ON CLASSICAL AND COMPLEX MULTIVARIATE
NORMAL DISTRIBUTION PROBLEMS *

by

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</table>
Page 42 Line 16 Change $z_1$

Page 44 Line 1, 2 Change $f_1 = \frac{2}{3} \cdot \frac{1}{4}$

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Page 45 Line 18 Change $f_1^* = n \cdot \frac{1}{2}$

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Page 58, 58 Line 28 Change $(y^2 - \Sigma_1^2)^{-1} \log V$ → $(v^2 - \Sigma_3^2)^{-1} \log V$

Page 63 Line 16 Change $N(H_2, \Sigma_2)$

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Page 70 Line 3 Change $\Sigma_1 \Sigma_2$

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Page 71 Line 10 Change $\Sigma_1 \Sigma_2^{-1} \Sigma_3^{-1}$

Page 83 Line 3 Change $|\beta_{4+4-2}|$
CHAPTER I

ON THE MOMENTS OF ELEMENTARY SYMMETRIC FUNCTIONS

OF THE ROOTS OF TWO MATRICES

1. Introduction and Summary

Let $\underline{A}_1$ and $\overline{A}_2$ be two symmetric matrices of order $p$, $\underline{A}_1$, positive definite and having a Wishart distribution [3], [33], with $f_1$ degrees of freedom and $\overline{A}_2$, at least positive semi-definite and having a non-central (linear) Wishart distribution [2], [4], [14], [33], [34] with $f_2$ degrees of freedom. Now let

$$\underline{A}_2 = \underline{C} \underline{Y} \underline{Y}' \underline{C}'$$

where $\underline{Y}$ is $p \times f_2$ and $\underline{C}$ is a lower triangular matrix such that

$$\underline{A}_1 + \underline{A}_2 = \underline{C} \underline{C}'$$

Now consider the $s (= \min (f_2, p))$ non-zero characteristic roots of the matrix $\underline{Y} \underline{Y}'$. It can be shown that the density function of the characteristic roots of $\underline{Y} \underline{Y}'$ for $f_2 \leq p$ can be obtained from that of the characteristic roots of $\underline{Y} \underline{Y}'$ for $f_2 > p$ if in the latter case the following changes are made [16], [33];

$$(1.1) \quad (f_1, f_2, p) \rightarrow (f_1 + f_2 - p, p, f_2)$$
Now define $U_i^{(p)} = \text{tr}_i(I_p - \tilde{\Lambda} \tilde{Y})^{-1} - p = \text{tr}_i(I_p - \tilde{Y} \tilde{Y}^{-1} - p_2$, where $\text{tr}_i\tilde{A}$ denotes the $i$th elementary symmetric function (esf) of the characteristic roots of $\tilde{A}$. In view of (1.1) we only consider $U_i^{(s)}$ when $s = p$, i.e. $U_i^{(p)}$ based on the density function [24] of $\tilde{L} = \tilde{Y} \tilde{Y}'$ for $p_2 \geq p$. Now define $V_i^{(p)} = \text{tr}_i\tilde{L}$ and further $\tilde{U} = (I_p - \tilde{Y} \tilde{Y}')^{-1} - p$. Khatri and Pillai [23] have obtained some results towards finding the moments of $U_i^{(p)}$ and $V_i^{(p)}$ and in this paper an attempt is made to give general expressions of the first three moments of $U_i^{(p)}$ and the first two moments of $V_i^{(p)}$. Further, the moments of the second esf of a matrix in the non-central means case (James [12]) have been considered and tabulation of certain constants made which arose in this context.

2. Results on the $i$th esf of the Roots of a Matrix

The lemma below is proved by Khatri and Pillai [23] and is used to obtain the results of Section 3.

Lemma: Let $L = \begin{pmatrix} L_{11} & L'_{12} \\ L_{21} & L_{22} \end{pmatrix}_{p \times p}$ be a symmetric matrix of order $p$, $L_{22} = L_{11} - L_{12}' / L_{11}$, $L_{p-1} - L_{22}$ be positive definite and $u = (L_{p-1} - L_{22})^{-\frac{1}{2}} / \{L_{11}(1-L_{11})\}^{\frac{1}{2}}$. Further let $\tilde{U} = (\tilde{L}_p - \tilde{L})^{-1} - \tilde{L}_p$ and $M = (\tilde{L}_p - \tilde{L}_p)^{-1} - \tilde{L}_p$. Then
\[
\text{tr}_i U = \lambda_{11}^{-1} \left( (1-\eta_{11})(1-\eta^*_{11}) \right)^{i-1} \text{tr}_{i-1} M + \text{tr}_i M \\
+ (1-\eta^*_{11})^{-1} \sum_{j=0}^{i-1} (-1)^j \eta^*_{11} (M^{j+1} + M^{j+1}) \eta (\text{tr}_{i-1-j} M) \text{ for } i < p \\
= \lambda_{11}^{-1} \left( (1-\eta_{11})(1-\eta^*_{11}) \right)^{-1} |M| \text{ for } i = p.
\]

Notice that the distributions of $\lambda_{11}$, $\eta$ and $L_{22}$ are available in [21], [22] except that the non-centrality parameter, which is involved in the density of $\lambda_{11}$ above, will be denoted here by $\lambda$ in place of $2\lambda^2$ given there.

3. Moments of $\text{tr}_i U$

Let $U_0$ be a $U$ matrix when $\lambda = 0$, let $L_{11,0}$ be the top left corner element of $L_0$ (L matrix where $\lambda = 0$) and let

\[(3.1) \quad y_1 = E(1-\eta^*_{11})^{-1} \left[ E[\lambda_{11}/(1-\lambda_{11})] - E[\lambda_{11,0}/(1-\lambda_{11,0})] \right] = \lambda/(a-1),\]

\[(3.2) \quad y_2 = E(1-\eta^*_{11})^{-2} \left[ E[\lambda_{11}/(1-\lambda_{11})]^2 - E[\lambda_{11,0}/(1-\lambda_{11,0})]^2 \right] = \left[2(\lambda^2+2)\lambda + \lambda^2\right]/\left[(a-1)(a-3)\right],\]

\[(3.3) \quad y_3 = E(1-\eta^*_{11})^{-3} \left[ E[\lambda_{11}/(1-\lambda_{11})]^3 - E[\lambda_{11,0}/(1-\lambda_{11,0})]^3 \right] = \left[3(\lambda^2+2)(\lambda^2+4)\lambda + 3(\lambda^2+4)\lambda^2 + \lambda^3\right]/\left[(a-1)(a-3)(a-5)\right],\]

and
(3.4) \[ y_4 = E(1-u^i u) E[\lambda_i (1-\lambda_i)]^4 - E[\lambda_{i,0} (1-\lambda_{i,0})]^{14} \]
\[ = \{4(f_2+2)(f_2+4)(f_2+6)\lambda + 6(f_2+4)(f_2+6)\lambda^2 + 4(f_2+6)\lambda^3 + \lambda^4 \} / \{(a-1)(a-3)(a-5)(a-7)\} \]

where \( a = f_1 - p \).

Now let

\[ \beta_i = tr_{i-1} M \quad \text{and} \quad \alpha_i = tr_{i-1} M + \sum_{j=0}^{i-1} (-1)^j (1-u^i u)^{j+1} M^{j+1} \]

Then

(3.5) \[ E[tr_{i-1}] = E[tr_{i-1} U] + y_1 E \beta_1 \]

(3.6) \[ E[tr_{i-1}]^2 = E[tr_{i-1} U]^2 + y_2 E \beta_1^2 + 2y_1 E \beta_1 \alpha_1 \]

(3.7) \[ E[tr_{i-1}]^3 = E[tr_{i-1} U]^3 + y_3 E \beta_1^3 + 3y_2 E \beta_1^2 \alpha_1 + 3y_1 E \beta_1 \alpha_1^2 \]

and

(3.8) \[ E[tr_{i-1}]^4 = E[tr_{i-1} U]^4 + y_4 E \beta_1^4 + 4y_3 E \beta_1^3 \alpha_1 + 6y_2 E \beta_1^2 \alpha_1^2 + 4y_1 E \beta_1 \alpha_1^3 \]
In order to evaluate the right sides of (3.5) - (3.8), it appears that general results are obtainable in terms of functions of esf's of latent roots of $M$. Hence we suggest the following general form for $E \beta_i \alpha_i$

$$E \beta_i \alpha_i = \frac{1}{a-1} \ E[tr_{i-1}M((p-1)tr_{i-1}M + (a+i-1)tr_iM)] .$$

The above result as well as others in this section and the next have been suggested by computing special cases for $i = 1, 2, 3, 4$ and further checking the result for $i = 5$.

Similarly

$$E \beta_i^2 \alpha_i = \frac{1}{a-1} \ E[ (tr_{i-1}M)^2 ((p-i)tr_{i-1}M + (a+i-1)tr_iM)] ,$$

and

$$E \beta_i^3 \alpha_i = \frac{1}{(a-1)(a-3)} \ E[tr_{i-1}M(p-i)(p-i+2)(tr_iM)^2] + 2[(a+i-3)(p-1)+2]tr_{i-1}Mtr_iM + (a+i-3)(a+i-1)$$

$$\cdot (tr_iM)^2 + \sum_{k=0}^{i-1} \sum_{j=2i-2-k}^{2i-k} a_{kj} tr_kMtr_jM] ,$$

where
\[ a_{kj} = \begin{cases} 
0 & \text{if } j-k \leq 1 \\
-2(j-k) & \text{if } j-k > 1 \text{ and } j-k \text{ is even} \\
4(j-k) & \text{if } j-k > 1 \text{ and } j-k \text{ is odd} 
\end{cases} \]

Now noting that \( E[\text{tr}_i U_j] \), \( E[\text{tr}_i U_j]^2 \) and \( E[\text{tr}_i U_j]^3 \) are available in Pillai [27], [28], using (3.9) - (3.11) in (3.5) - (3.7) and the fact that \( E \beta_i^j = E(\text{tr}_i M)^j \), we can obtain the first three moments of \( U_i^p = \text{tr}_i ((I-L)^{-1} - I) \) (which are suggested based on computations for \( i = 1, 2, 3, 4, 5 \)). Expected values of functions of \( \text{tr}_i M \) can be obtained in individual cases by use of zonal polynomials [13] or by Pillai's lemma on multiplication of a basic Vandermonde determinant by monomials of esf's [27].

