid \xi (0) = i \} \, dt = \frac{1}{\lambda} m_1 (\xi),
\]

Upon taking transforms in (11) we obtain:

\[
(14) \quad \tilde{R}_i (\xi, s_o, s_1, s_2, s_3, s_4) = \]

\[
\int_0^\infty e^{-\xi t} \mathbb{E}_t \left\{ e^{-s_o U_0 - s_1 U_1 - s_2 U_2 - s_3 U_3 - s_4 U_4} \right\} \, dt
\]
\[ \sim \rho_1 (s, s_0, s_1, s_2, s_3, s_4) \]

\[ + \gamma (\xi) \left\{ 1 + \lambda \rho_1 (s, s_0, s_1, s_2, s_3, s_4) \right\} \]

\[ \sim \rho_1 (s, s_0, s_1, s_2, s_3, s_4), i \geq 1, \text{ is given in (10).} \]

When \( 1 - \lambda \alpha > 0 \), \( \alpha \) being the mean of \( H (\cdot) \), the existence of a joint limiting distribution for \( U_j, j = 0, 1, 2, 3, 4 \) is guaranteed by the main limit theorem for regenerative processes and the existence of a stationary version of the imbedded Markov renewal process of the \( M|G|1 \) queue [3]. When \( 1 - \lambda \alpha \leq 0 \), the main limit theorem for regenerative processes guarantees that \( R_i (t, s_0, s_1, s_2, x_j, x_k) \) tends to zero for all \( i, x_j \geq 0, j = 0, \ldots, 4 \). Since the limiting distribution exists, when \( 1 - \lambda \alpha > 0 \), its transform is given by:

\[ \tilde{R} (s_0, s_1, s_2, s_3, s_4) = \lim_{\xi \to 0^+} \xi \tilde{\rho}_1 (\xi, s_0, s_1, s_2, s_3, s_4) \]

\[ = (1 - \lambda \alpha) \left\{ 1 + \lambda \tilde{\rho}_1 (0^+, s_0, s_1, s_2, s_3, s_4) \right\} . \]

4. The Joint Distribution of \( \bar{\eta} (t,x), \eta (t) \) and \( \bar{\eta} (t,x) \)

The random variables \( \bar{\eta} (t,x), \eta (t) \) and \( \bar{\eta} (t,x) \) are for each \( t > 0 \) related to the random variables \( U_j, j = 0, 1, 2, 3, 4 \) associated with the timeinstant \( t \) by:

\[ \bar{\eta} (t,x) = U_o + U_2 + U_4, \]

\[ \eta (t) = U_o + U_1 + U_2, \]

\[ \bar{\eta} (t,x) = U_o + U_1 + U_3 \]

That this is indeed so, we argue for \( \bar{\eta} (t,x) \). The other cases are similar. Consider a virtual customer with service time \( x \) arriving at time \( t \). He has to
wait until all customers of the present generation, if any, have been served. This is a length of time $U_0$. Next, in the next generation, all customers with service time greater than $x$ are served ahead of him. Regardless of the actual order of service the total amount of processing time required by all customers with service time exceeding $x$ is $U_2 + U_4$. $U_2$ is the processing time of those who preceded him and $U_4$ that of those who succeeded him in the arrival sequence. We have:

\[(17) \quad \bar{\eta}(t,x) = \eta(t) + \int e^{-\int_0^t e^{-\eta(t,x)} dt} E \left[ e^{-(\eta(t) + \eta(t,x)) t} \right] dt \]

which implies that:

\[(18) \quad \bar{S}_1(\xi, \zeta_1, \zeta_2, \zeta_3) = \int_0^\infty e^{-\xi t} E \left[ e^{-(\eta(t) + \eta(t,x)) t} \right] dt \]

\[= R_1(\xi, \zeta_1 + \zeta_2 + \zeta_3, \zeta_2 + \zeta_3, \zeta_1 + \zeta_2, \zeta_3, \zeta_1) \]

where $R_1(..., ..., ..., ...)$ is given by (14).

Formula (18) shows how the joint distribution of $\eta(t,x)$, $\eta(t)$ and $\eta(t,x)$ is related to the basic parameters of the $M|G|1$ queue. We discuss the limiting joint distribution of these three variables as $t \to \infty$, in some detail.

5. The Limiting Joint Distribution

The limiting joint distribution of the three virtual waiting times exists if and only if $1 - \lambda \alpha > 0$. Its Laplace-Stieltjes transform is given by:

\[(19) \quad \tilde{S}(\zeta_1, \zeta_2, \zeta_3) = \]

\[= (1 - \lambda \alpha) \left\{ 1 + \lambda R_1(0, \zeta_1 + \zeta_2 + \zeta_3, \zeta_2 + \zeta_3, \zeta_1 + \zeta_2, \zeta_3, \zeta_1) \right\} \]
\[ = (1 - \lambda \alpha) \left\{ 1 - \frac{\lambda}{\zeta_1 + \zeta_2 + \zeta_3 + \lambda \mathbb{H}(x) \left[ \mathbb{H}(\zeta_2 + \zeta_3) - \mathbb{H}(\zeta_3) \right] + \lambda \left[ 1 - \mathbb{H}(x) \right]} \right\} \]

\[ \times \sum_{n=0}^{\infty} \left\{ o(q_1) \left\{ o, h \left[ \lambda + \zeta_1 + \zeta_2 + \zeta_3 - \lambda \mathbb{H}(x) \mathbb{H}(\zeta_3) - \lambda \left[ 1 - \mathbb{H}(x) \right] \mathbb{H}(\zeta_1) \right] \right\} \right\} \cdot o(q_1) \left\{ o, h \left[ \lambda - \lambda \mathbb{H}(x) \mathbb{H}(\zeta_2 + \zeta_3) - \lambda \left[ 1 - \mathbb{H}(x) \right] \mathbb{H}(\zeta_1 + \zeta_2) \right] \right\} \}

