LIMIT LAWS FOR MAXIMA OF A SEQUENCE OF
RANDOM VARIABLES DEFINED ON A MARKOV CHAIN*

by

Sidney I. Resnick and Marcel F. Neuts**

Department of Statistics
Division of Mathematical Sciences
Mimeograph Series No. 201
August 1969

*This research was partly supported by the Office of Naval Research Contract NONR 1100(26)
and by the National Science Foundation Graduate Traineeship Program at Purdue University.
Reproduction in whole or in part is permitted for any purpose of the United States Government.

**The first author is from Cornell University. The second author was on sabbatical
leave at Cornell University during the academic year 1968-1969.
Limit Laws for Maxima of a Sequence of
Random Variables Defined on a Markov Chain*

by

Sidney I. Resnick and Marcel F. Neuts
Purdue University and Cornell University

Abstract

Consider the bivariate sequence of r.v.'s \((J_n, X_n), n \geq 0\) with \(X_0 = \infty\) a.s. The marginal sequence \(\{J_n\}\) is an irreducible, aperiodic, \(m\)-state M.C., \(m < \infty\), and the r.v.'s \(X_n\) are conditionally independent given \(\{J_n\}\). Furthermore \(P\{J_n = j, X_n \leq x | J_{n-1} = i\} = p_{ij}H_i(x) = Q_{ij}(x)\), where \(H_1(\cdot), \ldots, H_m(\cdot)\) are c.d.f.'s. Setting \(M_n = \max\{X_1, \ldots, X_n\}\), we obtain \(P\{J_n = j, M_n \leq x | J_0 = i\} = [Q^n(x)]_{i,j}\), where \(Q(x) = \{Q_{ij}(x)\}\).

The limiting behavior of this probability and the possible limit laws for \(\{M_n\}\) are characterized:

**Theorem:** Let \(\rho(x)\) be the Perron-Frobenius eigenvalue of \(Q(x)\) for real \(x\); then: a) \(\rho(x)\) is a c.d.f. b) if for a suitable normalization \(\{Q^n(\cdot; a_{ij}x + b_{ij})\}\) converges completely to a matrix \(\{U_{ij}(x)\}\) whose entries are nondegenerate distributions, then \(U_{ij}(x) = \pi_j \rho(x)\), where

---

* This research was partly supported by the Office of Naval Research Contract Nonr 1100(26) and by the National Science Foundation Graduate Traineeship Program at Purdue University. Reproduction in whole or in part is permitted for any purpose of the United States Government.

During the academic year 1968-1969 the authors were visiting in the Department of Operations Research at Cornell University, Ithaca, N.Y.
\[ \pi_j = \lim_{n \to \infty} p_{ij} \] and \( \rho_U(x) \) is an extreme value distribution. c) the normalizing constants need not depend on \( i,j \). d) \( \rho^n(a_n x + b_n) \) converges completely to \( \rho_U(x) \). e) The maximum \( M_n \) has a nontrivial limit law \( \rho_U(x) \) iff \( Q^n(x) \) has a nontrivial limit matrix \( \Pi(x) = \{ U_{ij}(x) \} = \{ \pi_j \rho_U(x) \} \) or equivalently iff \( \rho(x) \) or the c.d.f. \( \Pi H_i(x) \) is in the domain of attraction of one of the extreme value distributions. Hence the only possible limit laws for \( \{ M_n \} \) are the extreme value distributions which generalizes the results of Gnedenko for the i.i.d. case.
I. Introduction

The limit laws for the maxima of a sequence of independent, identically distributed (i.i.d.) random variables were fully characterized by B.V. Gnedenko [3]. They are the so-called extreme value distributions. Precisely, if \( \{X_n, n \geq 1\} \) is a sequence of i.i.d. random variables with distribution function \( F(\cdot) \), let \( M_n = \max (X_1, X_2, \ldots, X_n) \). Then if there exist normalizing constants \( a_n > 0 \) and \( b_n \) such that

\[
P[a_n^{-1}(M_n - b_n) \leq x] = F^n(a_n x + b_n) \xrightarrow{c} \phi(x) \text{ where } \phi(x) \text{ is a nondegenerate limiting distribution, then } \phi(x) \text{ belongs to the type of one of the following three distributions:}
\]

\[
\Lambda(x) = \exp\{-e^{-x}\} \quad -\infty < x < \infty
\]

\[
\phi_\alpha(x) = \begin{cases} 
0 & x < 0 \\
\exp\{-x^{-\alpha}\} & x > 0
\end{cases}
\quad x < 0
\]

\[
\psi_\alpha(x) = \begin{cases} 
\exp\{-(-x)^{\alpha}\} & x < 0 \\
1 & x > 0
\end{cases}
\]

where \( \alpha \) is a positive constant.

Consider the analogous problem for random variables defined on a finite Markov chain (M.C.) which are conditionally independent given the chain. Let \( \{J_n, n \geq 0\} \) be an \( m \)-state M.C. whose transition matrix \( P = \{P_{ij}\} \) is irreducible and aperiodic. The random variables \( X_n, n \geq 1 \), are conditionally independent given the M.C. \( \{J_n\} \) and \( P\{X_n \leq x|J_{n-1} = i\} = H_i(x) \).
The distributions $H_i(x)$, $i = 1, \ldots, m$ are assumed to be nondegenerate and honest ($H_i(\pm \infty) = \pm 1$). Let $M_n = \max \{X_1, \ldots, X_n\}$ and $Q(x) = \{p_{ij}H_i(x)\}$ for $i, j = 1, 2, \ldots, m$. The $Q$-matrix governs the system. (There is no loss of generality in allowing the distribution of $X_n$ to depend only on $J_{n-1}$ — Pyke [5, p 1751]. The case where the distribution of $X_n$ depends on the pair $(J_{n-1}, J_n)$ can be reduced to this case.)

By induction we establish that:

$$Q^n_{ij}(x) = H_i(x) \sum_{k_1=1}^{m} \sum_{k_2=1}^{m} \cdots \sum_{k_{n-1}=1}^{m} p_{ik_1}H_{k_1}(x)p_{k_1k_2}H_{k_2}(x)p_{k_2k_3}H_{k_3}(x)p_{k_3k_{n-2}}H_{k_{n-2}}(x)p_{k_{n-2}k_{n-1}}H_{k_{n-1}}(x)p_{k_{n-1}k_n}H_{k_n}(x)$$

where $Q^n(x) = \{Q^n_{ij}(x)\}$ is the $n$-th power of the $Q$-matrix (In this paper, we are not concerned with matrix-convolution powers.) Using this formula and the conditional independence of the $X_n$, we get

$$(1.1) \quad P\{J_n = j, M_n \leq x | J_0 = i\} = Q^n_{ij}(x).$$

We concern ourselves with the existence of normalizing constants $a_{ijn} > 0$ and $b_{ijn}$, $i, j = 1, \ldots, m$, $n \geq 1$ such that the expressions

$$P\{J_n = j, a_{ijn}^{-1}(M_n - b_{ijn}) \leq x | J_0 = i\} = Q^n_{ij}(a_{ijn}x + b_{ijn})$$

converge to nondegenerate mass-functions $U_{ij}(x)$ at all continuity points of the latter and such that $\sum_{j=1}^{m} U_{ij}(x)$, $i = 1, \ldots, m$, is an honest distribution function.