4. Moments of \( \text{tr}_i L \)

Khatri and Pillai have shown [23] that

\begin{align*}
E[\text{tr}_i L] &= E[\text{tr}_i L] + x_1 E \beta_1(i), \\
E[\text{tr}_i L]^2 &= E[\text{tr}_i L]^2 - x_2 E \beta_1^2(i) + 2x_1 E \alpha_1(i) \beta_1(i), \\
E[\text{tr}_i L]^3 &= E[\text{tr}_i L]^3 + x_3 \beta_1^3(i) - 3x_2 E \beta_1(i) \alpha_1(i) \\
&\quad + 3x_1 E \beta_1(i) \alpha_1^2(i),
\end{align*}

and
\[(4.4) \quad E[\text{tr}_i L_i]^{4} = E[\text{tr}_i L_i]^{4} - x_1 \beta_{1(i)} + 4x_3 \beta_{1(i)} \alpha_{1(i)} - 6x_2 \beta_{1(i)} \alpha_{1(i)}^2 - 4x_1 \beta_{1(i)} \alpha_{1(i)}^3\]

where \(x_1, x_2, x_3, x_4, \alpha_{1(i)}, \beta_{1(i)}\) and \(L_i\) are defined in [23] and are functions similar to \(y_i\)'s, \(\alpha_i\)'s and \(\beta_i\)'s in the preceding section.

Using the results of Section 2 of [22] and further computing as in the previous section we get

\[(4.5) \quad E[\beta_{1(i)}] = \frac{1}{f_{1}} E[(a+i) \text{tr}_{i-1} L_{22} + i \text{tr}_{i-1} L_{22}]\]

where \(a = f_{1} - p\) and \(\text{tr}_{i-2} L_{22} = 1\).

Similarly

\[(4.6) \quad E[\alpha_{1(i)} \beta_{1(i)}] = \frac{1}{f_{1}} E[(a+2i) \text{tr}_{i-1} L_{22} \text{tr}_{i} L_{22} + (a+i)(\text{tr}_{i-1} L_{22})^2 + i(\text{tr}_{i} L_{22})^2]\]

and

\[(4.7) \quad E[\beta_{1(i)}^2] = \frac{1}{f_{1}(f_{1} + 2)} E[(a+i)(a+i+2)(\text{tr}_{i-1} L_{22})^2 + [2(a+i+1) + 2(i-2)] \text{tr}_{i-1} L_{22} \text{tr}_{i} L_{22} + i(i+2)(\text{tr}_{i} L_{22})^2 + \sum_{k=0}^{i-1} \sum_{j=2i-2-k} a_{kj} \text{tr}_{k-2} L_{22} \text{tr}_{j} L_{22}]\]

where
\[(4.4) \quad E[\text{tr}_{i-1}L_i]_{\infty}^4 = E[\text{tr}_{i-1}L_i]_{\infty}^4 - x_4 \beta_{i-1}^h + 4x_4 \beta_{i-1}^h \alpha_{i-1}^h - 6x_4 \beta_{i-1}^h \alpha_{i-1}^h + 4x_4 \beta_{i-1}^h \alpha_{i-1}^h , \]

where \(x_1, x_2, x_3, x_4, \alpha_{i-1}, \beta_{i-1}\) and \(L_\infty\) are defined in [23] and are functions similar to \(y_1\)'s, \(\alpha_1\)'s and \(\beta_1\)'s in the preceding section.

Using the results of Section 2 of [22] and further computing as in the previous section we get

\[(4.5) \quad E[\beta_{i-1}] = \frac{1}{L_i} E[(a+i)\text{tr}_{i-1}L_{22} + i \text{tr}_{i-1}L_{22}] , \]

where \(a = f_{i-1} - p\) and \(\text{tr}_{\alpha_{22}} = 1\).

Similarly

\[(4.6) \quad E[\alpha_{i-1}^2 \beta_{i-1}] = \frac{1}{L_i} E[(a+2i)\text{tr}_{i-1}L_{22} \text{tr}_{i-1}L_{22} + (a+i)(\text{tr}_{i-1}L_{22})^2 \]

\[+ i(\text{tr}_{i-1}L_{22})^2 ] \]

and

\[(4.7) \quad E[\beta_{i-1}^2] = \frac{1}{L_i(f_{i-1}+2)} E[(a+i)(a+i+2)(\text{tr}_{i-1}L_{22})^2 \]

\[+ [2i(a+i+1) + 2(i-2)]\text{tr}_{i-1}L_{22} \text{tr}_{i-1}L_{22} + i(i+2)(\text{tr}_{i-1}L_{22})^2 \]

\[= \sum_{i=1}^{2i-2} \sum_{j=2i-2-k}^{2i} a_{kj} \text{tr}_{i-1}L_{22} \text{tr}_{i-1}L_{22} , \]

where
\[ a_{kj} = \begin{cases} 
0 & \text{if } j-k \leq 1 \\
2(j-k) & \text{if } j-k > 1 \text{ and even} \\
4(j-k) & \text{if } j-k > 1 \text{ and odd} 
\end{cases} \]

Now noting that \( E[\text{tr}_1L_0] \) and \( E[\text{tr}_1L_0]^2 \) are available in Pillai [27], [28] and using the above results, we can obtain the first two moments of \( V_i(p) = \text{tr}_1L \ (4.5) - (4.7) \) being suggested based on computations for \( i = 1, 2, 3, 4 \) and 5). Further expected values of functions of \( \text{tr}_iL_{22} \) can be obtained by methods suggested at the end of the preceding section.

5. Moments of the Second esf of a Matrix

Let \( X: \sim p \times f \) be a matrix variate (\( p \leq f \)) whose columns are independently normally distributed with \( E(X) = M \) and covariance matrix \( \Sigma \). Let \( w_1, \ldots, w_p \) be the characteristic roots of \( |X X' - w \Sigma| = 0 \), then the distribution of \( \sim W = \text{diag} (w_i) \) is given by James [12], [13]

\[
(5.1) \quad e^{-\frac{1}{2}\text{tr} \Omega} F_1\left(\frac{1}{2}I; \sim \Omega, \sim \right) e^{-\frac{1}{2}\text{tr} W} |W|^\frac{1}{2}(f-p-1) \Pi_{i>j} (w_i - w_j)
\]

\[ 0 < w_1 \leq \ldots \leq w_p < \infty \]

where

\[
(5.2) \quad \kappa(p,f) = \frac{1}{2p^2} / \left\{ 2^{\frac{1}{2}pf} \Gamma_p\left(\frac{1}{2}f\right) \Gamma_p\left(\frac{1}{2}p\right) \right\}
\]

\( \Omega = \text{diag} (w_i) \) where \( w_i, i = 1, \ldots, p \) are the characteristic roots of \( |M M' - w \Sigma| = 0 \), \( F_1 \) is the hypergeometric function of matrix argument and \( \Gamma_p(\cdot) \) is the multivariate gamma function defined in [13]. Now define \( w_{(2)}(p) \) as the second esf in \( \frac{1}{2}w_1, \ldots, \frac{1}{2}w_p \). Then from Gupta [11] we have
\[
E[W_2^3] = \frac{7}{64} L^{(3^2)} + \frac{1}{12} L^{(321)} + \frac{57}{320} L^{(2^21^2)} + \frac{3}{40} L^{(31^3)} + \frac{1}{8} L^{(2^3)}
+ \frac{9}{40} L^{(21^4)} + \frac{27}{64} L^{(1^6)}
\]

where \( L_k \) represents \( L_k(-2\Omega) \), which is the generalized Laguerre polynomial of the form [6], [13]

\[
L_k(-2\Omega) = (\nu + \frac{1}{2}(p+1))_k c_k(I) \sum_{n=0}^{k} \sum_{\nu} (-1)^n \left[ \frac{\alpha_{k,\nu}}{(\nu+\frac{1}{2}(p+1))_{\nu}} \right] c_{\nu}(-2\Omega)
\]

Pillai and Gupta [32] have evaluated the first two moments of \( \bar{W}_2(p) \) using \( a_{k,\nu} \) coefficients for \( k = 2, 4 \) available in [6].

Here we evaluate the third moment in (5.3) using the table of \( a_{k,\eta} \) coefficients presented in the next section.

\[
E[W_2^3] = I_3(0) \{ W_2(p) \} + \sum_{i=1}^{4} \sum_{j=1}^{3} \sum_{k=1}^{3} \sum_{l=0}^{2} a_{ijkl} b_i^k b_j^l
\]

where
\[ a_{1210} = 12\mu_3^{(0)}[W^{(p)}] / c_0, \quad a_{1220} = \frac{c_2 \{ c_2 \{ (342c_4 + 70c_0) + c_1 \{ (3d_2 + 175c_4) \} + 6d_2 \}}{13440}, \quad a_{2110} = \frac{c_2 \{ 4c_4 (175c_4 + 27c_1 + 25c_0) + 3c_1 d_2 \}}{40320}, \]
\[ a_{2120} = \frac{c_2 \{ 4d_0 + 290c_2 + 504c_1 \} + 7c_7 \{ 7c_7 - 8c_1 \}}{840}, \quad a_{3110} = \frac{c_7 \{ 5c_7 + 345c_4 + 135c_4 (39c_2 + 20c_5 - c_2 d_9 + 1120c_2 c_2) \}}{16800}, \]
\[ a_{1230} = \frac{c_2 d_3 + c_1 d_4 + 1120d_3 + 400d_5 + 23712c_3 + 7182c_2 + 35c_4 \{ 7c_3 - 184c_3 \}}{12600}, \quad a_{2311} = 18, \quad a_{1212} = \frac{(3/2)c_1 + 6}{(6c_1 - 1)}, \quad a_{1221} = \frac{c_2 \{ (d_2 + 2520c_1) \}}{25200}, \quad a_{1211} = \frac{c_2 \{ (18c_4 - 75c_0) \}}{2520}, \quad a_{1211} = \frac{c_2 \{ (805c_4 + 170c_4 + 2964c_1 - 2980c_2) \}}{25200}, \]
\[ a_{1311} = \frac{c_2 \{ (805c_4 + 170c_4 + 2964c_1 - 2980c_2) + 10c_4 \}}{25200}, \]
and all other \( a_{ijkl} = 0, \quad c_\alpha = (f + \alpha)(p + \alpha), \quad \mu_3^{(0)}[W^{(p)}] \) is the third moment in the central case [5] with \( 2m = f - p + 1 \) and

\[
\begin{align*}
\text{d}_3 &= 7c_1 c_3 c_4, \quad \text{d}_5 &= 19c_2 - 7c_5, \quad \text{d}_2 &= c_2 - c_1, \quad \text{d}_4 &= 35c_0 + 99c_4, \quad \text{d}_3 &= 3c_1 - 9c_2 \\
\text{d}_9 &= 6840c_1 + 1995c_4 + 2835c_4, \quad \text{d}_2 &= c_2 - 3c_1, \quad \text{d}_0 &= c_0 + 18c_4, \quad \text{d}_4 &= 152c_1 + 63c_4.
\end{align*}
\]
6. Results for $a_{\kappa, \tau}$