6. Moments of the Limiting Distribution of the Basic Variables \( U_j, j=0,1,2,3,4. \)

Let us denote:

\[ s = (s_0, s_1, s_2, s_3, s_4) \]

\[ \tilde{Q} = (0, 0, 0, 0, 0) \]

\[ \tilde{\theta} = \tilde{\theta} (x, \tilde{s}) \]

(20)

\[ = s_0 + \lambda \mathbb{H}(x) \left[ \mathbb{H}(s_1) - \mathbb{H}(s_3) \right] + \lambda \left[ 1 - \mathbb{H}(x) \right] \left[ \mathbb{H}(s_2) - \mathbb{H}(s_4) \right] \]

\[ \mathbb{Y} = \mathbb{Y} (x, \tilde{s}) \]

(21)

\[ = h \left[ \lambda + s_0 - \lambda \mathbb{H}(x) \mathbb{H}(s_3) - \lambda (1 - \mathbb{H}(x)) \mathbb{H}(s_4) \right] \]

\[ \tilde{\mathbb{Y}} = \tilde{\mathbb{Y}} (x, \tilde{s}) \]

(22)

\[ = h \left[ \lambda - \lambda \mathbb{H}(x) \mathbb{H}(s_1) - \lambda (1 - \mathbb{H}(x)) \mathbb{H}(s_2) \right] \]

\[ \psi_n = \psi_n (x, \tilde{s}) \]

(23)

\[ = h_n (0, \tilde{y}) - h_n (0, \tilde{y}), n \geq 0, \]

where the functions \( h_n (\cdot, \cdot), n \geq 0 \) are defined in [2].
From (15) the L.S. transform of the limiting joint distribution of
$U_j$, $j = 0, 1, 2, 3, 4$ is given by:

$$R(s) = (1 - 2\alpha) \left\{ 1 - \frac{\lambda}{\theta} \sum_{n=0}^{\infty} \left[ q_{\perp}^{(n)}(0, y) - q_{\perp}^{(n)}(0, \tilde{y}) \right] \right\},$$

(24)

The theorem in Appendix I of [2] implies:

$$q_{\perp}^{(n)}(0, y) - q_{\perp}^{(n)}(0, \tilde{y}) = h_n(0, y) - h_n(0, \tilde{y}) = \tilde{\psi}_n(x, s)$$

(25)

Substitution of (25) in (24) yields:

$$R(s) = (1 - 2\alpha) \left\{ 1 - \frac{\lambda}{\theta} \sum_{n=0}^{\infty} \tilde{\psi}_n \right\},$$

(26)

$E_{\infty} U$ denotes the expected value of the limiting distribution of $U$ and

further denote:

$$\alpha = \int_{0}^{\infty} u \, dH(u) \quad \alpha_x = \int_{0}^{x} u \, dH(u)$$

$$\beta = \int_{0}^{\infty} u^2 \, dH(u) \quad \beta_x = \int_{0}^{x} u^2 \, dH(u)$$

$$\gamma = \int_{0}^{\infty} u^3 \, dH(u) \quad \gamma_x = \int_{0}^{x} u^3 \, dH(u)$$

$$\alpha_x^* = \alpha - \alpha_x \quad \beta_x^* = \beta - \beta_x \quad \gamma_x^* = \gamma - \gamma_x$$

Then from (26) we have:

$$E_{\infty} U_j = - \left. \frac{\partial R}{\partial \tilde{s}_j} \right|_{\tilde{s} = 0}$$
\[(27) \quad E_{\infty} U_j = \lambda (1-\lambda \alpha) \sum_{n=0}^{\infty} \left[ \frac{\theta^2 \frac{\partial \psi_n}{\partial s_j} - \frac{\partial \theta}{\partial s_j} \frac{\partial \psi_n}{\partial s_j}}{\theta^2} \right] \quad \sim = 0\]

where term by term differentiation is justified as in [2]. Applying l'Hospital's rule twice on the right hand side of (27) we get:

\[(28) \quad E_{\infty} U_j = \frac{\lambda (1-\lambda \alpha)}{2} \sum_{n=0}^{\infty} \left[ \frac{\partial \theta}{\partial s_j} \frac{\partial^2 \psi_n}{\partial s_j^2} - \frac{\partial^2 \theta}{\partial s_j^2} \frac{\partial \psi_n}{\partial s_j} \right] \quad \sim = 0 , \quad j = 0, 1, 2, 3, 4 .