If such normalizing constants exist, what are the possible limit matrices $\{U_{ij}(\cdot)\}$?
Finally we establish basic properties of the normalizing constants $a_{ijn}$ and $b_{ijn}$ and discuss the limiting behavior of the marginal distribution of $\mathcal{M}_n$.

2. Preliminaries

A semi-Markov matrix (S.M.M.) $Q(x) = \{Q_{ij}(x)\}$ is a matrix whose entries $Q_{ij}(x), i,j = 1, \ldots, m$ are mass functions such that $\sum_{j=1}^{m} Q_{ij}(x) \leq 1$.

A S.M.M. is honest if for all $i = 1, \ldots, m$, equality holds, otherwise it is dishonest. Unless otherwise specified all distribution functions and S.M.M.'s are honest.

Let $\{Q(x)\}$ be a sequence of S.M.M.'s. The sequence of S.M.M.'s converges completely to a limit matrix $Q(x)$ iff $Q(x)$ is honest and for each $i,j$, $Q_{ij}(\cdot) \xrightarrow{w} Q_{ij}(\cdot)$, we write $Q(\cdot) \xrightarrow{c} Q'(\cdot)$.

A matrix analogue of the classical weak compactness theorem for distribution functions holds for S.M.M.s: Given a sequence of S.M.M.'s $Q(x)$, there exists a subsequence $n_k$ and a limit S.M.M. $\tilde{Q}(x)$ (not necessarily honest) such that $n_k Q(\cdot) \xrightarrow{w} \tilde{Q}(\cdot)$; that is, for $i,j = 1, \ldots, m$,

$$n_k Q_{ij}(\cdot) \xrightarrow{w} Q_{ij}(\cdot).$$

Two S.M.M.'s $\Upsilon(x), \Psi(x)$ are of the same type if there exist constants $A > 0$ and $B$ such that for each $i,j$, $V_{ij}(x) = U_{ij}(Ax + B)$. The following lemma of Khintchin is useful [2, p. 246]:

**Lemma (2.1)** Let $U(\cdot)$ and $V(\cdot)$ be two non-degenerate distribution functions. If for a sequence $\{F_n(\cdot)\}$ of distribution functions and constants $a_n > 0, b_n$ and $\alpha_n > 0, \beta_n$: 

[End of text]
(2.2) \[ F_n(a_n x + b_n) \xrightarrow{w^*} U(x) , F_n(\alpha_n x + \beta_n) \xrightarrow{w^*} V(x) \]

Then:

(2.3) \[ \frac{\alpha_n}{a_n} \rightarrow A \neq 0 , \quad \frac{\beta_n - b_n}{a_n} \rightarrow B \]

and then

(2.4) \[ V(x) = U(Ax + B) \]

Conversely if (2.3) holds, then each of the two relations (2.2) implies the other and (2.4).

The set of normalizing constants \( a_n > 0, b_n, n \geq 1 \) is asymptotically equivalent to the set of normalizing constants \( \alpha_n > 0, \beta_n, n \geq 1 \) iff

\[ \frac{\alpha_n}{a_n} \rightarrow 1 , \quad \frac{\beta_n - b_n}{a_n} \rightarrow 0 \]

A S.M.M. \( Q(x) \) is a non-negative matrix for every \( x \); hence the Perron-Frobenius theory is applicable. For a matrix \( \mathcal{A} \) with real entries, we write \( \mathcal{A} \succeq Q (\succeq Q) \) if \( a_{ij} \geq 0 \) (\( a_{ij} > 0 \)) for each \( i,j \). For a complex matrix \( \mathcal{B} = \{b_{ij}\} \), \( |\mathcal{B}| \) denotes the matrix \( \{|b_{ij}|\} \). We use the following theorem [6, p.30]:

**Theorem 2.5** Let \( \mathcal{A} \succeq Q \) be an irreducible \( m \times m \) matrix. Then:

1. \( \mathcal{A} \) has a simple, positive eigenvalue equal to its spectral radius \( \rho_{\mathcal{A}} \).
2. To the eigenvalue \( \rho_{\mathcal{A}} \) corresponds a positive eigenvector \( \chi > 0 \).
3. \( \rho_{\mathcal{A}} \) increases when any entry of \( \mathcal{A} \) increases. (If \( \mathcal{A} \) is reducible, then \( \rho_{\mathcal{A}} \) does not decrease when any entry of \( \mathcal{A} \) increases.)
Theorem (2.6) [6, pp. 28,47]: Let $\mathbf{A}$ and $\mathbf{B}$ be two $m \times m$ matrices with $\rho \leq |\mathbf{B}| \leq \mathbf{A}$. Then $\rho_{\mathbf{B}} \leq \rho_{\mathbf{A}}$. If $\mathbf{A}$ is irreducible then $\rho_{\mathbf{B}} = \rho_{\mathbf{A}}$ implies that $|\mathbf{B}| = \mathbf{A}$.

Theorem (2.7) [6, p. 13]: If $\mathbf{A}$ is an $m \times m$ complex matrix, then $\mathbf{A}^n + \mathbf{Q}$ entrywise iff $\rho_{\mathbf{A}} < 1$.

For fixed $x$, $\mathbf{Q}(x)$ is a positive matrix whose spectral radius we denote by $\rho(x)$. $\rho(x)$ is a distribution function. $\mathbf{Q}(\cdot + \infty)$ is stochastic; hence $\rho(\cdot + \infty) = 1$. $\mathbf{Q}(\cdot - \infty) = \mathbf{Q}$; hence $\rho(\cdot - \infty) = 0$. $\rho(x)$ is nondecreasing by Theorem (2.5-3).

Furthermore:

Lemma (2.8) (1) If $\mathbf{Q}(x)$ is (right, left) continuous at $x_0$, then $\rho(x)$ is (right, left) continuous at $x_0$.

(2) If $\rho(x)$ is right continuous at $x_0$ and $\mathbf{Q}(x_0)$ is irreducible, then $\mathbf{Q}(x)$ is right continuous at $x_0$. If $\rho(x)$ is left continuous at $x_0$ and $\mathbf{Q}(x)$ is irreducible for $x > x_0 - \varepsilon$ for some $\varepsilon > 0$, then $\mathbf{Q}(x)$ is left continuous at $x_0$.

Proof: (1) If $\mathbf{Q}(x)$ is left continuous at $x_0$, select a sequence $x_n \uparrow x_0$. Then $\mathbf{Q}(x_n) \rightarrow \mathbf{Q}(x)$ and hence $\rho(x_n) + \rho(x)$. Hence $\rho(x)$ is left continuous at $x_0$. Similarly for right continuity.

(2) Suppose $\rho(x)$ is left continuous at $x_0$. Choose a sequence $(x_n)$ such that $x_0 - \varepsilon < x_n \downarrow x_0$. Then $\mathbf{Q}(x_n) \rightarrow \mathbf{Q}(x_0^-) \leq \mathbf{Q}(x_0)$. If there exists $(i,j)$ such that $Q_{ij}(x_0^-) < Q_{ij}(x_0)$ then $\rho(x_0^-) < \rho(x_0)$ by Theorem (2.5-3), contradicting the left continuity of $\rho(x)$ at $x_0$. Similarly for right continuity.
Lemma (2.9): Let \( \{nQ(.)\} \) be a sequence of S.M.M.'s and \( nQ(.) \underset{c}{\rightarrow} Q(.) \). Then \( \rho_n(.) \rightarrow C \rho(.) \) where \( \rho(x) \) and \( \rho_n(x) \) are the spectral radii of \( Q(x) \) and \( nQ(x) \) respectively.