The $a_{\kappa, \tau}$'s are constants \([6]\) satisfying the equality

\[
(6.1) \quad C_k(A + I) / C_k(I) = \sum_{\tau=0}^{k} \sum_{\tau} a_{\kappa, \tau} C_\tau(A)/C_\tau(I)
\]

where $\tau$ is a partition of $t$. The following are suggested based on the available results. For $\kappa = k-j, j^d$

\[
(6.2) \quad a_{\kappa, \tau} = \begin{cases} 
  j(2k-(j+1))/(2k-(j+2)) & \text{if } \tau = k-j, j^{d-1} \\
  (2k-j)(k-(j+1))/(2k-(j+2)) & \text{if } \tau = k-j-1, j^d 
\end{cases}
\]

Also for $\kappa = k-j, j, k \geq 2j$

\[
(6.3) \quad a_{\kappa, \tau} = \begin{cases} 
  j(2k-(4j-2))/(2k-(4j-1)) & \text{if } \tau = k-j, j-1 \\
  (k-2j)(2k-(2j-1))/(2k-(4j-1)) & \text{if } \tau = k-j-1, j 
\end{cases}
\]

For $\kappa = k, \tau = k-j$

\[
(6.4) \quad a_{\kappa, \tau} = k! / (j!(k-j)!) .
\]

As previously stated the $a_{\kappa, \tau}$ for $k = 1, 2, 3, 4$ are available in \([6]\) and for $k = 5, 6, 7, 8$ now follow.
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* \( a_{k, \tau} = 1 \) when \( \tau = k \)
7. Further Uses of $a_{k,r}$

Pillai [29] has shown that

\begin{equation}
E[e^{t \text{trL}}] = e^{-\frac{1}{2}t\Omega} \sum_{k=0}^{\infty} \sum_{n=0}^{k} \left(\frac{1}{2}\nu f_2\right)_k \left(\frac{1}{2}\nu\right)_n a_{k,n} t^{k-n} c_k(I) c_n(\Omega) \frac{(1/2\nu)_k (1/2\nu f_2)_n}{(1/2\nu)_n (1/2\nu f_2)_k n! c_{n}(I)} \right)
\end{equation}

where \( \Omega, \nu, f_1 \) and \( f_2 \) are defined in [29].

From (7.1) we get the moments of $\text{trL}$ by differentiation with respect to $t$ and letting $t = 0$. Thus

\begin{equation}
\frac{\partial^r}{\partial t^r} E[e^{t \text{trL}}] = e^{-\frac{1}{2}t\Omega} \sum_{k=0}^{\infty} \sum_{n=0}^{k} \left(\frac{1}{2}\nu f_2\right)_k \left(\frac{1}{2}\nu\right)_n a_{k,n} \cdot \frac{(k-n)(k-n-1)...(k-n-r+1)t^{k-n-r} c_k(I) c_n(\Omega)}{(1/2\nu)_k (1/2\nu f_2)_n n! c_n(I)}
\end{equation}

and hence

\begin{equation}
E[\text{trL}]^r = e^{-\frac{1}{2}t\Omega} \sum_{k=0}^{\infty} \sum_{n=0}^{k} \left(\frac{1}{2}\nu f_2\right)_k \left(\frac{1}{2}\nu\right)_n a_{k,n} \frac{r! c_k(I) c_n(\Omega)}{(1/2\nu)_k (1/2\nu f_2)_n n! c_n(I)}
\end{equation}

where $\eta$ is a partition of $n = k - r$.

Pillai [29] also gives in the case of canonical correlation

\begin{equation}
E[e^{t \text{trR}^2}] = E[e^{t \text{trL}}] \left|_{r=2} \right.
\end{equation}

\begin{equation}
E[\text{trR}^2] = E[\text{trL}] \left|_{r=2} \right.
\end{equation}

from which, as before, we obtain the $r$th moment,
where $\eta$ is as above, and $X^2$ and $F^2$ are defined in [29].

Further, Khatri [18] has obtained the moment generating function of $V(p)$ associated with the test $\lambda \sum_1^2 = \sum_2^2$ as

$$E[e^{tV(p)}] = [\lambda\lambda]^{-\frac{1}{2}t} \sum_{k=0}^{\infty} \sum_{K} \sum_{n=0}^{\infty} \sum_{\eta} \frac{(\frac{1}{2}t)_K (\frac{1}{2}t)_\eta a_{K,\eta}^{-1} \eta^{k-n} c_K (\eta) c_{\eta} (\eta^{-1})}{(\frac{1}{2}t)_K k! c_{\eta} (\eta^{-1})}.$$  

We get the $r$th moment of $V(p)$ in this case as

$$E[V(p)^r] = [\lambda\lambda]^{-\frac{1}{2}tr} \sum_{k=0}^{\infty} \sum_{K} \sum_{\eta} \frac{(\frac{1}{2}t)_K (\frac{1}{2}t)_\eta a_{K,\eta}^{-1} \eta^{r} c_K (\eta) c_{\eta} (\eta^{-1})}{(\frac{1}{2}t)_K k! c_{\eta} (\eta^{-1})},$$

where $\eta$ is a partition of $n = k-p$ and $\lambda$ is defined in [18], [29].
CHAPTER II

ON THE NON-CENTRAL DISTRIBUTIONS OF WILKS' - Λ

FOR TESTS OF THREE HYPOTHESES

1. Introduction and Summary

In multivariate analysis we are interested in testing three hypotheses, namely

1) that of equality of the dispersion matrices of two p-variate normal populations,

2) that of equality of the p-dimensional mean vectors for k p-variate normal populations having a common covariance matrix and

3) that of independence between a p-set and a q-set of variates in a (p+q)-variate normal population, with p ≤ q. We obtain the non-central distribution of Wilks' criterion $\Lambda = W(p) = \sum_{i=1}^{p} (1-c_i)$ in each of the above cases, where the $c_i$'s are functions of the characteristic roots of the appropriate matrices. The density functions for Case 2 have been obtained by Pillai and Al-Ani [30] for p = 2, 3, 4 and here we obtain the density functions for all three cases for general p in terms of Meijer's G-function [25] with special cases being explicitly evaluated. In this connection a theorem has been proved using some results on Mellin transforms [7], [8], [9]. Also the cumulative distribution function (or cdf) of $W(p)$ is obtained for p = 2 in the above three cases. The densities in all cases may be put in a single general form given by
\[(1,1)\quad f(W(p)) = \frac{\Gamma_p(\delta)}{\Gamma_p(\frac{1}{2}Y)} \alpha[W(p)]^\frac{1}{2}(Y-p-1)\]

\[\cdot \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\delta)_k \beta}{k!} \kappa(C_{\kappa}(M))^p, Q(p), p(W(p) | a_1, a_2, \ldots, a_p)_{b_1, b_2, \ldots, b_p}\]

where

\[a_i = \frac{1}{2}(2\delta - \gamma + k_{p-i+1} + b_i) \quad \text{and} \quad b_i = (i-1)/2\]

and

Case 1
\[\gamma = n_2\]
\[\delta = \frac{1}{2}n\]
\[\beta = \left(\frac{1}{2}n_1\right)_\kappa\]
\[\alpha = |\lambda \Lambda|^{-\frac{1}{2}n}\]
\[M = I_p - (\lambda \Lambda)^{-1}\]

Case 2
\[\gamma = t\]
\[\delta = \nu\]
\[\beta = \frac{1}{2}n\]
\[\alpha = e^{-\frac{1}{2}n}\]
\[M = I_p - (\lambda \Lambda)^{-1}\]

Case 3
\[\gamma = n-q\]
\[\delta = \frac{1}{2}n\]
\[\beta = \left(\frac{1}{2}n_1\right)_\kappa\]
\[\alpha = e^{-\frac{1}{2}n}\]
\[M = I_p - (\lambda \Lambda)^{-1}\]

See the following sections for definitions of the parameters as well as the G-function.

2. Preliminary Results

Some results on Mellin transforms [7], [8], [9] and Meijer's G-function [25] useful in proving the theorem below will now be given.

Lemma 1. If \( s \) is any complex variate and \( f(x) \) is a function of a real variable \( x \), such that
(2.1) \[ F(s) = \int_0^\infty x^{s-1} f(x) \, dx \]

exists, then under certain regularity conditions

(2.2) \[ f(x) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} x^{-s} F(s) \, ds . \]

\( F(s) \) is called the Mellin transform of \( f(x) \) and \( f(x) \) is the inverse Mellin transform of \( F(s) \).