Similarly:

\[(29) \quad E_{\infty} U_j^2 = \frac{\partial^2 R}{\partial s_j^2} \quad \sim = 0 \]

and

\[(30) \quad E_{\infty} (U_i U_j) = \frac{\partial^2 R}{\partial s_i \partial s_j} \quad \sim = 0 \]
\[ \begin{align*} &+ 5 \frac{\partial^3 \theta}{\partial s_i^3} \frac{\partial \psi}{\partial s_j} \frac{\partial \psi}{\partial s_i} - 3 \frac{\partial \theta}{\partial s_i} \cdot \frac{\partial^2 \theta}{\partial s_i^2} \cdot \frac{\partial^2 \psi}{\partial s_i \partial s_j} \\ &+ 3 \frac{\partial \theta}{\partial s_j} \cdot \frac{\partial^2 \theta}{\partial s_i^2} \cdot \frac{\partial^2 \psi}{\partial s_i} + 3 \left( \frac{\partial \theta}{\partial s_i} \right)^2 \frac{\partial^3 \psi}{\partial s_i \partial s_j^2} \right|_{s=0} \end{align*} \]

To compute (28), (29) and (30) we need the following:

\[ \frac{\partial \theta}{\partial s_0} = 1 \quad , \quad \frac{\partial^2 \theta}{\partial s_0^2} = 0 \quad , \quad \frac{\partial^3 \theta}{\partial s_0^3} = 0 \]

\[ \frac{\partial \psi}{\partial s_1} \bigg|_{s_1=0} = - \lambda \alpha_x \quad , \quad \frac{\partial^2 \theta}{\partial s_1^2} \bigg|_{s_1=0} = \lambda \beta_x \]

\[ \frac{\partial^3 \theta}{\partial s_1^3} \bigg|_{s_1=0} = - \lambda \gamma_x \quad , \quad \frac{\partial \psi}{\partial s_2} \bigg|_{s_2=0} = - \lambda \alpha_x^* \]

\[ \frac{\partial^2 \theta}{\partial s_2^2} \bigg|_{s_2=0} = \lambda \beta_x^* \quad , \quad \frac{\partial^3 \theta}{\partial s_2^3} \bigg|_{s_2=0} = - \lambda \gamma_x^* \]

\[ \frac{\partial \theta}{\partial s_3} \bigg|_{s_3=0} = - \frac{\partial \theta}{\partial s_1} \bigg|_{s_1=0} \quad , \quad \frac{\partial \theta}{\partial s_4} \bigg|_{s_1=0} = \frac{\partial \theta}{\partial s_2} \bigg|_{s_2=0} \]

\[ r = 1, 2, 3 . \]
\[(31) \quad \frac{\partial \psi_n}{\partial s_j} \bigg|_{s = 0} = h_n'(0,1) \left[ \frac{\partial \psi}{\partial s_j} - \frac{\partial \tilde{\psi}}{\partial s_j} \right] \bigg|_{s = 0} = 0, \quad j = 0, 1, 2, 3, 4. \]

\[\frac{\partial \psi_n}{\partial s_0} \bigg|_{s = 0} = -\alpha h_n'(0,1) \]

\[\frac{\partial \psi_n}{\partial s_1} \bigg|_{s = 0} = \lambda \alpha \nu h_n'(0,1) \]

\[\frac{\partial \psi_n}{\partial s_2} \bigg|_{s = 0} = \lambda \alpha \nu h_n'(0,1) \]

\[\frac{\partial \psi_n}{\partial s_3} \bigg|_{s = 0} = -\frac{\partial \psi_n}{\partial s_1} \bigg|_{s = 0}, \quad \frac{\partial \psi_n}{\partial s_4} \bigg|_{s = 0} = -\frac{\partial \psi_n}{\partial s_2} \bigg|_{s = 0} \]

\[(32) \quad \frac{\partial^2 \psi_n}{\partial s_j^2} \bigg|_{s = 0} = h_n'(0,1) \left[ \frac{\partial^2 \psi}{\partial s_j^2} - \frac{\partial^2 \tilde{\psi}}{\partial s_j^2} \right] \bigg|_{s = 0} = 0,\]

\[+ h_n''(0,1) \left[ \left( \frac{\partial \psi}{\partial s_j} \right)^2 - \left( \frac{\partial \tilde{\psi}}{\partial s_j} \right)^2 \right] \bigg|_{s = 0}, \quad j = 0, 1, 2, 3, 4 \]

\[\frac{\partial^2 \psi_n}{\partial s_0^2} \bigg|_{s = 0} = \beta h_n'(0,1) + \alpha^2 h_n''(0,1) \]
\[
\frac{\partial^2 \psi_n}{\partial s_1^2} = - \left[ \lambda \alpha_x^2 + \lambda \alpha_x^* \right] h_n'(0,1) - \lambda^2 \alpha_x^2 h_n''(0,1)
\]

\[
\frac{\partial^2 \psi_n}{\partial s_2^2} = - \left[ \lambda \alpha_x^2 \beta + \lambda \alpha_x^* \right] h_n'(0,1) - \lambda^2 \alpha_x^2 h_n''(0,1)
\]

\[
\frac{\partial^2 \psi_n}{\partial s_3^2} = - \frac{\partial^2 \psi_n}{\partial s_4^2} \Bigg|_{s=0} \frac{\partial^2 \psi_n}{\partial s_2^2} \Bigg|_{s=0} = - \frac{\partial^2 \psi_n}{\partial s_2^2} \Bigg|_{s=0}
\]

\[
\frac{\partial^2 \psi_n}{\partial s_1 \partial s_j} \Bigg|_{s=0} = h_n'(0,1) \left[ \frac{\partial^2 Y}{\partial s_i \partial s_j} - \frac{\partial^2 Y}{\partial s_i \partial s_j} \right] \Bigg|_{s=0}
\]

\[
+ h_n''(0,1) \left[ \frac{\partial^2 Y}{\partial s_i \partial s_j} - \frac{\partial^2 Y}{\partial s_i \partial s_j} \right] \Bigg|_{s=0}
\]