Proof: Weak convergence of distribution functions is equivalent to pointwise convergence on a set everywhere dense on the real line, so \( nQ(.) \rightarrow C Q(.) \) implies that for \( x \in D \), \( nQ(x) \rightarrow Q(x) \); \( D \) is an everywhere dense subset of \( R \). Hence for \( x \in D \), \( \rho_n(x) \rightarrow \rho(x) \) and hence \( \rho_n(.) \rightarrow (.) \). But \( Q(.) \) is honest, so \( Q(+\infty) \) is stochastic. Thus \( \rho(+\infty) = 1 \) and \( \rho_n(.) \rightarrow C \rho(.) \).

We can say more about the spectral properties of a S.M.M. \( Q(x) \). Suppose there exists \( x_0 < \infty \) such that for \( x > x_0 \) \( Q(x) \) is irreducible. Now let \( \xi(x) = (\xi_1(x), \ldots, \xi_m(x)) \), \( \xi_0(x) = (\xi_1(x), \ldots, \xi_m(x)) \) be right and left eigenvectors of \( Q(x) \) corresponding to \( \rho(x) \). The components of \( \xi(x) \) and \( \xi_0(x) \) can be chosen to be non-negative and for \( x > x_0 \) all components are then strictly positive (2.5-2). As functions of \( x \), \( \xi(x) \) and \( \xi_0(x) \) are only determined up to arbitrary factors, since for any scalar functions \( k_1(x) \) and \( k_2(x) \), \( k_1(x) \xi(x) \) and \( k_2(x) \xi_0(x) \) are also eigenvectors. In order to discuss continuity properties and limiting behavior of \( \xi(x) \) and \( \xi_0(x) \) we must specify a version of the eigenvectors.

Lemma (2.10): Let \( Q(x) \), \( \xi(x) \), \( \xi_0(x) \) be as above. Restrict attention to the domain \( x > x_0 \) where \( Q(x) \) is irreducible. We normalize \( \xi(x) \) and \( \xi_0(x) \) by: \( \sum_{i=1}^{m} r_i(x) = \sum_{i=1}^{m} \ell_i(x) = 1 \). Suppose \( \xi_0 = Q(+\infty) \) is primitive.
We have

\[ \lim_{x \to \infty} \chi(x) = (m^{-1}, \ldots, m^{-1}) \]

\[ \lim_{x \to \infty} \xi(x) = (\pi_1, \ldots, \pi_m) \quad \text{where} \quad (\pi_1, \ldots, \pi_m) \quad \text{are the stationary} \]
probabilities associated with \( \rho \). Also \( \rho \prod \to \xi \) where \( \prod_{ij} = \pi_j \)

(2) If \( \xi(x) \) is (right, left) continuous at \( x_1 > x_0 \), then \( \chi(x) \) and \( \xi(x) \) are (right, left) continuous at \( x_1 \).

Proof: (1) \( \chi(x) \) is in a compact set. For any sequence \( x_n \to \infty, \{r(x_n)\} \) must have a convergent subsequence, say \{r(x_{n_k})\}. Suppose

\[ \lim_{k \to \infty} \chi(x_{n_k}) = r = (r_1, \ldots, r_m) \]  

Since \( \sum_{i=1}^{m} r_i = 1 \), not all components of \( \chi \) can vanish. Then \( \lim_{k \to \infty} \rho(x_{n_k}) \chi(x_{n_k}) = \lim_{k \to \infty} \rho(x_{n_k}) \chi(x_{n_k}) \), so \( \lim_{k \to \infty} \rho^n \chi(x) = \chi \).

Since \( P \) is stochastic and irreducible, its right eigenvector corresponding to Perron-Frobenius eigenvalue 1 is uniquely determined up to a factor and hence \( r_i = m^{-1}, i = 1, \ldots, m \). Since every convergent subsequence of \( \{\chi(x_n)\} \) converges to the same limit, \( \lim_{n \to \infty} \chi(x_n) = (m^{-1}, \ldots, m^{-1}) \). Similarly for \( \xi(x) \).

(2) Suppose \( Q(x) \) is left continuous at \( x_1 \). Pick any sequence \{x_n\} such that \( x_0 < x_n \to x_1 \). Then \( Q(x_n) \to Q(x_1) \) and \( \rho(x_n) \to \rho(x_1) \).

By compactness, these exist a subsequence \( n_k \) and \( s = (s_1, \ldots, s_m) \) such that \( \sum_{i=1}^{m} s_i = 1 \) and \( \lim_{k \to \infty} \chi(x_{n_k}) = \xi \).

Hence \( \lim_{k \to \infty} Q(x_{n_k}) \chi(x_{n_k}) = \lim_{k \to \infty} \rho(x_{n_k}) \chi(x_{n_k}) \), i.e. \( Q(x_1) \xi = \rho(x_1) \xi \). But since \( Q(x_1) \) is irreducible \( \xi = \xi(x_1) \). All convergent subsequences have
the same limit; hence  \( \lim_{r \to \infty} x_n(x) = x_1(x) \). Similarly for  \( x(x) \) and for right continuity.

Now let  \( \Omega(x) = \{ p_{ij}H_i(x) \} \)  \( i,j = 1, \ldots, m \) where  \( p = \{ p_{ij} \} \) is an irreducible, aperiodic, stochastic matrix and  \( p^n \to \Omega \) and  \( H_1(\cdot), \ldots, H_m(\cdot) \) are nondegenerate distribution functions. There exist an integer  \( k' \) such that  \( p^k > \Omega \) for  \( k > k' \) and a real number  \( x_0 \), such that for  \( x > x_0 \)
m in  \( \{ H_1(x), \ldots, H_m(x) \} > 0 \). We may limit ourselves to the domain  \( x > x_0 \)
where  \( \eta^k(x) > \Omega \).

The conditions  \( \sum_{i=1}^m \lambda_i(x)r_i(x) = 1 \) and  \( \sum_{i=1}^m r_i(x) = 1 \) determine a version of the right and left eigenvectors possessing the continuity properties and limiting behavior discussed in Lemma (2.10). This version can be obtained from the one satisfying  \( \sum_{i=1}^m r_i(x) = \sum_{i=1}^m \lambda_i(x) = 1 \) through the transformations  \( r_i(x) \to \frac{r_i(x)}{\sum_{i=1}^m r_i(x)\lambda_i(x)} \),  \( i = 1, \ldots, m \). We assume henceforth that  \( x(x) \) and  \( \lambda(x) \) are so normalized.

Form the matrix  \( M(x) = \{ r_i(x) \lambda_j(x) \} \),  \( i,j = 1, \ldots, m \). It is known [4, p.248]:

\[
\text{(2.11)} \quad \lim_{x \to \infty} M(x) = \Omega
\]

\[
\text{(2.12)} \quad M^2(x) = M(x)
\]

\[
\text{(2.13)} \quad \text{For any vector } \eta = (V_1, \ldots, V_m) \text{ we have:}
\]

\[
M(x)\eta = (\lambda(x), \lambda(x))\eta(x) \quad \text{and} \quad \eta M(x) = (\eta, \lambda(x))\eta(x).
\]
(2.14) \( Q(x)M(x) = M(x)Q(x) = \rho(x)M(x) \)

(2.15) \( \lim_{n \to \infty} \rho_n^{-n} Q(x) = M(x) \).