**Lemma 2.** If \( f_1(x) \) and \( f_2(x) \) are the inverse Mellin transforms of \( F_1(s) \) and \( F_2(s) \) respectively, then the inverse Mellin transform of \( F_1(s) F_2(s) \) is

(2.3) \[ (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} x^{-s} F_1(s)F_2(s) \, ds = \int_0^\infty f_1(u)f_2(x/u) \, du/u . \]

Meijer [25] defined the \( G \)-function by

(2.4) \[ G_{m,n}^{p,q} (x| a_1, a_2, \ldots, a_p ; b_1, b_2, \ldots, b_q) = (2\pi i)^{-1} \int_C x^s \, ds , \]

\[ (2\pi i)^{-1} \int_C \frac{\Pi_{j=1}^m \Gamma(b_j-s) \Pi_{j=1}^n \Gamma(1-a_j+s)}{\Pi_{j=m+1}^q \Gamma(b_j-s) \Pi_{j=n+1}^p \Gamma(a_j-s)} x^s \, ds , \]

where an empty product is interpreted as unity and \( C \) is a curve separating the singularities of \( \Pi_{j=1}^m \Gamma(b_j-s) \) from those of \( \Pi_{j=1}^n \Gamma(1-a_j+s) \), \( q \geq 1, 0 \leq n \leq p \leq q, 0 \leq m \leq q; x \neq 0 \) and \( |x| < 1 \) if \( q = p; \)
\( x \neq 0 \) if \( q > p \). It is easily verified that

\[
(2.5) \quad g_{2,2}^{2,0}(x | b_1, b_2 | a_1, a_2) = \frac{x^{\frac{1}{2} (1-x)}}{\Gamma(a_1 + a_2, b_1 - b_2)}
\]

\( \text{Beta}(a_2 - b_2, a_1 - b_2; a_1 + a_2, b_1 - b_2; 1 - x) \quad 0 < x < 1 \)

where the generalized hypergeometric function \( \text{Beta} \) is given by James [13].

The \( G \)-function of (2.4) can be expressed as a finite number of generalized hypergeometric functions as follows [26],

\[
G_{p,q}^{m,n}(x | b_1, \ldots, b_q | a_1, \ldots, a_p) = \sum_{h=1}^{m} \frac{\prod_{j=1}^{m} \Gamma(b_j - b_h) \prod_{j=1}^{n} \Gamma(1+b_h - a_j)}{\prod_{j=m+1}^{p} \Gamma(1+b_h - b_j) \prod_{j=n+1}^{q} \Gamma(a_j - b_h)} x^{b_h}
\]

\[
\text{Beta}(1+b_h - a_1, \ldots, 1+b_h - a_p; 1+b_h - b_1, \ldots, 1+b_h - b_q; (-1)^{p-m-n} x)
\]

where the asterisk denotes that the number \( 1+b_h - b_h \) is omitted in the sequence \( 1+b_h - b_1, \ldots, 1+b_h - b_q \). Although the following theorem gives a more complicated form for expressing the \( G \)-function, it is useful in that expression (2.4) of Consul [9] and Lemma 1 of Pillai and Al-Ani [30] are special cases.

**Theorem 1.** If \( x \) is a complex variate, \( a_i, b_i, i = 1,2, \ldots, p \) are reals, then for \( p \geq 3 \)
(2.6) \[ g_{p,0}(x|_{b_1,b_2,\ldots,b_p}) = \]

\[ \frac{x^p(1-x)^{c-1}}{\Gamma(c_1+c_2+c_3)} \prod_{i=1}^{p-3} \left( \sum_{j_i=0}^{\infty} \frac{(b_{i+1}+c_{i+1})_{j_i}}{(j_i)!} \right) \]

\[ \prod_{j=0}^{p-3} \frac{\left( c_1 \right)_{j}}{(c_1+c_2+c_3+j)!} \frac{\left( b_1 \right)_{j}}{(1-x)^{j}} \frac{\left( \Gamma(g_{\ell}+g_{\ell}) \right)}{\Gamma(h_{\ell})} \]

\[ \cdot \prod_{\ell=1}^{p-2} \left( b_{3+\ell}+c_{3+\ell}-b_{1+\ell}, f_1, f_2, \ldots, f_{p-2}; g_1, g_2, \ldots, g_{p-2}; 1-x \right) \]

\[ 0 < x < 1 \]

where for notational convenience \( c_i = a_i - b_i \), \( c = \sum_{i=1}^{p} c_i \),

\( f_\ell = \sum_{i=1}^{\ell+1} c_i + \sum_{i=1}^{\ell-1} j_i + j \), \( g_\ell = \sum_{i=1}^{\ell+2} c_i + \sum_{i=1}^{\ell-1} j_i + j \), \( h_\ell = \sum_{i=1}^{\ell+3} c_i + \sum_{i=1}^{\ell} j_i + j \) and \( (a)_k = a(a+1)\ldots(a+k-1) \).

**Proof.** Using mathematical induction starting with \( p = 3 \), we see making the substitution \( (a,b,c,m,n,p) \rightarrow (b_3,b_2,b_1,c_3,c_2,c_1) \) in (2.4) of Consul [8] that

(2.7) \[ g_{3,0}(x|_{b_1,b_2,b_3}) = \frac{x^3(1-x)^{c_1+c_2+c_3-1}}{\Gamma(c_1+c_2+c_3)} \cdot \sum_{j=0}^{\infty} \frac{(c_1)_{j}(b_2+c_2-b_1)_{j}}{(c_1+c_2+c_3+j)!} \cdot \left( b_3+c_3-b_2, c_1+c_2+j; c_1+c_2+c_3+j; 1-x \right) \]

\[ 0 < x < 1 \]
which is (2.6) with \( p = 3 \). Now assuming (2.6) is true for \( p = n \), we show it holds for \( p = n+1 \). Applying Lemma 2 with

\[
F_1(s) = \frac{\prod_{i=1}^{n} \Gamma(s+a_i)}{\prod_{i=1}^{n} \Gamma(s+a_i)} \quad \text{and} \quad F_2(s) = \frac{\Gamma(s+b_{n+1})}{\Gamma(s+a_{n+1})}
\]

we have \( f_1(x) \) is (2.6) with \( p = n \) and \( f_2(x) = \frac{x^{b_{n+1}}(1-x)^{c_{n+1}-1}}{\Gamma(c_{n+1})} \)

and it follows that

\[
G^{n+1,0}_{n+1,n+1}(x|a_1,a_2,\ldots,a_{n+1};b_1,b_2,\ldots,b_{n+1}) = \frac{x^{b_{n+1}}}{\Gamma(c_1+c_2+c_3)\Gamma(c_{n+1})} \int_0^1 \frac{u^{b_{n+1}-b_{n+1}}}{\Gamma(c_{n+1})} \prod_{i=1}^{n-3} \left( \sum_{j_1=0}^{\infty} \frac{(b_{n-i+1}+c_{n-i+1}+b_{n-i})_j}{(j_1)_j!} \right) \frac{1}{(1-u)^{c_1+c_2+c_3}} \frac{1}{\prod_{j=1}^{n-3} (c_1+c_2+c_3)_j} \sum_{j=0}^{\infty} \frac{(c_1)_j(b_2+c_2-b_1)_j}{(c_1+c_2+c_3)_j} \frac{1}{(1-u)^{p-3} \prod_{j=1}^{p-3} \Gamma(q_j+j_1)} \frac{1}{\prod_{j=1}^{p-3} \Gamma(h_j)} \frac{1}{(u-x)^{c_{n+1}-1}} du.
\]

Expanding \( u^{b_{n+1}-b_{n+1}} \) in powers of \( 1-u \) when \( b_{n+1}+c_{n+1} > b_n \), letting \( u = x+(1-x)t \) and integrating with respect to \( t \), the result is the same as (2.6) with \( p = n+1 \).
It is easily verified that Lemma 1 of Pillai and Al-Ani [30] is a special case of (2.6) with \( p = 4 \) by making the following substitution

\[
(b_1, b_2, b_3, b_4, c_1, c_2, c_3, c_4) \rightarrow (c, b, a, d, p, n, m, \ell).
\]

It should be mentioned that this theorem doesn't apply when \( p = 1,2 \). This is due to the fact that a simplification in the form of the \( G \)-function for \( p = 3 \) reduces the hypergeometric function involved from \( 3F_2 \) to \( 2F_1 \). A general form for all \( p \) can be given as below, but we see it is more cumbersome to use because we have \( F_{p-1}^F \) rather than \( p-1^F_{p-2} \) as in (2.6)

\[
(2.9) \quad G_{p,0}^{b_1, b_2, \ldots, b_p} = \frac{x}{\Gamma(c)} \frac{\Gamma(a_i)}{\prod_{i=1}^{p-3} \Gamma(c_i)} \sum_{\ell=0}^{p-1} \frac{\ell!}{(p-\ell)!} \sum_{r=0}^{p-1} \frac{b_1(p+r) \cdots b_{p-1}(p+r)}{r! (c_i)_{\ell+r}} 
\]

\[
= \frac{p^p p-1^p F_{p-1}^{b_{p-1}} p^p p-1^p F_{p-1}^{c_{p-1}} (1-x)}{(1-x)^{\ell+r}} \quad 0 < x < 1
\]

where

\[
\ell = \sum_{i=1}^{p-3} \ell_i, f_1 = \sum_{j=p-i}^{p} c_j, g_1 = \sum_{j=p-i}^{p} c_j + \sum_{j=p-i-1}^{p} \ell_j + r, c = \sum_{i=1}^{p} c_i.
\]

It follows that letting \( p = 2 \) we get (2.5) and \( p = 1 \) gives
\[ a_{1,1}^0(\frac{x^a}{b}) = x^b(1-x)^{c-1}/\Gamma(c) \cdot \]

3. The Non-Central Distribution of $W(p)$ in Case 1

Let $X: p \times n_1$ and $Y: p \times n_2$, $p \leq n_i$, $i = 1, 2$, be independent matrix variates with the columns of $\sim X$ independently distributed as $\sim N(0, \Sigma_1)$ and those of $\sim Y$ independently distributed as $\sim N(0, \Sigma_2)$. Hence $\sim S_1 = XX'$ and $\sim S_2 = YY'$ are independently distributed as Wishart $(n_1, p; \Sigma_1), i = 1, 2$. Let $0 < f_1 < f_2 < \ldots < f_p < \infty$ be the characteristic (ch.) roots of the determinantal equation

\[(3.1) \quad |S_1 - f S_2| = 0\]

and $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_p < \infty$ be the ch. roots of

\[(3.2) \quad |\Sigma_1 - \gamma \Sigma_2| = 0.\]

For testing the hypothesis $H_0: \lambda \sim \Lambda = I_p$, $\lambda > 0$ being given, we will use

\[(3.3) \quad W(p) = \prod_{i=1}^{p} (1-w_i)\]

where

$\Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_p)$, $w_i = \lambda_i f_i/(1+\lambda_i f_i)$ $i = 1, 2, \ldots, p$.  