\[
i \neq j = 0, 1, 2, 3, 4
\]

\[
\frac{\partial^2 \psi_n}{\partial s_1 \partial s_2} \Bigg|_{s=0} = - \lambda^2 \beta \alpha_x^2 \alpha_x^* h_n'(0,1) - \lambda^2 \alpha_x^2 \alpha_x^* h_n''(0,1)
\]

\[
\frac{\partial^2 \psi_n}{\partial s_1 \partial s_3} \Bigg|_{s=0} = 0
\]

\[
\frac{\partial^2 \psi_n}{\partial s_1 \partial s_4} \Bigg|_{s=0} = 0
\]

\[
\frac{\partial^2 \psi_n}{\partial s_2 \partial s_3} \Bigg|_{s=0} = 0
\]

\[
\frac{\partial^2 \psi_n}{\partial s_2 \partial s_4} \Bigg|_{s=0} = 0
\]

\[
\frac{\partial^2 \psi_n}{\partial s_3 \partial s_4} \Bigg|_{s=0} = - \frac{\partial^2 \psi_n}{\partial s_1 \partial s_2} \Bigg|_{s=0}
\]
\[
\frac{\partial^3 \psi_n}{\partial s_i^2 \partial s_j} \bigg|_{s=0} = \frac{\partial^3 \psi}{\partial s_i^2 \partial s_j} \bigg|_{s=0} = 0
\]

\[
+ h_n^{(0,1)} \left[ 2 \frac{\partial^2 \psi}{\partial s_i \partial s_j} - \frac{\partial \psi}{\partial s_i} \frac{\partial \psi}{\partial s_j} \right] \bigg|_{s=0} = 0
\]

\[
+ h_n^{(0,1)} \left[ \left( \frac{\partial \psi}{\partial s_i} \right)^2 - \left( \frac{\partial \psi}{\partial s_j} \right)^2 \right] \bigg|_{s=0} = 0
\]

\[
i \neq j = 0, 1, 2, 3, 4.
\]

\[
\frac{\partial^3 \psi_n}{\partial s_0^2 \partial s_1} \bigg|_{s=0} = 0 , \quad \frac{\partial^3 \psi_n}{\partial s_0^2 \partial s_2} \bigg|_{s=0} = 0
\]

\[
\frac{\partial^3 \psi_n}{\partial s_0^2 \partial s_3} \bigg|_{s=0} = 0
\]

\[
= - \lambda \alpha_x \gamma h_n^{(0,1)} - 3 \lambda \alpha_x \beta h_n^{''(0,1)} - \lambda \alpha_x^3 h_n^{''''(0,1)}
\]

\[
\frac{\partial^3 \psi_n}{\partial s_0^2 \partial s_4} \bigg|_{s=0} = 0
\]

\[
\frac{\partial^3 \psi_n}{\partial s_1^2 \partial s_2} \bigg|_{s=0} = 0
\]

\[
= \lambda^2 \alpha_x^* (\lambda \alpha_x^2 + \beta \alpha_x) h_n^{(0,1)} + \lambda^2 \alpha_x^* (3 \lambda \alpha_x^2 \beta + \alpha \beta \alpha_x) h_n^{''(0,1)}
\]

\[
+ \lambda^3 \alpha_x^3 \alpha_x^2 h_n^{''''(0,1)}
\]

\[
\frac{\partial^3 \psi_n}{\partial s_1^2 \partial s_3} \bigg|_{s=0} = 0 , \quad \frac{\partial^3 \psi_n}{\partial s_1^2 \partial s_4} \bigg|_{s=0} = 0
\]
\[
\frac{\partial^3 \psi_n}{\partial s_2^2 \partial s_3} \bigg|_{s=0} = 0, \quad \frac{\partial^3 \psi_n}{\partial s_2^2 \partial s_4} \bigg|_{s=0} = 0
\]

\[
\frac{\partial^3 \psi_n}{\partial s_3^2 \partial s_4} \bigg|_{s=0} = \frac{\partial^3 \psi_n}{\partial s_1^2 \partial s_2} \bigg|_{s=0} = 0
\]

(34) \[
\frac{\partial^3 \psi_n}{\partial s_j^3} \bigg|_{s=0} = h_n^{(0,1)} \left[ \frac{\partial^3 \gamma}{\partial s_j^3} - \frac{\partial^3 \gamma}{\partial s_j^2} \right]_{s=0} + 3 h_n^{(0,1)} \left[ \frac{\partial \gamma}{\partial s_j} \frac{\partial^2 \gamma}{\partial s_j^2} - \frac{\partial \gamma}{\partial s_j} \frac{\partial^2 \gamma}{\partial s_j} \right]_{s=0} + h_n^{(0,1)} \left[ \frac{\partial \gamma}{\partial s_j^3} - \frac{\partial \gamma}{\partial s_j^3} ight]_{s=0}, \quad j = 0, 1, 2, 3, 4.
\]

\[
\frac{\partial^3 \psi_n}{\partial s_0^3} \bigg|_{s=0} = -\gamma h_n^{(0,1)} - 3 \alpha \beta h_n^{(0,1)} - \alpha^3 h_n^{(0,1)}
\]