We examine (2.15) in detail. Set \( R(x) = Q(x) - \rho(x)M(x) \). Then by (2.12) and (2.14), we have \( R^n(x) = Q^n(x) - \rho^n(x)M(x) \).

**Theorem (2.15):** Let \( Q(x) = \{p_{ij}H_i(x)\} \), \( M(x) \), \( R(x) \) be as above. There exists a real number \( M \) such that \( \lim_{n \to \infty} R^n(x) = \lim_{n \to \infty} [Q^n(x) - \rho^n(x)M(x)] = \emptyset \)

uniformly in \( x > M \). Equivalently:

(2.17) \( Q^n(x) = \rho^n(x)M(x) + Q(1) \) where \( \lim_{n \to \infty} Q(1) = \emptyset \) uniformly in \( x > M \).

**Proof:** We can show by induction that \( |R^n| \leq |R|^n \) for integral \( n \).

Let \( E \) be the \( m \times m \) matrix \( E_{ij} = 1 \) and \( R(x) = \{E_{ij}(x)\} \). Fix \( N \), a positive integer such that \( \max_{i,j} |p_{ij}^N - \pi_j| < m^{-1} \). Set \( \alpha = \max_{i,j} |E_{ij}^N - \pi_j| \).

Pick \( \epsilon > 0 \) such that \( \alpha + \epsilon < m^{-1} \). Since \( \lim_{x \to \infty} R^n(x) = R^n \), there exists \( M_N \) such that for \( x > M_N \), \( |R^n_{ij}(x)| \leq \alpha + \epsilon, i,j=1,\ldots,m \). Then

\( |R^n_{ij}(x)| = \{|E_{ij}^n(x)|\} \leq (\alpha + \epsilon)E \leq m^{-1}E \). The spectral radius of \( E \) is \( m \) so the spectral radius of \( (\alpha + \epsilon)E \) is strictly less than \( 1 \); hence

\( ((\alpha + \epsilon)E)^n \to Q \) as \( n \to \infty \) by Theorems (2.6), (2.7). So for \( x > M_N \),

\( |R^n_n(x)| \to \emptyset \) uniformly in \( x \) and since \( |R^n_n(x)| \geq |R^n_n(x)| \) we have that

\( |R^n_n(x)| \to Q \) uniformly in \( x > M_N \).

Now for any \( n \), write

\( |E^n_n(x)| = |R^n_n(x)E^n_n N(x)| \leq |R^n_n(x)| |R^n_n N(x)| \).
For any \( n \), the second factor is one of the following: \(|p_0^n(x)|, |p_1^n(x)|, \ldots, |p_{N-1}^n(x)|\). For \( k = 1, 2, \ldots, N-1 \) there exist real numbers \( M_1, \ldots, M_{N-1} \) such that \( x > M_k \) implies \(|p_k^n(x)| \leq E\). So for \( X > M = \max\{M_1, \ldots, M_{N-1}, M_N\} \) the second factor is bounded by \( E \); the first factor approaches \( Q \) uniformly in \( x > M \). This completes the proof.

We use the following lemma [1]:

Lemma (2.18): Let \( P = \{p_{ij}\} \) be an \( m \times m \), irreducible, aperiodic, stochastic matrix such that \( \lim_{n \to \infty} P^n = \pi \). Suppose there are constants \( c_{ijn} \) with \( 0 \leq c_{ijn} \leq 1 \), \( n \geq 1 \), \( i,j = 1, 2, \ldots, m \), such that \( \lim_{n \to \infty} (c_{ijn})^n = \phi_{ij} \).

Then:

\[
\lim_{n \to \infty} (c_{ijn}p_{ij})^n = \left[ \prod_{j=1}^{m} \phi_{ij} \right] \pi
\]

3. Limit Laws

Theorem (3.1): Limit Laws for the Q-Matrix:

Let \( Q(x) = \{q_{ij}H_i(x)\} \) where \( P = \{p_{ij}\} \) is irreducible, aperiodic, stochastic,

\[
\lim_{n \to \infty} P^n = \pi \quad \text{and} \quad H_1(\cdot), \ldots, H_m(\cdot) \quad \text{are nondegenerate, honest distribution functions.}
\]

If there exist \( a_{ijn} > 0 \) and \( b_{ijn} \), \( i,j = 1, 2, \ldots, m \) and \( n \geq 1 \), such that

\[
\{P[J_n = j, a_{ijn}^{-1}(M_n - b_{ijn}) \leq x | J_0 = i] = \{Q_{ij}^n(a_{ijn}x + b_{ijn}) \leq U_{ij}(x) \}.
\]

where \( U_{ij}(x) \) is nondegenerate, then
(1) \( U_{ij}(x) \) is independent of \( i \) and is given by \( \rho_U(x) \pi_j \); \( \rho_U(x) \) is an honest, nondegenerate distribution function, the Perron-Frobenius eigenvalue of \( \{U_{ij}(x)\} \).

(2) \( \rho_U(x) \) is an extreme value distribution. In fact for all \( i,j \)
\( \rho^n(a_{ijn}x + b_{ijn}) \xrightarrow{c} \rho_U(x) \)

(3) \( a_{ijn} \) and \( b_{ijn} \) may be chosen independently of \( i,j \). \( \rho_U(x) \) is of the form \( \prod_{i=1}^{m} \phi_i^{\pi_i}(x) \) where \( \phi_i^{\pi_i}(x) \) is an honest distribution function such that \( H_{i\kappa}(a_{ijn}x + b_{ijn}) \xrightarrow{c} \phi_i(x) \) for some subsequence \( n_{\kappa} \).

(4) The domain of attraction of \( \rho_U(x) \) includes also \( \prod_{i=1}^{m} H_{i\kappa}^{\pi_i}(x) \).

The proof of part (2) requires a lemma. We state it now but defer its proof until after the proof of Theorem 3.1. Recall the representation
\( G^n(x) = \rho^n(x)G(x) + G(1) \) where \( \lim_{n \to \infty} G(1) = 0 \) uniformly in \( x \in [K, \infty] \) for a suitably chosen \( K \).

**Lemma 3.2:** If \( \rho_U(x) > 0 \) then: \( \lim_{n \to \infty} M_{ij}(a_{ijn}x + b_{ijn}) = \pi_j \) for all \( i,j \).

We can show more. If \( \rho_U(x) > 0 \) then:

(a) \( \lim_{n \to \infty} H_{i\kappa}(a_{ijn}x + b_{ijn}) = 1 \)

(b,1) If there exists some \( i_0 \) such that \( H_{i_0}(x) < 1 \) for all \( x \), then
\( \lim_{n \to \infty} a_{ijn}x + b_{ijn} = +\infty \) for all \( i,j \).
(b,2) If $H_i(x) = 1$ and $H_i(x_i - \epsilon) < 1$ for all $\epsilon > 0$, 
for $i = 1,2, \ldots, m$ and $x_0 = \max \{x_1, \ldots, x_m\} < \infty$, then for $x$ fixed 

either (b,2,i) $a_{ijn}x + b_{ijn} > x_0$ for finitely many $n$ and 
\[
\lim_{n \to \infty} a_{ijn}x + b_{ijn} = x_0
\]
or (b,2,ii) $a_{ijn}x + b_{ijn} > x_0$ infinitely often and
\[
\{\phi^n(a_{ijn}x + b_{ijn})\} \to \phi \text{ and } \rho_{\phi}(x) = 1.
\]
(Note in $\{\phi^n(a_{ijn}x + b_{ijn})\}$ we evaluate each 
component $\phi^n(\cdot)$ at $a_{ijn}x + b_{ijn}$ for 
k, $k = 1,2, \ldots, m$.)