Khatri [18] has shown that
\begin{align*}
(3.4) \quad f(w_1, w_2, \ldots, w_p) &= c [\lambda \Lambda]^{-\frac{1}{2}n_1} \left| W \right|^{\frac{1}{2}(n_1 - p - 1)} \prod_{i < j} \left| I_{w_i - w_j} \right|^{\frac{1}{2}(n_2 - p - 1)} \Gamma_0 \left( \frac{3}{2} n_i; (\lambda \Lambda)^{-1}, W \right) \\
\text{where} \\
W &= \text{diag}(w_1, w_2, \ldots, w_p), \quad n = n_1 + n_2, \quad \Gamma_p(t) = \pi^{p(p-1)/4} \prod_{j=1}^{P} \Gamma(t - \frac{1}{2}j + \frac{1}{2}), \\
c &= \pi^{-2p} \Gamma_p\left( \frac{1}{2}n \right) \left[ \Gamma_p\left( \frac{1}{2}n_1 \right) \Gamma_p\left( \frac{1}{2}n_2 \right) \right]^{-1}.
\end{align*}

To find \( E[W(p)]^h \) we multiply (3.4) by \( [I_{w_i - w_i}]^h = \prod_{i=1}^{p} (1 - w_i)^h \), transform \( \sim \rightarrow \sim \sim \sim \), where \( \sim \) is an orthogonal and \( \sim \) is a symmetric matrix, integrate out \( H \) and \( V \) using (4.4) and (22) of Constantine [5] and we find

\begin{align*}
(3.5) \quad E[W(p)]^h &= \frac{\Gamma_p\left( \frac{1}{2}n \right) \Gamma_p\left( \frac{1}{2}n_1 + h \right)}{\Gamma_p\left( \frac{1}{2}n_2 \right) \Gamma_p\left( \frac{1}{2}n + h \right)} \left| \lambda \Lambda \right|^{-\frac{1}{2}n_1} \Gamma_p\left( \frac{1}{2}n_1; \frac{1}{2}n_2, \frac{1}{2}n + h; I_{\sim}^{\sim} \left( \lambda \Lambda \right)^{-1} \right).
\end{align*}

Using Lemma 1, the density of \( f(w(p)) \) has the form

\begin{align*}
(3.6) \quad f(w(p)) &= c_p \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\left( \frac{1}{2}n \right)_k \left( \frac{1}{2}n_1 \right)_k}{k!} c_k \Gamma_p\left( \frac{1}{2}n_1, \frac{1}{2}n_2, \frac{1}{2}n + h; I_{\sim}^{\sim} \left( \lambda \Lambda \right)^{-1} \right) [w(p)]^{\frac{1}{2}(n_2 - p - 1)} \\
\quad \cdot (2\pi)^{-1} \int_{c-i\infty}^{c+i\infty} [w(p)]^{r} \prod_{i=1}^{p} \frac{\Gamma(r + b_i)}{\Gamma(r + a_i)} dr.
\end{align*}

where
\[ r = \frac{1}{2}n_2 + h - \frac{1}{2}(p-1), \quad b_i = \frac{1}{2}(i-1), \quad a_i = \frac{1}{2}n_1 + k_{p-i+1} + b_i, \]

\[ C_p = \frac{\Gamma_p \left( \frac{1}{2}n \right)}{\Gamma_p \left( \frac{1}{2}n_2 \right)} |\lambda\Lambda|^{-\frac{1}{2}n_1}, \quad (a)_k = \prod_{i=1}^{p} \Gamma(a - \frac{1}{2}(i-1))_{i}^{-1}, \]

\[ (a)_k = a(a+1) \cdots (a+k-1), \]

the \( \sum \) is the sum over all partitions \( \kappa \) of the integer \( k \) where

\[ \kappa = (k_1, k_2, \ldots, k_p), \quad k_1 \geq k_2 \geq \cdots \geq k_p > 0, \quad \sum_{i=1}^{p} k_i = k, \text{ and } \]

\( C_\kappa^{(\lambda \Lambda)} \) is a zonal polynomial; see James [13].

Noting that the integral in (3.6) is in the form of Meijer's G-function we can write the density of \( W(p) \) as

(3.7) \[ f(W(p)) = C_p [W(p)] \frac{1}{2}n_2 - p - 1 \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\Gamma_p(\frac{1}{2}n_1)_k}{k!} \cdot \]

\[ \cdot C_\kappa^{(\lambda \Lambda)} g_p, p \left( \begin{array}{c} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_p \end{array} \right). \]

Letting \( p = 2 \) in (3.7) and using (2.5) we obtain

(3.8) \[ f(W(2)) = C_2 [W(2)] \frac{1}{2}n_2 - 3 \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\Gamma_p(\frac{1}{2}n_1)_k}{k!} \cdot \]

\[ \cdot C_\kappa^{(\lambda \Lambda)} \left( \begin{array}{c} a_1 + a_2 - b_1 - b_2 - 1 \\ 1 - W(2) \end{array} \right) \frac{\Gamma(a_1 + a_2 - b_1 - b_2)}{\Gamma(a_1 + a_2 - b_1 - b_2)} \cdot \]

\[ \cdot \left( \begin{array}{c} \gamma \cdot \beta \cdot \alpha \cdot \delta \\ \gamma \cdot \beta \cdot \alpha \cdot \delta \end{array} \right), \]

\[ _2F_1(a_2 - b_2, a_1 - b_2, a_1 + a_2 - b_1 - b_2; 1 - W(2)). \]
The probability that \( w^{(2)} \leq w \leq 1 \) can be obtained by integrating (3.8) by parts \( n_1 \) times when \( n_1 \) is even. Using the relation [8]

\[
(3.9) \quad \left( \frac{d^n}{dz^n} \right)[z^{c-n} F_1(a,b;c;z)] = (c-n)_n z^{c-n-1} \frac{z^{n-1} F_1(a,b;c-n;z)}{2^{c-1}},
\]

and recalling that \( \kappa = (k_1,k_2) \), we obtain the cdf of \( W^{(2)} \) in terms of \( a_i \)'s and \( b_i \)'s as

\[
(3.10) \quad Pr[W^{(2)} \leq w] = |\lambda a|^{-\frac{1}{2}n_1} \sum_{k=0}^{\infty} \sum_{K} \frac{(\frac{1}{2}n_1)_k C_k (I_2(k,\lambda a)^{-1})}{k!} \sum_{r=0}^{a} \frac{(a_1+a_2-b_1-b_2-r)_r}{[\frac{1}{2}(n_2-1)]_{r+1}}
\]

\[
\cdot \frac{\Gamma_2(\frac{1}{2}n_1)}{\Gamma_2(\frac{1}{2}n_2) \Gamma(a_1+a_2-b_1-b_2)} \cdot \frac{\Gamma_2(\frac{1}{2}n_2)}{\Gamma_2(\frac{1}{2}n_2) \Gamma(a_1+a_2-b_1-b_2)} \cdot \frac{\Gamma_2(\frac{1}{2}n_2)}{\Gamma_2(\frac{1}{2}n_2) \Gamma(a_1+a_2-b_1-b_2)}
\]

\[
\cdot 2^{n_1} F_1(a_1-b_2,a_1-b_2,a_1+a_2-b_1-b_2-r;1-w)+I_w(\frac{1}{2}n_2,b)
\]

where \( I_w(a,b) \) denotes the incomplete beta function, \( a_i, b_i \) are defined in (3.6), \( a = a_1-1 \) and \( b = a_2-b_2 \). When \( n_1 \) is odd, after integrating (3.8) by parts \( n_2 \) times, the cdf of \( W^{(2)} \) is (3.10) with \( a = a_2-1 \) and \( b = a_1-b_2 \). Letting \( p = 3 \) in (3.7) we have

\[
(3.11) \quad f(W^{(3)}) = \frac{\Gamma_3(\frac{1}{2}n_1)}{\Gamma_3(\frac{1}{2}n_2)} \frac{\lambda^{\frac{1}{2}n_1}}{[W^{(3)}]^{\frac{1}{2}(n_2-4)}}
\]

\[
\cdot \sum_{k=0}^{\infty} \sum_{K} \frac{(\frac{1}{2}n_1)_k C_k (I_3(k,\lambda a)^{-1})}{k!} \sum_{r=0}^{a_1} \frac{(a_1+a_2+a_3)_r}{[\frac{1}{2}(n_2-1)]_{r+1}}
\]

\[
\cdot 2^{n_1} F_1(a_1-b_2,a_1-b_2,a_1+a_2-b_1-b_2-r;1-w)+I_w(\frac{1}{2}n_2,b)
\]

where \( a_i \) and \( b_i \) are defined in (3.6).
It is clear \( g_{3,3,3}^{3,0}(w(3)|_{b_1,b_2,b_3}^{a_1,a_2,a_3}) \) could be written out in terms of the hypergeometric function using Theorem 1, for computation purposes. Also letting \( p = 4 \) in (3.7) yields

\[
(3.12) \quad f(w^{(4)}) = \frac{\Gamma_4(\frac{3}{2}n)}{\Gamma_4(\frac{3}{2}n_2)} |\lambda\lambda|^{-\frac{3}{2}n} |w^{(4)}|^{\frac{1}{2}(n_2-5)} \cdot \sum_{k=0}^{\infty} \sum_{k=0}^{\infty} \frac{(\frac{3}{2}n)_k (\frac{3}{2}n_2)_k}{k!} c_k (\frac{1}{2} -\{\lambda\lambda\}^{-1}) G_4^{(4)}(w^{(4)}|_{b_1,b_2,b_3,b_4})
\]

where \( a_i \)'s and \( b_i \)'s are defined in (3.6).