\[
\frac{\partial^3 \psi_n}{\partial s_1^3} \bigg|_{s=0} = h_n^{(0,1)} \left[ \lambda^3 \alpha_x^3 \gamma + 3 \lambda^2 \alpha_x \beta x + \lambda \alpha \gamma x \right]_{s=0} + 3 h_n^{(0,1)} \left[ \lambda^2 \alpha \beta_x + \lambda \beta_x \right]_{s=0} + h_n^{(0,1)} \left( \lambda \alpha_x \lambda \alpha_x \right)^3
\]

\[
\frac{\partial^3 \psi_n}{\partial s_2^3} \bigg|_{s=0} = h_n^{(0,1)} \left[ \lambda^3 \alpha_x^3 \gamma + 3 \lambda \alpha_x \beta x^* + \lambda \alpha \gamma x^* \right]_{s=0}
\]
\[ + 3 h''_{n}(0,1) \lambda \alpha_{x}^{*} \left[ \lambda^{2} \alpha_{x}^{*2} \beta + \lambda \alpha \beta_{x}^{*} \right] \]

\[ + h'''_{n}(0,1) (\lambda \alpha \alpha_{x}^{*})^{3} \]

\[ \frac{\partial^{3} \psi_{n}}{\partial s_{3}^{3}} \bigg|_{s = 0} = - \frac{\partial^{3} \psi_{n}}{\partial s_{1}^{3}} \bigg|_{s = 0} = 0 \]

\[ \frac{\partial^{3} \psi_{n}}{\partial s_{4}^{3}} \bigg|_{s = 0} = 0 \]

\[ \frac{\partial^{3} \psi_{n}}{\partial s_{2}^{3}} \bigg|_{s = 0} = 2 \]

And from reference [2] we have:

\[ \sum_{n=0}^{\infty} h'_{n}(0,1) = \frac{1}{1 - \lambda \alpha} \]

\[ \sum_{n=0}^{\infty} h''_{n}(0,1) = \beta \lambda^{2} \left(1 - \lambda \alpha\right) \left(1 - \lambda^{2} \alpha^{2}\right) \]

\[ \sum_{n=0}^{\infty} h'''_{n}(0,1) = \frac{1}{1 - \lambda \alpha} \left[ \frac{\lambda \alpha^{3}}{1 - \lambda^{3} \alpha^{3}} + \frac{3 \lambda^{5} \alpha \beta^{2}}{(1 - \lambda^{2} \alpha^{2})(1 - \lambda^{3} \alpha^{3})} \right] \]

Substituting the above calculations in (28), (29) and (30) and simplifying, we get

\[ E_{\infty} U_{0} = \frac{\lambda (1 - \lambda \alpha)}{2} \sum_{n=0}^{\infty} \left[ \beta h'_{n}(0,1) + \alpha^{2} h''_{n}(0,1) \right] \]

\[ = \frac{\lambda \beta}{2 (1 - \lambda^{2} \alpha^{2})} \]

\[ E_{\infty} U_{1} = \frac{\lambda^{2} (1 - \lambda \alpha) \alpha_{x}}{2} \sum_{n=0}^{\infty} \left[ \beta h'_{n}(0,1) + \alpha^{2} h''_{n}(0,1) \right] \]

\[ = \frac{\lambda^{2} \alpha_{x} \beta}{2 (1 - \lambda^{2} \alpha^{2})} \]
\[ E_\infty U_2 = \frac{\lambda^2 (1-\lambda \alpha) \alpha_x^*}{2} \sum_{n=0}^{\infty} \left[ b \, h_n'(0,1) + \alpha^2 \, h_n''(0,1) \right] \]

By Symmetry it follows that,

\[ E_\infty U_3 = \frac{\lambda^2 \alpha_x^* \beta}{2 \left(1 - \lambda^2 \alpha^2\right)} \]

and

\[ E_\infty U_4 = \frac{\lambda^2 \alpha_x^* \beta}{2 \left(1 - \lambda^2 \alpha^2\right)} \]

Denote:

\[ U = \begin{pmatrix} U_0 \\ U_1 \\ U_2 \\ U_3 \\ U_4 \end{pmatrix} \]

Then:

\[ E U = \begin{pmatrix} a \\ \lambda \alpha_x^* a \\ \lambda \alpha_x^2 a \\ \lambda \alpha_x a \\ \lambda \alpha_x^* a \end{pmatrix} \]

where \( a = \frac{\lambda \beta}{2 \left(1 - \lambda^2 \alpha^2\right)} \)

\[ E U_0 = \frac{\lambda \left(1-\lambda \alpha\right)}{3} \sum_{n=0}^{\infty} \left[ \gamma \, h_n'(0,1) + 3 \alpha \beta \, h_n''(0,1) + \alpha^3 \, h_n'''(0,1) \right] \]

\[ = \frac{\lambda}{3 \left(1 - \lambda^3 \alpha^3\right)} \left[ \gamma + \frac{3 \lambda^2 \alpha \beta^2}{1 - \lambda^2 \alpha^2} \right] \]
\[ E_{\infty} U_1^2 = \frac{\lambda}{6} \sum_{n=0}^{\infty} \left[ (2 \lambda^2 \alpha_x^2 \gamma + 3 \lambda \beta \beta_x) \ h_n'(0, 1) + 3 \lambda \alpha \beta \beta_x \ h_n''(0, 1) + 2 \lambda \alpha^3 \beta_x^2 \ h_n'''(0, 1) \right] \]

\[ = \frac{\lambda^3 \alpha_x^2}{3 \left(1 - \lambda^3 \alpha^3\right)} \left( \gamma + \frac{3 \lambda^2 \alpha \beta^2}{1 - \lambda^2 \alpha^2} \right) + \frac{\lambda^2 \beta \beta_x}{2 \left(1 - \lambda^2 \alpha^2\right)} \]