Proof of Theorem 3.1: (1) We have:
\[
\{Q_{ij}(a_{ijn}x + b_{ijn})\}^n = \{p_{ij}H_i(a_{ijn}x + b_{ijn})\}^n.
\]
There exists a subsequence $n_k$ such that for all $i,j$
\[
H_i(a_{ijn_k}x + b_{ijn_k})^n \xrightarrow{w} \phi_{ij}(x) \text{ for distributions } \phi_{ij}(x) \text{ by the}
\]
weak compactness theorem. For a given $x$, if there exists an index pair $(i,j)$ 
such that $\phi_{ij}(x) = 0$, then $\{p_{ij}H_i(a_{ijn_k}x + b_{ijn_k})^n \to Q$
[1]. Since $\{p_{ij}H_i(a_{ijn_k}x + b_{ijn_k})^n \to \{U_{ij}(x)\}$ we have that, if for any $(i,j)$
$U_{ij}(x) > 0$, then for all $i,j \phi_{ij}(x) > 0$. For $x$ such that $\phi_{ij}(x) > 0$
for all $i,j$ we have
\[
\{p_{ij}H_i(a_{ijn_k}x + b_{ijn_k})\}^n \to [\prod_{i,j=1}^m \phi_{ij}]^n
\]
by (2.18) and also
\[ \{p_{ij} H_i(a_{ijn} x + b_{ijn})^n k \} \rightarrow \{U_{ij}(x)\} \]
so that
\[ U_{ij}(x) = \prod_{i,j=1}^{m} \phi_{ij}^{\pi_i P_{ij}} \] and \( U_{ij}(x) \) is independent of \( i \).

Set \( \rho_U(x) = \prod_{i,j=1}^{m} \phi_{ij}^{\pi_i P_{ij}}(x) \). Then \( \rho_U(x) \) is independent of the choice of subsequence and \( \rho_U(x) = \sum_{j=1}^{m} U_{ij}(x) \) for all \( i \) and

\[ (3.3) \quad \rho_U(x) > 0 \text{ implies that } \phi_{ij} > 0 \text{ for all } i,j \]

Since \( \sum_{j=1}^{m} U_{ij}(x) = \rho_U(x) \), we have \( \rho_U(x) \) is honest and nondegenerate (by definition of complete convergence of S.M.M.'s). If \( P_{ij} > 0 \) for all \( i,j \), we see that none of the \( \phi_{ij}(\cdot) \) can be dishonest. This will be seen to hold true even if some of the \( P_{ij} \)'s vanish. Hence

\[ H_i(a_{ijn} x + b_{ijn})^n k \rightarrow \phi_{ij}(x) \cdot \]

At least one of the \( \phi_{ij}(\cdot) \) is nondegenerate since if this were not the case \( \rho_U(x) \) would be degenerate.

\[ (3.4) \quad \text{Furthermore: } [ \prod_{i,j=1}^{m} H_i^{\pi_i P_{ij}(a_{ijn} x + b_{ijn})^n} ] \rightarrow \rho_U(x) \]
since every convergent subsequence will converge to \( \rho_U(x) \).
(2) For $x$ such that $\rho_U(x) > 0$, we have

$$\lim_{n \to \infty} Q_{ij}^n(a_{ijn}x + b_{ijn}) = \lim_{n \to \infty} [\rho^n(a_{ijn}x + b_{ijn}) M_{ij}(a_{ijn}x + b_{ijn}) + o(1)].$$

Therefore

$$\rho_U(x) \pi_j = \lim_{n \to \infty} \rho^n(a_{ijn}x + b_{ijn}) \pi_j$$

by Lemma (3.2) and

$$\rho_U(x) = \lim_{n \to \infty} \rho^n(a_{ijn}x + b_{ijn}) \text{ for all } i,j.$$ (3.5)

Therefore $\rho_U(x)$ is an extreme value distribution [3].

(3) Since (3.5) holds for all $i,j$ $a_{ijn}$ and $b_{ijn}$ may be chosen independently of $i$ and $j$ (Lemma 2.1)

For a suitably chosen subsequence $n_k$, we have that

$$H_i(a_{ijn_k}x + b_{ijn_k})^{n_k} \xrightarrow{w} \phi_{ij}(x).$$

Since $a_{ijn}$ and $b_{ijn}$ need not depend on $i,j$,

$$H_i(n_k x + b_{n_k})^{n_k} \xrightarrow{w} \phi_{ij}(x);$$

Therefore $\phi_{ij}(\cdot)$ is independent of $j$. This implies that

$$\rho_U(x) = \prod_{i,j=1}^m \phi_{ij}(x) = \prod_{i=1}^m \phi_i(x).$$

So each $\phi_i(\cdot)$ is honest and $H_i(n_k x + b_{n_k})^{n_k} \xrightarrow{c} \phi_i(x).$
(4) Since \[ \prod_{i,j=1}^{m} H_{i}^{\pi_{i}^{ij}}(a_{ijn} + b_{ijn}) \overset{c}{\longrightarrow} \rho_{U}(x) \]

by (3.4), we have

\[ \prod_{i=1}^{m} H_{i}^{\pi_{i}^{ij}}(a_{ijn} + b_{ijn}) = \prod_{i=1}^{m} H_{i}^{\pi_{i}}(a_{n} + b_{n}) \overset{c}{\longrightarrow} \rho_{U}(x) . \]

So \[ \prod_{i=1}^{m} H_{i}^{\pi_{i}}(.) \text{ is in the domain of attraction of } \rho_{U}(x) . \]

It only remains to prove Lemma (3.2):

**Proof of Lemma (3.2):** (a) We fix \( x \) such that \( \rho_{U}(x) > 0 \) and pick a subsequence \( n_{k} \) such that \( H_{i}(a_{ijn_{k}} + b_{ijn_{k}}) \) converges. Suppose that \( \lim_{k \to \infty} H_{i}(a_{ijn_{k}} + b_{ijn_{k}}) = \& . \) There exists a further subsequence \( n'_{k} \)

such that

\[ H_{i}(a_{ijn_{k}} + b_{ijn_{k}}) \overset{n'_{k}}{\longrightarrow} \psi_{ij}(x) \]

and because of (3.3) and the assumption that \( \rho_{U}(x) > 0 \) we have \( \psi_{ij}(x) > 0 . \)

So taking logarithms:

\[ n'_{k} \log H_{i}(a_{ijn_{k}} + b_{ijn_{k}}) \longrightarrow \log \psi_{i}(x) \]

and therefore

\[ \log H_{i}(a_{ijn_{k}} + b_{ijn_{k}}) \longrightarrow 0 \]

and

\[ H_{i}(a_{ijn_{k}} + b_{ijn_{k}}) \longrightarrow 1 . \]

This identifies \( \& = 1 \) and since any convergent subsequence must converge to 1 we have the desired result.
(b,1) If \( H_{i_0}(x) < 1 \) for all \( x \) then \( \rho(x) < 1 \) for all \( x \) by (2.6) and for all \( x \) \( \lim_{n \to \infty} Q_n(x) = Q_0 \) by (2.7). Suppose \( a_{ijn} x + b_{ijn} \) does not converge to \( +\infty \). Then there is a subsequence \( n_k \) and a real number \( k^0 \) such that \( a_{ijn_k} x + b_{ijn_k} < k^0 < +\infty \) for all \( k \).