4. The Non-Central Distribution of \( W^{(p)} \) in Case 2

Let \( \Lambda = W^{(p)} = \prod_{i=1}^{p} (1-\lambda_i) \) where \( \lambda_1, \lambda_2, \ldots, \lambda_p \) are the ch. roots of the determinantal equation

\[
(4.1) \quad |S_1 - \lambda(S_1 + S_2)| = 0
\]

where \( S_1 \) is a \((p \times p)\) matrix distributed as non-central Wishart with \( s \) degrees of freedom, \( \Omega \) is a matrix of non-centrality parameters and \( S_2 \) has the Wishart distribution with \( t \) degrees of freedom, the covariance matrix in each case being \( \Sigma \). Pillai and Al-Ani [30] obtained the density of \( W^{(p)} \) for \( p = 2,3,4 \). Here we obtain the density of \( W^{(p)} \) in general in terms of Meijer's G-functions. As in Section 3, applying Lemma 1 to the expression for \( \text{E}[W^{(p)}]_h \) obtained by Al-Ani [1] and using (2.4) we find
\[ r(w(p)) = c_p \left[ t - p \right]^{1/2} \sum_{k=0}^{\infty} \left( v \right)_k c_k \left( \Omega \right) \left( t - p - 1 \right) \sum_{k=0}^{\infty} \frac{(v)_k}{k!} \]

\[ \mathcal{P}_{w(p)} \left( a_1, a_2, \ldots, a_p \right) \]

where

\[ v = \frac{1}{2}(s+t), \quad c_p = \frac{\Gamma_p(v)}{\Gamma_p(\frac{1}{2}t)} e^{-t \Omega}, \quad b_i = \frac{1}{2}(i-1), \quad a_i = \frac{1}{2}s + k \cdot p - i + 1 + b_i. \]

The probability that \( W^{(2)} \leq w \leq 1 \) can be obtained by using (2.5) in (4.2), integrating by parts \( a_1 \) times when \( s \) is even, then using (3.9) we get the cdf of \( W^{(2)} \) as

\[ \Pr[W^{(2)} \leq v] = e^{-t \Omega} \sum_{k=0}^{\infty} \sum_{k=0}^{\infty} \frac{c_k(\Omega)}{k!} \frac{w^{(t-1)}}{\Gamma_2(v) \Gamma(a_1+a_2-b_1-b_2)} \]

\[ \cdot \sum_{r=0}^{a_2} \frac{(a_1+a_2-b_1-b_2-r)}{(r!)} w^{r} (1-w)^{a_1+a_2-b_1-b_2-r-1} \]

\[ \cdot 2F_1(a_2-b_2, a_1-b_2; a_1+a_2-b_1-b_2-r; 1-w^{(2)} + \frac{1}{2}(s+t,b)) \]

where \( a = a_1 - 1, b = a_2 - b_2 \) and the \( a_1 \)'s and \( b_1 \)'s are defined in (4.2). When \( s \) is odd, we integrate (4.2) by parts \( a_2 \) times and find the cdf is (4.3) with \( a = a_2 - 1, b = a_1 - b_2 \). The densities of \( W^{(3)} \) and \( W^{(4)} \) obtained by Pillai and Al-Ani [30] are special cases of (4.2) as can be verified by letting \( p = 3, 4 \) in (4.2), applying Theorem 1 and making the substitution.
\[(a_1, a_3, b_1, b_3) \rightarrow (a_3, a_1, b_3, b_1).\]

5. **The Non-Central Distribution of** \(W(p)\) **in Case 3**

Let the columns of \((X_{1})_{\sim 2}^{\sim}\) be independent normal \((p+q)\)-variates \((p \leq q, p+q \leq n, n\) is the sample size) with zero means and covariance matrix

\[
(5.1) \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{12}^t & \Sigma_{22} \end{pmatrix}.
\]

Let \(\mathbf{z}^2 = \text{diag}(r_1^2, r_2^2, \ldots, r_p^2)\) where \(r_i^2\) are the ch. roots of

\[
(5.2) \quad |X_{\sim 2} X_{\sim 2}' (X_{\sim 2} X_{\sim 2}')^{-1} X_{\sim 2} X_{\sim 1}' - r^2 X_{\sim 1} X_{\sim 1}'| = 0
\]

and \(\mathbf{z}^\rho = \text{diag}(\rho_1^2, \rho_2^2, \ldots, \rho_p^2)\) where \(\rho_i^2\) are the ch. roots of

\[
(5.3) \quad |\Sigma_{12} \Sigma_{12}^{-t} \Sigma_{22}^{-1} = \rho^2 \Sigma_{11}| = 0.
\]

Constantine [5] obtained the density of \(r_1^2, r_2^2, \ldots, r_p^2\) as

\[
(5.4) \quad f(r_1^2, r_2^2, \ldots, r_p^2) = C |_{\mathbf{z}^2} - |^{\frac{1}{2}n} |R^2|^{\frac{1}{2}(q-p-1)} |_{\mathbf{z}^\rho} - |^{\frac{1}{2}(n-q-p-1)}
\]

\[
\cdot \prod_{i<j}(r_i^2 - r_j^2) \sum_{k=0}^{\infty} \frac{(\frac{1}{2}n)_{k} (\frac{3}{2}n)_{k} C_{k}(R^2)}{(\frac{3}{2}q)_{k} C_{k}(I_{\rho})} k!
\]

where

\[
C = \pi^\frac{3p}{2} \Gamma_p(\frac{1}{2}n)[\Gamma_p(\frac{3}{2}q)\Gamma_p(\frac{1}{2}(n-q))\Gamma_p(\frac{3}{2}p)]^{-\frac{1}{2}}.
\]
To find $E[W(p)]^h$, $W(p) = \frac{p}{n} (1 - \frac{q}{i})$, we multiply (5.4) by $|I_p - R^2|^h$,

proceed as in Section 3 for Case 1 and we find

\[(5.5) \quad E[W(p)]^h = \frac{\Gamma_p(\frac{1}{2}n) \Gamma_p(\frac{1}{2}(n-q)+h)}{\Gamma_p(\frac{1}{2}(n-q)) \Gamma_p(\frac{1}{2}n+h)} |I_p - R^2|^\frac{1}{2}n \sum_{k=0}^{\infty} \frac{(\frac{1}{2}n)_k (\frac{1}{2}n)_k}{k!} R^2 (\frac{1}{2}n, \frac{1}{2}n; \frac{1}{2}n+h; R^2). \]

Noting that (5.5) can be obtained from (3.5) by substituting

\[(5.6) \quad (n_2, n, (\lambda)^{-1}) \rightarrow (n-q, n, I_p - R^2) \]

it can be verified that the density of $W(p)$ in this case is

\[(5.7) \quad f(W(p)) = C_p [W(p)]^{\frac{1}{2}(n-q-p+l)} \sum_{k=0}^{\infty} \frac{(\frac{1}{2}n)_k (\frac{1}{2}n)_k}{k!} R^2 (\frac{1}{2}n, \frac{1}{2}n; \frac{1}{2}n+h; R^2) \]

where

\[C_p = \frac{\Gamma_p(\frac{1}{2}n)}{\Gamma_p(\frac{1}{2}(n-q))} |I_p - R^2|^\frac{1}{2}n, \quad a_i = \frac{1}{2}q + k_{p-i+1} + b_i, \quad b_i = \frac{1}{2}(i-1). \]

The cdf of $W(2)$ is obtained from (3.10) when $q$ is even by substituting as in (5.6) and using the $a_i$'s as just defined. For $q$ odd the cdf of $W(2)$ follows from that of Case 1 for $n_1$ odd by making the substitution (5.6) and using the $a_i$'s just defined. The densities of $W(p)$ for $p = 2, 3, 4$ follow from (3.8), (3.11), (3.12) respectively making substitution (5.6).
CHAPTER III
EXACT DISTRIBUTION OF THE LIKELIHOOD RATIO CRITERION
FOR TESTING INDEPENDENCE OF SETS OF VARIATES
UNDER THE NULL HYPOTHESIS

1. Introduction and Summary

Let the p-component vector $X$ be distributed according to $N(\mu, \Sigma)$.

We partition $X$ into $q$ subvectors with $P_1, P_2, \ldots, P_q$ components respectively, that is

$$X \sim \begin{pmatrix}
X^{(1)} \\
\vdots \\
X^{(q)}
\end{pmatrix}.$$

(1.1)

The vector of means $\mu$ and the positive definite covariance matrix $\Sigma$ are partitioned similarly

$$\mu \sim \begin{pmatrix}
\mu^{(1)} \\
\vdots \\
\mu^{(q)}
\end{pmatrix} \quad \text{and} \quad \Sigma \sim \begin{pmatrix}
\Sigma_{11} & \Sigma_{12} & \cdots & \Sigma_{1q} \\
\Sigma_{21} & \Sigma_{22} & \cdots & \Sigma_{2q} \\
\vdots & \vdots & \ddots & \vdots \\
\Sigma_{q1} & \Sigma_{q2} & \cdots & \Sigma_{qq}
\end{pmatrix}.$$
The hypothesis \( H_0 \) to be tested is whether the \( q \) sets are mutually independent. Thus we test the hypothesis

\[
H_0: N(x|\mu, \Sigma) = \prod_{i=1}^{q} N(x^{(i)}|\mu^{(i)}, \Sigma^{(i)})
\]

or equivalently \( H_0: \Sigma_{ii} = 0, \ i \neq j \), against \( H_1: \Sigma_{ii} \neq 0, \ i \neq j \).

If \( x_1, x_2, \ldots, x_N \) is a sample of \( N \) observations drawn from \( N(\mu, \Sigma) \), where \( x, \mu, \Sigma \) are partitioned as above, then the likelihood ratio criterion is a monotonic increasing function of \( V \) defined by [37]

\[
V = |A| / \prod_{i=1}^{q} |A_{ii}|
\]

where

\[
A = \sum_{i=1}^{N} (x_i - \bar{x})(x_i - \bar{x})'
\]

and is partitioned in the same manner as \( \Sigma \). The corresponding matrix \( A_{ii} \) is defined and partitioned similarly.