\[ E_{\infty} U_2^2 = \frac{\lambda(1-\lambda\alpha)}{6} \sum_{n=0}^{\infty} \left[ (2 \lambda^2 \alpha_x^* \gamma + 3 \lambda \beta \beta_x^*) \ h_n'(0, 1) + 3 \lambda \alpha \beta \beta_x^* \ h_n''(0, 1) + 2 \lambda \alpha^3 \beta_x^* \ h_n'''(0, 1) \right] \]

\[ = \frac{\lambda^3 \alpha_x^*}{3 \left(1 - \lambda^3 \alpha^3\right)} \left( \gamma + \frac{3 \lambda^2 \alpha \beta^2}{1 - \lambda^2 \alpha^2} \right) + \frac{\lambda^2 \beta \beta_x^*}{2 \left(1 - \lambda^2 \alpha^2\right)} \]

\[ E_{\infty} U_3 = E_{\infty} U_1, \quad E_{\infty} U_4 = E_{\infty} U_2 \]

\[ E_{\infty}(U_0 U_1) = \frac{\lambda^2 (1-\lambda \alpha) \alpha_x}{6} \sum_{n=0}^{\infty} \left[ \gamma h_n'(0, 1) + 3 \alpha \beta h_n''(0, 1) + \alpha^3 h_n'''(0, 1) \right] \]

\[ = \frac{\lambda^2 \alpha_x}{6 \left(1 - \lambda^3 \alpha^3\right)} \left[ \gamma + \frac{3 \lambda^2 \alpha \beta^2}{1 - \lambda^2 \alpha^2} \right] \]

\[ E_{\infty}(U_0 U_2) = \frac{\lambda^2 (1-\lambda \alpha) \alpha_x^*}{6} \sum_{n=0}^{\infty} \left[ \gamma h_n'(0, 1) + 3 \alpha \beta h_n''(0, 1) + \alpha^3 h_n'''(0, 1) \right] \]

\[ = \frac{\lambda^2 \alpha_x^*}{6 \left(1 - \lambda^3 \alpha^3\right)} \left[ \gamma + \frac{3 \lambda^2 \alpha \beta^2}{1 - \lambda^2 \alpha^2} \right] \]
$$E_{\infty}(U_{0}U_{3}) = \frac{\lambda^2 (1-\lambda \alpha') \alpha_x}{3} \sum_{n=0}^{\infty} \left[ \gamma h_n'(0,1) + 3 \alpha \beta h_n''(0,1) + \alpha^3 h_n'''(0,1) \right]$$

$$= \frac{\lambda^2 \alpha_x}{3(1-\lambda^3 \alpha')^3} \left[ \gamma + \frac{3 \lambda^2 \alpha \beta^2}{1 - \lambda^2 \alpha^2} \right]$$

$$E_{\infty}(U_{0}U_{4}) = \frac{\lambda^2 (1-\lambda \alpha') \alpha_x^*}{3} \sum_{n=0}^{\infty} \left[ \gamma h_n'(0,1) + 3 \alpha \beta h_n''(0,1) + \alpha^3 h_n'''(0,1) \right]$$

$$= \frac{\lambda^2 \alpha_x^*}{3(1-\lambda^3 \alpha')^3} \left[ \gamma + \frac{3 \lambda^2 \alpha \beta^2}{1 - \lambda^2 \alpha^2} \right]$$

$$E_{\infty}(U_{1}U_{2}) = \frac{\lambda^3 (1-\lambda \alpha') \alpha_x \alpha_x^*}{3} \sum_{n=0}^{\infty} \left[ \gamma h_n'(0,1) + 3 \alpha \beta h_n''(0,1) + \alpha^3 h_n'''(0,1) \right]$$

$$= \frac{\lambda^3 \alpha_x \alpha_x^*}{3(1-\lambda^3 \alpha')^3} \left[ \gamma + \frac{3 \lambda^2 \alpha \beta^2}{1 - \lambda^2 \alpha^2} \right]$$

$$E_{\infty}(U_{1}U_{3}) = \frac{\lambda^3 (1-\lambda \alpha') \alpha^2}{6} \sum_{n=0}^{\infty} \left[ \gamma h_n'(0,1) + 3 \alpha \beta h_n''(0,1) \right]$$

$$= \frac{\lambda^3 \alpha^2}{6(1-\lambda^3 \alpha')^3} \left[ \gamma + \frac{3 \lambda^2 \alpha \beta^2}{1 - \lambda^2 \alpha^2} \right]$$

$$E_{\infty}(U_{1}U_{4}) = \frac{\lambda^3 (1-\lambda \alpha') \alpha_x \alpha_x^*}{6} \sum_{n=0}^{\infty} \left[ \gamma h_n'(0,1) + 3 \alpha \beta h_n''(0,1) + \alpha^3 h_n'''(0,1) \right]$$

$$= \frac{\lambda^3 \alpha_x \alpha_x^*}{6(1-\lambda^3 \alpha')^3} \left[ \gamma + \frac{3 \lambda^2 \alpha \beta^2}{1 - \lambda^2 \alpha^2} \right]$$
\( E_\infty(U_2 U_4) = E_\infty(U_1 U_4) \)

\[
E_\infty(U_2 U_4) = \frac{\lambda^3 (1-\lambda \alpha) \sigma_x^2}{6} \sum_{n=0}^{\infty} \left[ \gamma n'(0,1) + 3\alpha^2 h_n''(0,1) + \alpha^3 h_n''''(0,1) \right]
\]

\[
= \frac{\lambda^3 \sigma_x^2}{6 (1-\lambda \alpha^3)} \left[ \gamma + \frac{3 \lambda^2 \alpha \beta^2}{1 - \lambda^2 \alpha^2} \right]
\]