Then

\[
Q_n(a_{ijn_k} x + b_{ijn_k}) \leq Q_n(k^0) \to 0 \text{ as } k \to \infty.
\]

In particular

\[
Q_{ij}(a_{ijn_k} x + b_{ijn_k}) \to 0.
\]

Since

\[
Q_{ij}(a_{ijn_k} x + b_{ijn_k}) \to \rho_U(x) \pi_j > 0
\]

we have a contradiction.

For this case, since \( \lim_{n \to \infty} a_{ijn} x + b_{ijn} = +\infty \), we have immediately from (2.10) and the fact that \( M_{ij}(x) = r_i(x) \ell_j(x) \) that

\[
\lim_{n \to \infty} M_{ij}(a_{ijn} x + b_{ijn}) = \pi_j.
\]

(b,2,1) If \( a_{ijn} x + b_{ijn} > x_0 \) for only finitely many \( n \) then there exists a positive integer \( N_x \) such that if \( n > N_x \) then

\[
a_{ijn} x + b_{ijn} \leq x_0.
\]

Pick a convergent subsequence \( n_k \) and suppose

\[
a_{ijn_k} x + b_{ijn_k} \to x' \leq x_0 \text{ as } k \to \infty.
\]

If \( x' < x_0 \) then there is an \( \varepsilon > 0 \) such that \( x' < x_0 - \varepsilon \). Then for all \( n_k \) sufficiently large
\begin{align*}
\mathcal{Q}_k(a_{ijn}^n x + b_{ijn}^n) & \leq \mathcal{Q}_k(x_0^n - \varepsilon) \to \mathcal{Q}_0 \text{ as } k \to \infty \text{ but also } \\
\mathcal{Q}_k^{n}(a_{ijn}^n x + b_{ijn}^n) & \to \rho_j(x) \pi_j > 0 \text{ which gives a contradiction.}
\end{align*}

Hence \( x^* = x_0 \). Since any convergent subsequence converges to \( x_0 \), the sequence converges to \( x_0 \).

Hence for \( n > N_x, x_0 = a_{ijn} x + b_{ijn} \to x_0 \), \( n \to \infty \); we have

\[ H_i(a_{ijn} x + b_{ijn}) \to H_i(x_0^-) . \]

However from

\[ (3.9) \quad H_i(a_{ijn} x + b_{ijn}) \to 1, \text{ whence } H_i(x_0^-) = 1 = H_i(x_0) . \] So

\[ H_i(\cdot), i = 1, \ldots, m. \text{ are continuous at } x_0 \text{ and hence so is } \mathcal{Q}(\cdot) . \]

By Lemma (2.10) \( \rho(\cdot), \xi(\cdot), \xi(\cdot) \) and hence \( \mu(\cdot) \) are all continuous at \( x_0 \). Therefore \( \lim_{n \to \infty} M_{ij}(a_{ijn} x + b_{ijn}) = M_{ij}(x_0) = \pi_j . \)

(b,2,ii) If \( a_{ijn} x + b_{ijn} > x_0 \), for infinitely many \( n \), then for infinitely many \( n \)

\[ \mathcal{Q}_0^n(a_{ijn} x + b_{ijn}) = \rho^n . \]

Hence \( \mathcal{Q}_j^n(a_{ijn} x + b_{ijn}) \to \pi_j \)

and by Theorem (3.8) this suffices for \( \mathcal{Q}_0^n(a_{ijn} x + b_{ijn}) \to \mu_0 \) and

\[ \rho^n(a_{ijn} x + b_{ijn}) \to 1 \text{ as } n \to \infty . \] So \( \lim_{n \to \infty} Q_j^n(a_{ijn} x + b_{ijn}) = \]

\[ = \lim_{n \to \infty} \rho^n(a_{ijn} x + b_{ijn}) M_{ij}(a_{ijn} x + b_{ijn}) + o(1) \right\}, \text{ whence } \pi_j = \lim_{n \to \infty} M_{ij}(a_{ijn} x + b_{ijn}) . \]

The lemma is completely proved.

If there are constants \( a_{ijn} > 0, b_{ijn}, n \geq 1, i,j = 1, \ldots, m \) for
which \( \{Q_{ij_n}(a_{ijn}^n + b_{ijn})\} \longrightarrow U(x) \) with \( U_{ij}(x) \) nondegenerate, then by

Theorem (3.1), part (2), for fixed \( (i_0, j_0) \) the set of constants \( a_{i_0j_0n} > 0 \), \( b_{i_0j_0n} \), \( n > 1 \) is asymptotically equivalent to each of the sets \( a_{k\&n} > 0 \), \( b_{k\&n} \) for \( k, \ell = 1, \ldots, m \). Without loss of generality we henceforth assume that normalizing constants are chosen independently of \( i \) and \( j \).

**Corollary (3.6) Convergence to Types:** If for given constants \( \alpha_n > 0 \), \( \beta_n \) and \( a_n > 0 \), \( b_n \):

\[
\{Q_{ij_n}(\alpha_n x + \beta_n)\} \longrightarrow U(x) = \{V_{ij}(x)\} \quad \text{and} \quad \{Q_{ij_n}(\alpha_n x + b_n) \longrightarrow U(x) = \{U_{ij}(x)\}
\]

where \( U_{ij}(x) \), \( V_{ij}(x) \) are nondegenerate for each \( (i, j) \), then \( U(x) \) and \( V(x) \) are of the same type. There exist \( A > 0 \) and \( B \) such that

\[
A = \lim_{n \to \infty} \alpha_n^{-1} a_n \quad \text{and} \quad B = \lim_{n \to \infty} \alpha_n^{-1} (\beta_n - b_n) \quad \text{and}
\]

\[
\{V_{ij}(x)\} = \{U_{ij}(Ax + B)\} = \{U_{ij}(Ax + B)\}. \quad \text{Furthermore} \quad U(x) = \rho_{U(x)} U \quad \text{and}
\]

where \( \rho_U(x) \) is an extreme value distribution and \( V(x) = \rho_{U(Ax + B)} U \).