In 1935 Wilks [37] obtained the distributions of the likelihood ratio criterion \( V \), in the following special cases; for \( q = 3 \),

i) all values of \( p_3, p_1 = p_2 = 1 \), ii) \( p_1 = 1, p_2 = 2, p_3 = 3 \),

iii) \( p_1 = 1, p_2 = 2, p_3 = 2,4 \) iv) \( p_1 = p_2 = 2, p_3 = 2,3 \), each expression being a finite series involving incomplete beta functions. Wald and Brookner [36] gave a method for obtaining the exact distributions of \( V \), if not more than one \( p_1 \) was odd, but exact distributions weren't explicitly obtained. Here we give expressions for the distributions of the
likelihood ratio criterion for the cases: \( q = 3 \); all values of \( p_1 \) and

i) \( p_2 = p_3 = 1 \), ii) \( p_2 = 2, p_3 = 1 \), iii) \( p_2 = p_3 = 2 \), iv) \( p_2 = 3, p_3 = 1 \),

v) \( p_3 = 3, p_2 = 2 \), vi) \( p_2 = 4, p_3 = 1 \), vii) \( p_2 = 4, p_3 = 2 \), (See Section 3).

\( q = 4 \), all values of \( p_1 \) and i) \( p_2 = p_3 = p_4 = 2 \), ii) \( p_2 = p_3 = 2, p_4 = 1 \),

iii) \( p_2 = 2, p_3 = p_4 = 1 \), iv) \( p_2 = p_3 = p_4 = 1 \), (See Section 4).

Consul [8] has obtained the cdf's in all cases of Section 3, except

case vi), expressing his results as infinite series using Mellin's inversion

theorem. Using a transformation suggested by Schatzoff [35] and

some results of Gupta [11] many of the cdf's given here are in finite

series form. Exact lower 1% and 5% points for Case ii), iii) and v) of

Section 3, are given in Tables 5 - 10.

2. Some Preliminary Results

The following results, available in Gupta [11] will be noted. Let

\( X_j \) be a beta random variable with

\[
(2.1) \quad X_j \sim \beta(\frac{f_1-j+1}{2}, \frac{f_2}{2}) = k_j x_j^{(\frac{f_1-j-1}{2})} (1-x_j)^{\frac{f_2-2}{2}} \quad 0 < X_j < 1
\]

\( f_1 \geq j \)

where

\[
(2.2) \quad k_j = (1/\beta(\frac{f_1-j+1}{2}, \frac{f_2}{2}))
\]

When \( f_2 \) is even, \( (f_2-2)/2 \) is an integer and (2.1) can be expanded

using the binomial theorem, giving
\[ (2.3) \quad \beta[(f_1 - j + 1)/2, f_2/2] = K_j \sum_{\ell=0}^{b} (-1)^{\ell} x_j^{f_1 - j + 2\ell - 1)/2} \]

where

\[ (2.4) \quad b = (f_2 - 2)/2 . \]

Making the transformation

\[ (2.5) \quad Y_j = -\log X_j, \quad dY_j = -dX_j/X_j \]

where

\[ \log X = \log_e X , \]

we find the density of \( Y_j \) is given by

\[ (2.6) \quad Y_j \sim K_j \sum_{\ell=0}^{b} (-1)^{\ell} \binom{b}{\ell} e^{-Y_j(f_1 - j + 2\ell + 1)/2} Y_j > 0 . \]

Considering the relationship of Theorems 8.5.1 and 8.5.2 in Chapter 8 of Anderson [3] we see that for

\[ (2.7) \quad Z_j = X_{2j-1} \cdot X_{2j} \quad \text{where} \quad X_j \quad \text{has density (2.1)} \]

\[ (2.8) \quad Z_j \sim C_j \cdot Z_j^{(f_1 - 2j - 1)/2} (1 - \sqrt{Z_j})^{f_2 - 1} \]

where

\[ (2.9) \quad C_j = \beta[(f_1 - 2j + 1), f_2]^{-1} . \]
If we make the further transformation as in Schatzoff [35]

\[(2.10) \quad Y_j' = - \log Z_j\]

then expanding, using the binomial theorem, we get the density of \(Y_j'\) as

\[(2.11) \quad Y_j' \sim C_j \sum_{\ell=0}^{f_2-1} (-1)^\ell \binom{f_2-1}{\ell} \exp[-2f_1+\ell-2j+1]/2], \quad Y_j' > 0.\]

Now consider the density and distribution of random variables like \(V = V_1 + V_2\) where apart from normalizing constants

\[(2.12) \quad V_1 \sim \frac{k}{k} e^{\frac{k}{k} V_1}, \quad V_1 > 0, \quad k \text{ a non-negative integer}\]

and

\[(2.13) \quad V_2 \sim e^{\frac{b}{b} V_2}, \quad V_2 > 0.\]

The density function of \(V\) is easily found by forming the convolution integral

\[(2.14) \quad V_1^k e^{\frac{k}{k} V_1} \ast e^{\frac{b}{b} V_2} = \int_0^V V_1^k e^{\frac{k}{k} V_1} e^{\frac{b}{b} (V-V_1)} \, dv_1 = e^{\frac{b}{b} V} \int_0^V V_1^k e^{\frac{(a-b) V_1}{V}} \, dv_1\]

where the asterisk denotes the convolution operator. We have two cases

1) \(a = b\), then (2.14) is

\[(2.15) \quad e^{\frac{b}{b} V} \int_0^V V_1^k \, dv_1 = e^{\frac{b}{b} V} \frac{k^k}{k+1}\]
ii) \( a \neq b \) is

\[
e^{bv} \int_{v_1}^{v} e^{k \frac{(a-b)v_1}{1}} dv_1 =
\]

\[
e^{av} \left[ \sum_{r=1}^{k+1} (-1)^{r+1} \frac{k! v^{k-r+1}}{(k-r+1)! (a-b)^r} \right] + e^{bv} \left( \frac{-1}{a-b} \right)^{k+1} k!.
\]

Now, more explicitly we will denote \( V \) by \( V_{p_1, p_2, \ldots, p_q; N} \) in the case where we have \( q \) sets of variates with \( p_i \) variates in the \( i \)th set and \( N \) observations have been taken on \( X \). Also notice that \( V \) is unchanged by permutation of the \( p_i \)s. Anderson [3] shows that

\[
V_{p_1, p_2, \ldots, p_q; N} \sim \prod_{i=2}^{q} \prod_{j=1}^{p_i} X_{ij}
\]

where \( X_{ij} \) are independent and have density

\[
X_{ij} \sim \mathcal{G}[(n-p_{i-j+1})/2; \bar{p}_i/2]
\]

where \( \bar{p}_i = p_1 + p_2 + \ldots + p_{i-1} \) and \( n = N-1 \).

3. Exact Distributions of \( V \) When \( q = 3 \)

Distribution of \( V_{p_1, 1, 1; N} \)

For any \( p_1 \) value, \( p_2 = p_3 = 1 \) we have from (2.17)

\[
V_{p_1, 1, 1; N} = X_{21} \cdot X_{31}
\]
Denote $X_{2j}$ by $S_j$ and $X_{3j}$ by $T_j$. Thus

\begin{equation}
(3.2) \quad S_j \sim \beta[(f_1-j+1)/2, f_2/2]
\end{equation}

where

\begin{equation}
(3.3) \quad f_1 = n-p_2 \quad \text{and} \quad f_2 = p_2
\end{equation}

and

\begin{equation}
(3.4) \quad T_j \sim \beta[(f_1'-j+1)/2, f_2'/2]
\end{equation}

where

\begin{equation}
(3.5) \quad f_1' = n-p_3 = f_1 - 1 \quad \text{and} \quad f_2' = p_3 = f_2 + 1
\end{equation}

Now letting

\begin{equation}
(3.6) \quad Y_1 = -\log S_1 \quad \text{and} \quad Y_1^* = -\log T_1
\end{equation}

we have

\begin{equation}
(3.7) \quad -\log V_{1,1,1}; N = \log S_1 - \log T_1
\end{equation}

\[ = Y_1 + Y_1^* = W_1 \quad \text{(say)} \]

It follows from (2.6) that

\begin{equation}
(3.8) \quad Y_1 \sim (\beta[f_1/2, f_2/2])^{-1} \sum_{\ell=0}^{b} (-1)^{\ell} \binom{\beta}{\ell} e^{-Y_1(f_1+2\ell)/2} \quad Y_1 > 0
\end{equation}
where

\[ b = (f_2 - 2)/2 \]  

and also

\[ Y_1^* \sim (\beta[(f_1 - 1)/2, (f_2 + 1)/2])^{-1} \sum_{m=0}^{b'} (-1)^m \binom{b'}{m} e^{-Y_1^*(f_1 + 2\ell - 1)/2} \]

where

\[ b' = (f_2 - 1)/2 \]

Now

\[ W_1 = -\log V_{p_1, 1, 1; N} \sim \left\{ (\beta[(f_1/2, f_2/2])^{-1} \sum_{\ell=0}^{b} (-1)^\ell \binom{b}{\ell} e^{-Y_1^*(f_1 + 2\ell)/2} \right\}^* \]

\[ \left\{ (\beta[(f_1 - 1)/2, (f_2 + 1)/2])^{-1} \sum_{m=0}^{b'} (-1)^m \binom{b'}{m} e^{-Y_1^*(f_1 + 2m - 1)/2} \right\} \]

where * denotes the convolution operator. Thus we get the probability density function of \(-\log V_{p_1, 1, 1; N}\) using (2.14) and (2.16), is

\[ W_1 = -\log V_{p_1, 1, 1; N} \sim 2p_1 \left[ \sum_{\ell=0}^{(f_2-2)/2} \sum_{m=0}^{(f_2-1)/2} (-1)^{\ell+m} e_1(\ell, m) \cdot \left\{ e^{-W_1(f_1 + 2\ell)/2} - e^{-W_1(f_1 + 2m - 1)/2} \right\} \right]. \]
where
\begin{equation}
(3.14)
\begin{aligned}
\varepsilon_1(\ell, m) &= \binom{(f_2 - 2)/2}{\ell} \binom{(f_2 - 1)/2}{m} \\
\end{aligned}
\end{equation}

and
\begin{equation}
(3.15)
\begin{aligned}
\beta_1 &= \left( \beta\frac{f_1}{2}, \frac{f_2}{2} \right) \beta\left( \frac{(f_1 - 1)/2}{(f_2 + 1)/2} \right)^{-1} \\
\end{aligned}
\end{equation}