\( E_\infty(U_3 U_4) = E_\infty(U_1 U_2) \)

\[
E_\infty(U U') = \begin{pmatrix}
\frac{\lambda}{2} \alpha_x b & \lambda^2 \alpha_x^* b + \lambda \beta_x a \\
\lambda \alpha_x b & \lambda^2 \alpha_x \alpha_x^* b + \lambda^2 \alpha_x^* b + \lambda \beta_x a
\end{pmatrix}
\]

\( (36) \)

where \( a = E_\infty(U_0), \quad b = E_\infty(U_0)^2 \).

That is, \( a \) is the steady state expected residual life length of the generation serving at time \( t \), and \( b - a^2 \), its variance.

Remark:

For \( x = 0^+ \) the second and the fourth rows as well as the second and the fourth columns of the matrix \( E_\infty(U U') \) tend to zero. While for sufficiently large \( x \), the third and the fifth rows and columns tend to zero.
If we define:

\[
    r_{ij} = \frac{E_x(E_{U_i} U_j) - (E_x E U_i)(E_x E U_j)}{\sqrt{E_x E U_i^2} - (E_x E U_i)^2} \frac{\sqrt{E_x E U_j^2} - (E_x E U_j)^2}{,}
\]

\[i, j = 0, 1, 2, 3, 4\]

then in the case of an \( M^1 | M^1 | 1 \) queue as the traffic intensity tends to one it can be seen that \( r_{23} \) and \( r_{14} \) are negative, and \( r_{04} > r_{03} > r_{02} > 0 \)

7. Moments of the Limiting Distribution of the Virtual Waiting Times

\[\eta(t, x), \eta(t) \text{ and } \overline{\eta}(t, x)\]

Let us denote:

\[
    \Lambda = \begin{pmatrix} \eta(t, x) \\ \eta(t) \\ \overline{\eta}(t, x) \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}
\]

Then from (16) we get

\[
    \Lambda = A \Lambda' \sim
\]

(37) \( E_\infty \Lambda = A E_\infty \sim \)

(38) \( E_\infty (\Lambda \Lambda') = A E_\infty (\sim U \sim') \Lambda' \)

Direct computation of \( E_\infty \Lambda \) was given in [2]. Further a direct computation of \( E_\infty A \Lambda' \) is possible from (19). However, here we compute (37) and (38) in terms of the steady state moments of the basic variables \( U \)'s. Substituting (35) in (37) and (36) in (38) and simplifying we get:

(39) \( E_\infty \Lambda = \begin{pmatrix} (1 + 2 \lambda \alpha_x) a \\ (1 + \lambda \alpha) a \\ (1 + 2 \lambda \alpha_x^*) a \end{pmatrix} \)
\[ E_\infty(\Lambda \Lambda^*) = \begin{cases} 
 b(1 + 3b + 3a^2) 
 + 2 \lambda a 
 \end{cases} 
\]

\[ (40) \]

We define:

\[ (41) \quad \rho_{12} = \left[ E_x (E \Pi (\infty, X) \eta (\infty)) - (E_x E \Pi (\infty, X)) (E \eta (\infty)) \right] \]

\[ \left[ E_x (E \Pi (\infty, X)) - (E_x E \Pi (\infty, X))^2 \right]^{-\frac{1}{2}} \]

\[ \left[ E \eta^2 (\infty) - (E \eta (\infty))^2 \right]^{-\frac{1}{2}} \]

\[ (42) \quad \rho_{13} = \left[ E_x (E \Pi (\infty, X) \bar{\Pi} (\infty, X)) - (E_x E \Pi (\infty, X)) (E_x E \bar{\Pi} (\infty, X)) \right] \]

\[ \left[ E_x (E \Pi (\infty, X)) - (E_x E \Pi (\infty, X))^2 \right]^{-\frac{1}{2}} \]

\[ \left[ E_x (E \bar{\Pi} (\infty, X)) - (E_x E \bar{\Pi} (\infty, X))^2 \right]^{-\frac{1}{2}} \]

\[ (43) \quad \rho_{23} = \left[ E_x (E \eta (\infty) \bar{\Pi} (\infty, X)) - (E \eta (\infty)) (E_x E \bar{\Pi} (\infty, X)) \right] \]

\[ \left[ E \eta^2 (\infty) - (E \eta (\infty))^2 \right]^{-\frac{1}{2}} \]

\[ \left[ E_x (E \bar{\Pi} (\infty, X)) - (E_x E \bar{\Pi} (\infty, X))^2 \right]^{-\frac{1}{2}} \]
In the case of an M\|M\|1 queue when the traffic intensity tends to one it can be shown that:

(44) \( \rho_{12} = 0.79, \rho_{23} = 0.90 \) and \( \rho_{13} \sim \rho_{12} \rho_{23} \)

8. The Probability \( P \{ J(t,x) < \eta(t) \} \)

From (16) we have:

(45) \( P \{ J(t,x) < \eta(t) \} = P \{ U_3 < U_2 \} \)

\[
= \int_0^\infty \int_0^{x_2} dx_2 dx_3 \Lambda_1(t, x_2, x_3)
\]

where \( \Lambda_1(t, x_2, x_3) \) is the joint distribution of \( U_2 \) and \( U_3 \) given \( t \).