**Corollary (3.7): Asymptotic Independence:** Given

\[
\{P[J_n = j, a_n^{-1}(M_n - b_n) \leq x | J_0 = i]\} \longrightarrow \{U_{ij}(x)\} = \rho_{U(x)} U \quad \text{then}
\]

\[
P[a_n^{-1}(M_n - b_n) \leq x] \rightarrow \rho_U(x) \quad \text{and} \quad \lim_{n \to \infty} P[J_n = j, a_n^{-1}(M_n - b_n) \leq x] =
\]

\[
= \lim_{n \to \infty} P[J_n = j] \lim_{n \to \infty} P[a_n^{-1}(M_n - b_n) \leq x].
\]
Proof: We have that
\[
\lim_{n \to \infty} P[J_n = j, a_n^{-1}(M_n - b_n) \leq x \mid J_0 = i] = \rho_U(x) \pi_j
\]
so
\[
\lim_{n \to \infty} P[a_n^{-1}(M_n - b_n) \leq x \mid J_0 = 1] = \rho_U(x)
\]
and
\[
\lim_{n \to \infty} P[a_n^{-1}(M_n - b_n) \leq x] = \rho_U(x). \quad \text{Therefore } M_n \text{ has a limiting distribution which is an extreme value distribution. Next we have that}
\]
\[
\lim_{n \to \infty} P[J_n = j, a_n^{-1}(M_n - b_n) \leq x] =
\]
\[
= \lim_{n \to \infty} P[J_n = j, a_n^{-1}(M_n - b_n) \leq x \mid J_0 = 1] = \pi_j \rho_U(x) =
\]
\[
= \lim_{n \to \infty} P[J_n = j] \lim_{n \to \infty} P[a_n^{-1}(M_n - b_n) \leq x] \quad \text{which completes the proof.}
\]
That the norming constants can be chosen to be independent of \(i, j\) is not surprising. When we take the \(n^{th}\) power of the Q-matrix we sum over all paths of length \(n\) starting at \(i\) and ending at \(j\). This entails sufficient mixing of the distributions involved so that the effects of the endpoints \(i\) and \(j\) become negligible for large \(n\).

A further reflection of this thorough mixing when taking powers of the Q-matrix is given in:

**Theorem (3.8):** There exist norming constants \(a_n > 0, b_n, n > 1\) and an index pair \((i_0, j_0), 1 \leq i_0, j_0 < m\), such that

\[
Q_{i_0 j_0}^n (a_n x + b_n) \xrightarrow{c} U_{i_0 j_0}(x).
\]
with $U_{i_0j_0}(x)$ nondegenerate iff:

$$Q^m(a_nx + b_n) \overset{c}{\to} \{U_{ij}(x)\}$$

where $U_{ij}(x) = \rho_U(x)\pi_j$ and $\rho_U(x)$ is an extreme value distribution and 

$$\pi^{-1}_{i_0j_0} U_{i_0j_0}(x) = \rho_U(x).$$

Proof: We need only show that (3.9) implies convergence of the

$Q$-matrix. Focus attention on any $(i,j) \neq (i_0,j_0)$. Pick a convergent

subsequence $n_k$ and suppose $Q_{ij}^{n_k}(a_{n_k}x + b_{n_k}) \overset{w}{\to} U_{ij}(x)$. We wish to

identify $U_{ij}(x)$ and so we select a further subsequence $n'_k$ such that

$$H_i^{n_k}(a_{n_k}x + b_{n_k}) \overset{w}{\to} \phi_i(x), \quad 1 \leq i \leq m; \quad \phi_i(x) \text{ is a mass function.}$$

Hence $Q_{ij}^{n'_k}(a_{n'_k}x + b_{n'_k}) \overset{w}{\to} \prod_{i=1}^m \pi_i(x)$ by Lemma (2.18), which identifies

$$U_{ij}(x) = \prod_{i=1}^m \pi_i(x)\pi_j. \quad \text{But } [\prod_{i=1}^m \pi_i(x)\pi_j]_{i_0j_0} = U_{i_0j_0}(x) \text{ and}$$

therefore $[\prod_{i=1}^m \pi_i(x)]_{i_0j_0} = \pi^{-1}_{i_0j_0} U_{i_0j_0}(x)$; this is a nondegenerate, honest

probability distribution function, since the convergence in (3.9) is complete.

So $\lim_{k \to \infty} Q_{ij}^{n_k}(a_{n_k}x + b_{n_k}) = U_{ij}(x) = [\pi^{-1}_{i_0j_0} U_{i_0j_0}(x)]\pi_j$. Since this holds for

any convergent subsequence

$$\lim_{n \to \infty} Q_{ij}^n(a_nx + b_n) = [\pi^{-1}_{i_0j_0} U_{i_0j_0}(x)]\pi_j. \quad \text{The pair } (i,j) \text{ is arbitrary,}$$

which completes the proof.
Our results are related to those of Gnedenko by the following theorem.

**Theorem (3.10):** There exist norming constants \( a_n > 0, b_n, n \geq 1 \) such that
\[
P[a_n^{-1}(M_n - b_n) \leq x] \xrightarrow{c} \rho_U(x)
\]
where \( \rho_U(x) \) is a nondegenerate distribution function iff
\[
Q_0^n(a_n x + b_n) \xrightarrow{c} \rho_U(x)_{\infty}.
\]
Hence \( \rho_U(x) \) is an extreme value distribution and the only possible limiting distributions for the sequence \( \{M_n\} \) are the extreme value types.

**Proof:** Given the convergence of the \( Q \)-matrix, the desired result follows from (3.1) and (3.6).

Now we suppose that
\[
\lim_{n \to \infty} P[a_n^{-1}(M_n - b_n) \leq x] = \rho_U(x).
\]
For some initial distribution \( (p_i) \), \( i = 1, \ldots, m \) we have from (1.1) that
\[
\lim_{n \to \infty} P[a_n^{-1}(M_n - b_n) \leq x] = \lim_{n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{m} Q_{ij}^n(a_n x + b_n) p_i = \rho_U(x).
\]

By the weak compactness theorem for S.M.M.'s we can select a subsequence \( n_k \) such that, for some limit:
\[
\mathbb{U}(x) = \{U_{ij}(x)\}, \lim_{k \to \infty} \{Q_{ij}^{n_k}(a_{n_k} x + b_{n_k})\} = \{U_{ij}(x)\}.
\]
We will identify \( \{U_{ij}(x)\} \). From (3.11) we have:
\[
\sum_{k=1}^{m} \sum_{\ell=1}^{m} U_{k\ell}(x)p_k = \rho_U(x).
\]
There exists a further subsequence \( n'_k \) such that
\[
H_{n'_k}^{n_k}(a_{n_k} x + b_{n_k}) \xrightarrow{w} \phi_i(x)
\]
with the \( \phi_i(x) \) mass functions. We have
\[
Q_0^{n'_k}(a_{n'_k} x + b_{n'_k}) \xrightarrow{c} \mathbb{U}(x)\]
also \( Q_n^k(a_n, x + b_n) \rightarrow \prod_{i=1}^m \phi_i^{\pi_i(x)} \) by (2.18).

So \( U_{ij}(x) = [\prod_{i=1}^m \phi_i^{\pi_i(x)}]^{n_{ij}} \) and from (3.12)

\[
\rho_{U}(x) = \sum_{k=1}^m \sum_{i=1}^m [\prod_{i=1}^m \phi_i^{\pi_i(x)}]^{n_{ij}} p_k = \prod_{i=1}^m \phi_i^{\pi_i(x)} .
\]

Therefore \( U_{ij}(x) = \rho_{U}(x) \pi_j \) and \( Q_n^k(a_n x + b_n) \rightarrow \rho_{U}(x) \). Since this holds for any convergent subsequence we have \( Q_n^k(a_n x + b_n) \xrightarrow{c} \rho_{U}(x) \).

By (3.1) \( \rho_{U}(x) \) is an extreme value distribution.

Criteria for the existence of a limiting distribution for \( \{M_n\} \) are given in

**Theorem (3.13):** There exist constants \( a_n > 0 \), \( b_n \), \( n \geq 1 \) such that:

\[
Q_n^k(a_n x + b_n) \xrightarrow{c} \rho_{U}(x) \quad \text{where } \rho_{U}(x) \text{ is a nondegenerate (extreme value) distribution function}
\]

or:

\[
\prod_{i=1}^m H_i(a_n x + b_n) \xrightarrow{c} \rho_{U}(x) . \quad \text{It follows that } M_n \text{ has a limiting extreme value distribution } \rho_{U}(x) \iff \rho(x) \text{ or equivalently}
\]

\[
\prod_{i=1}^m H_i(x) \text{ are in the domain of attraction of } \rho_{U}(x) .
\]

**Proof:** Given (3.14), the latter two statements follow from theorem (3.1).
Assuming (3.15), there are two cases:

Case I: If \( \rho(x) < 1 \), \( x < \infty \), (3.15) implies \( \rho(a_n x + b_n) \to 1 \), \( n \to \infty \), for all \( x \) such that \( \rho(x) > 0 \) and \( a_n x + b_n \to \infty \). Hence

\[
\lim_{n \to \infty} M_{ij}(a_n x + b_n) = \pi_j. \quad \text{Therefore}
\]

\[
\lim_{n \to \infty} \theta^n(a_n x + b_n) = \lim_{n \to \infty} [\rho^n(a_n x + b_n)M(a_n x + b_n) + \theta(1)] \quad \text{and}
\]

\[
\lim_{n \to \infty} \theta^n(a_n x + b_n) = \rho(x)^n. \quad \text{Therefore}
\]

Case II: There exists \( x_0 < \infty \) such that \( \rho(x_0) = 1 \) and \( \rho(x_0 - \varepsilon) < 1 \) for all \( \varepsilon > 0 \). For a fixed \( x \) such that \( \rho(x) > 0 \), suppose \( a_n x + b_n > x_0 \) for only finitely many \( n \), then for \( n \) sufficiently large \( a_n x + b_n \leq x_0 \).

In fact \( a_n x + b_n \to x_0 \) as \( n \to \infty \). To show this, suppose there is a subsequence \( n_k \) with \( a_{n_k} x + b_{n_k} \to x' < x_0 \) as \( k \to \infty \).

Then for some \( \varepsilon > 0 \), \( x' < x_0 - \varepsilon \). Now \( \lim_{n \to \infty} \rho(a_n x + b_n) = 1 \) [3. p. 439] and

\[
\lim_{k \to \infty} \rho(a_{n_k} x + b_{n_k}) = 1. \quad \text{But}
\]

\[
\lim_{k \to \infty} \rho(a_{n_k} x + b_{n_k}) \leq \rho(x') \leq \rho(x_0 - \varepsilon) < 1
\]

yielding a contradiction. There are no subsequential limits less than \( x_0 \) and hence \( a_n x + b_n \to x_0 \). Thus \( \rho(a_n x + b_n) \to \rho(x_0 - \varepsilon) \) and since also \( \rho(a_n x + b_n) \to 1 \), \( \rho(x_0 - \varepsilon) = 1 = \rho(x_0) \) and \( \rho(\cdot) \) is continuous at \( x_0 \).

So \( Q(\cdot) \), \( r(\cdot) \), \( \theta(\cdot) \), \( M(\cdot) \) are all continuous at \( x_0 \) (2.8-2), 2.10-2) and

\[
\lim_{n \to \infty} M_{ij}(a_n x + b_n) = \pi_j. \quad \text{Therefore}
\]
\[
\lim_{n \to \infty} \mathcal{Q}_n(a_n x + b_n) = \lim_{n \to \infty} \mathcal{Q}_n(a_n x + b_n) \mathcal{M}(a_n x + b_n) + o(1) \]
and
\[
\lim_{n \to \infty} \mathcal{Q}(a_n x + b_n) = \rho_U(x) \mu_c.
\]

Suppose \(a_n x + b_n > x_0\) for infinitely many \(n\), then \(\rho_U(x) = 1\)
and \(\mathcal{Q}_n(a_n x + b_n) = \mu^n\) for such \(n\). If \(a_n x + b_n \leq x_0\) for only finitely
many \(n\), then \(\lim_{n \to \infty} \mathcal{Q}_n(a_n x + b_n) = \mu_c\), as was to be proved. If \(a_n x + b_n \leq x_0\)
for infinitely many \(n\), then we partition the set of positive integers into
sets \(\{n_1\}\) and \(\{n_2\}\) such that \(a_{n_1} x + b_{n_1} \leq x_0\) for all \(n_1\) and
\(a_{n_2} x + b_{n_2} > x_0\) for all \(n_2\). As above \(a_{n_1} x + b_{n_1} \to x_0\) as \(n_1 \to +\infty\)
and \(\mathcal{M}(\cdot)\) is continuous at \(x_0\), so
\[
\lim_{n_1 \to \infty} \mathcal{Q}_n(a_{n_1} x + b_{n_1}) = \lim_{n_1 \to \infty} \mathcal{Q}_{n_1}(a_{n_1} x + b_{n_1}) \mathcal{M}(a_{n_1} x + b_{n_1}) + o(1) \] and
\[
\lim_{n_1 \to \infty} \mathcal{Q}_{n_1}(a_{n_1} x + b_{n_1}) = \mu_c. \] Since \(\mathcal{Q}_{n_2}(a_{n_2} x + b_{n_2}) = \mu_c\) for all \(n_2\), we have
\[
\lim_{n \to \infty} \mathcal{Q}_n(a_n x + b_n) = \mu_c\] as was to be shown.

Now assume (3.16). By the weak compactness theorem for S.M.M.'s
we can select a convergent subsequence \(n_k\) such that
\[\{U_{ij}(a_{n_k} x + b_{n_k})\} \to \{U_{ij}(x)\}.\] To identify \(U_{ij}(x)\) as \(\rho_U(x)^{n_j}\),
we select a further subsequence \(n'_k\) such that for
\[1 \leq i \leq m, H_i (a_{n_k} x + b_{n_k}) \to \phi_i(x)\] with the \(\phi_i(x)\) mass functions,
and therefore
\[ Q^n_k(a_{n_k} x + b_{n_k}) \rightarrow [ \prod_{i=1}^{m} \pi_i(x)]_{n_k} \text{ by (2.18)}. \] But

\[ \prod_{i=1}^{m} \pi_i(a_{n_k} x + b_{n_k})^{n_k} \rightarrow \prod_{i=1}^{m} \pi_i(x) \text{ and also} \]

\[ \prod_{i=1}^{m} H_i \pi_i(a_{n_k} x + b_{n_k})^{n_k} \rightarrow \rho_U(x) \text{ so } \prod_{i=1}^{m} \pi_i(x) = \rho_U(x) \text{ and} \]

\[ \{Q^n_{ij}(a_{n_k} x + b_{n_k})\} \rightarrow \{U_{ij}(x)\} = \rho_U(x)^n. \]

This holds for all convergent subsequences, and hence for the full sequence.

Remark: Minor difficulties of a technical nature arise when \( \rho \) may be reducible and/or periodic. The details are forthcoming.