As in § 2, if given \( t \) and \( t' \), the joint distribution of \( U_2 \) and \( U_3 \) is given by:

(46) \( \Phi(t, t'; x_2, x_3) = \sum_{j_2=0}^{\infty} \sum_{j_3=0}^{\infty} e^{-\lambda t (1-H(x)) - \lambda t' H(x)} \frac{\lambda t [1-H(x)]}{j_2!} \frac{\lambda t' H(x)}{j_3!} \frac{(j_2)}{\mathcal{H}(x_2)} \frac{(j_1)}{\mathcal{H}(x_3)} \)

Let \( \Lambda_1^*(t, x_2, x_3) \) be the probability that in \( (0,t) \) the queue has never become empty and that the variables \( U_2 \) and \( U_3 \) associated with the time point \( t \) satisfy \( U_2 \leq x_2, U_3 \leq x_3 \), given that at \( t = 0 \) there were \( i \geq 1 \) customers in the queue, one of who was beginning his service at that time. Then:

(47) \( \Lambda_1^*(t, x_2, x_3) = \sum_{n=0}^{\infty} \sum_{\nu=1}^{n} \int_0^t \int_0^t d_o Q_i(\tau) d\mathcal{H}(\nu) (t+t' - \tau) \)
\[
\phi(t-\tau, t', x_2, x_3)
= \sum_{\nu=1}^{\infty} \int_0^t d_{oR_i\nu}(\tau) F_{\nu}(t-\tau, x_2, x_3),
\]

where \(oR_i\nu(\cdot) = \sum_{n=0}^{\infty} Q_{i\nu}(n) oQ_{i\nu}(\cdot)\) and:

\[
F_{\nu}(t-\tau, x_2, x_3) = \int_0^{\infty} dH^{(\nu)}(t+t'-\tau) \phi(t-\tau, t', x_2, x_3) \]

\[\quad (t')\]

By the renewal argument as in (11):

\[(49) \quad \Lambda_1(t, x_2, x_3) = \Lambda_1^*(t, x_2, x_3) + \int_0^t \Lambda_1^*(t-u, x_2, x_3) dM_1(u)\]

\[\quad + P \{ \xi(t) = 0 \left| \xi(0) = i \right. \} U(x_2, x_3)\]

where:

\[(50) \quad U(x_2, x_3) = 1 \text{ if } x_2 \geq 0 \text{ and } x_3 \geq 0,\]

\[= 0 \text{ otherwise.}\]

Substitution of (47) in (49) yields:

\[(51) \quad \Lambda_1(t, x_2, x_3) = \sum_{\nu=1}^{\infty} \int_0^t d_{oR_i\nu}(\tau) F_{\nu}(t-\tau, x_2, x_3)\]

\[\quad + \int_0^t \Lambda_1^*(t-u, x_2, x_3) dM_1(u)\]

\[\quad + P \{ \xi(t) = 0 \left| \xi(0) = i \right. \} U(x_2, x_3)\]
If \( \Lambda(x_2, x_3) \) is the limiting value of \( \Lambda_i(t, x_2, x_3) \) as \( t \to \infty \), then similar to the renewal argument in appendix II of [2] we have:

\[
\Lambda(x_2, x_3) = (1 - \lambda \alpha) \left\{ U(x_2, x_3) + \lambda \sum_{j=1}^{\infty} R_{ij}(+\infty) \int_{o}^{\infty} F_j(\tau, x_2, x_3) \, d\tau \right\}
\]

if \( 1 - \lambda \alpha > 0 \),

= 0 otherwise.

Finally from (45) it follows that:

\[
\lim_{t \to \infty} P \{ \tau(t, \mathbf{x}) < \tau(t) \} = (1 - \lambda \alpha) \left\{ 1 + \lambda \sum_{j=1}^{\infty} R_{ij}(+\infty) \int_{o}^{\infty} \int_{o}^{x_2} \int_{o}^{x_3} d_{x_2 x_3} F_j(\tau, x_2, x_3) \, d\tau \right\}
\]
REFERENCES


3. REPORT TITLE
An Exact Comparison of the Waitingtimes under Three Priority Rules

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)
Technical Report, December 1969

5. AUTHOR(S) (Last name, first name, initial)
Nair, S. Sreekantan and Neuts, Marcel F.

6. REPORT DATE
December 1969

6a. CONTRACT OR GRANT NO.
NONR 1100 (26)

6b. PROJECT NO.
c.

d.

7a. TOTAL NO. OF PAGES
30

7b. NO. OF REFERENCES
4

8a. ORIGINATOR'S REPORT NUMBER(S)
Mimeograph Series No. 213

8b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

9a. SPONSORING MILITARY ACTIVITY
Office of Naval Research
Washington, D.C.

9b. NONR 1100 (26)

10. AVAILABILITY/LIMITATION NOTICES
Distribution of this document is unlimited.

11. SUPPLEMENTARY NOTES

12. ABSTRACT
The waitingtimes of customers under three priority rules are compared. The first is the order of arrival rule. The second is the shortest processing time (within a generation) first and the third is longest processing time (within a generation) first.

In particular the covariance matrix of the trivariate distribution of the equilibrium waitingtimes is obtained explicitly.