Now substituting
\begin{equation}
(3.16)
\begin{aligned}
W_1 &= -\log V \\
dW_1 &= -\frac{dV}{V} \\
\end{aligned}
\end{equation}
in (3.13) we obtain
\begin{equation}
(3.17)
\begin{aligned}
V_{p_1, l, 1; N} &\sim 2\beta_1 \sum_{\ell=0} \sum_{m=0} (-1)^{\ell+m} \frac{\varepsilon_1(\ell, m)}{2m-2\ell-1} \\
&\cdot \left\{ \frac{(f_1 + 2\ell - 2)/2}{V} - \frac{(f_1 + 2m - 3)/2}{V} \right\} .
\end{aligned}
\end{equation}

To find the cdf of $V_{p_1, l, 1; N}$ we integrate between the limits $(0,v)$ in (3.17), $0 \leq v \leq 1$ obtaining
\begin{equation}
(3.18)
\begin{aligned}
\Pr[V_{p_1, l, 1; N} \leq v] &= 4\beta_1 \sum_{\ell=0} \sum_{m=0} (-1)^{\ell+m} \frac{\varepsilon_1(\ell, m)}{2m-2\ell-1} \\
&\cdot \left\{ \frac{(f_1 + 2\ell)/2}{v} - \frac{(f_1 + 2m - 1)/2}{v} \right\} .
\end{aligned}
\end{equation}

which is an infinite series for $f_2$ even or odd.
For any \( p_1 \) value, \( p_2 = 2, p_3 = 1 \) we have from (2.17)

\[
(3.19) \quad V_{p_1,2,1;N} = X_{21} \cdot X_{22} \cdot X_{31}
\]

where, as in Case i), with \( S_j = X_{2j} \) and \( T_j = X_{3j}, S_j \) and \( T_j \) are distributed as (3.2) and (3.4) respectively, but now

\[
(3.20) \quad f_1' = f_1^{-2} \quad \text{and} \quad f_2' = f_2^{+2}.
\]

Now letting

\[
(3.21) \quad Z_1' = S_1 \cdot S_2,
\]

and using (2.7) and (2.8) it follows

\[
(3.22) \quad Z_1' \sim (2B[f_1^{-1}, f_2])^{-1} \int_{Z_1}^{(f_1^{-3})/2} (1 - \frac{Z_1'}{Z_1})^{f_2^{-1} - 1} Z_1^{-1}.
\]

Applying (2.10) and (2.11) we have the density of \( Y_1' = -\log Z_1' \) as

\[
(3.23) \quad Y_1' \sim (2B[f_1^{-1}, f_2])^{-1} \sum_{\ell=0}^{f_2^{-1} - 1} (-1)\ell (\frac{f_2^{-1}}{\ell}) e^{-Y_1'(f_1 + \ell - 1)/2}, \quad Y_1' > 0.
\]

Further defining \( Y_1^* \) as in (3.6) and using (2.6) we find its density is

\[
(3.24) \quad Y_1^* \sim \frac{1}{\delta[(f_1^{-2})/2, (f_2^{-2})/2]} \sum_{m=0}^{f_2^{-2}} (-1)^m \frac{f_2^{-2}}{m} e^{-Y_1^*(f_1 + 2m - 2)/2}, \quad Y_1^* > 0.
\]

Since

\[-\log V_{p_1,2,1;N} = -\log Z_1 - \log T_1 = Y_1' + Y_1^* = W_2 \quad \text{(say)}\]
\[ (3.25) \quad W_2 \sim \left\{ (2\beta[f_1-1,f_2])^{-1} \sum_{\ell=0}^{f_2/2} (-1)^\ell \binom{f_2/2}{\ell} e^{-Y_1(f_1+\ell-1)/2} \right\} \]

\[ \quad \quad \times \left\{ (\beta[(f_1-2)/2,(f_2+2)/2])^{-1} \sum_{m=0}^{f_2/2} (-1)^m \binom{f_2/2}{m} e^{-Y_1(f_1+2m-2)/2} \right\}. \]

Now using (2.14) - (2.16) we get the density of \( W_2 \) as

\[ (3.26) \quad -\log V_{p_1,2,1; N} \sim 2\beta_2 \left[ \sum_{m=1}^{f_2/2} (-1)^{m+1} g_2(2m-1,m) W_2 e^{-W_2(f_1+2m-2)/2} \right. \]

\[ + 2 \sum_{\ell=0}^{f_2-1} \sum_{m=0}^{f_2/2} (-1)^{\ell+m} \binom{f_2/2}{\ell} e^{-W_2(f_1+\ell-1)/2} \frac{g_2(\ell,m)}{2m-\ell-1} \left( e^{-W_2(f_1+2m-2)/2} - e^{-W_2(f_1+2m-2)/2} \right) \]

where

\[ (3.27) \quad \beta_2 = (\beta[f_1-1,f_2]\beta[(f_1-2)/2,(f_2+2)/2])^{-1} \]

and

\[ (3.28) \quad g_2(\ell,m) = \binom{f_2/2}{\ell} \binom{f_2/2}{m}. \]

Substituting in (3.26)

\[ (3.29) \quad W_2 = -\log V \quad dW_2 = -\frac{dV}{V} \]

we obtain the density of \( V_{p_1,2,1; N} \) as
\( V_{p_1, 2, 1}; N \sim 2p_2 \left[ \sum_{m=1}^{f_2/2} (-1)^{m+1} g_2(2m-1, m)(-\log V) V^{\frac{1}{2}(f_1+2k-4)} \right. \\
+ 2 \sum_{l=0}^{f_2-1} \sum_{m=0}^{f_2/2} (-1)^{l+m} g_2(l, m) \left\{ \frac{(f_1+2k-3)/2}{2m-l-1} \left( V^{(f_1+2k-4)/2} - V^{-v} \right) \right\} \right].

Integrating (3.30) between the limits \((0, v), 0 \leq v \leq 1\), we obtain the c.d.f. of \( V_{p_1, 2, 1}; N \) as

\[
\begin{align*}
\Pr[V_{p_1, 2, 1}; N \leq v] &= p_2 \left[ \sum_{m=1}^{f_2/2} (-1)^{m+1} \frac{g_2(2m-1, m)}{a^2} v^{a/2} \{2-a \log v\} \\
+ 2 \sum_{l=0}^{f_2-1} \sum_{m=0}^{f_2/2} (-1)^{l+m} g_2(l, m) \left\{ \frac{(f_1+2k-3)/2}{2m-l-1} \left( \frac{v}{f_1+2k-1} - \frac{a/2}{a} \right) \right\} \right],
\end{align*}
\]

where

\[
a = f_1+2m-2,
\]

and (3.31) is a finite series for \( f_2 \) even and an infinite series for \( f_2 \) odd.

It can be shown that the expression obtained by Wilks [37] for the c.d.f. of \( V \) when \( p_1 = 1, p_2 = p_3 = 2 \) is a special case of (3.31) with \( p_1 = 2 \). This follows by letting \( f_1 = N-3 \) and \( f_2 = 2 \) in (3.31) and letting
\[ (3.33) \quad \beta_W(\frac{N-5}{2}, 2) = \frac{(N-3)(N-4)}{4} \left[ 2^N(\frac{N-5}{2})/N-5 - 2^N(\frac{N-3}{2})/N-5 + 2^N(\frac{N-3}{2})/N-5 \right] \]

in Wilks' expression and simplifying.

Distribution of \( V_{\beta_1, 2, 2}; N \)

For any \( \beta_1 \) value \( \beta_2 = \beta_3 = 2 \), from (2.17) we have

\[ (3.34) \quad V_{\beta_1, 2, 2}; N = X_{21} \cdot X_{22} \cdot X_{31} \cdot X_{32} \]

where again \( S_j \) and \( T_j \) are defined and distributed as in (3.2) and (3.4) and \( f_1^*, f_2^* \) are as in (3.20) and \( f_1 \) and \( f_2 \) as in (3.3). Defining \( Z_1^* \) as in (3.21) we have the density of \( Y_1^* = -\log Z_1^* \) is (3.23).

Now letting

\[ (3.35) \quad Z_1 = T_1 \cdot T_2 \]

and

\[ (3.36) \quad U_1 = -\log Z_1 \]

from (2.11) we have

\[ (3.37) \quad U_1 \sim (2 \beta[f_1-1, f_2^1])^{-1} \sum_{m=0}^{f_2^1-1} (-1)^m \binom{f_2^1-1}{m} e^{-U_1(f_1+m-1)/2} \quad U_1 > 0 \]

or

\[ (3.38) \quad U_1 \sim (2 \beta[f_1-3, f_2+2])^{-1} \sum_{m=0}^{f_2+1} (-1)^m \binom{f_2+1}{m} e^{-U_1(f_1+m-3)/2} \quad U_1 > 0. \]
Now

\[(3.39)\quad -\log V_{p,1,2,2; \ N} = -\log Z_1^f -\log Z_1 = Y_1^f + U_1 = W_3 \text{ (say)},\]

thus

\[(3.40)\quad W_3 \sim \left\{ (2\beta[f_{1,f_2}])^{-1} \sum_{\ell=0}^{f_2-1} (-1)^{\ell} \binom{f_2-1}{\ell} e^{-\ell(f_1+\ell-1)/2} \right\}^{f_2+1} \]
\[\times \left\{ (2\beta[f_{1-3,f_2+2}])^{-1} \sum_{m=0}^{f_2+1} (-1)^m \binom{f_2+1}{m} e^{-U_1(f_1+m-3)/2} \right\} .\]

Using (2.14) - (2.16) we obtain the density function of \( W_3 \) as

\[(3.41)\quad W_3 \sim \beta_3 \left[ \sum_{\ell=0}^{f_2-1} \frac{e^{-\ell^2/2}}{\sqrt{2\pi}} \sum_{m=0}^{f_2+1} (-1)^m \binom{f_2+1}{m} \right. \]
\[\left. + 2 \sum_{\ell=0}^{f_2-1} \sum_{m=0}^{f_2+1} (-1)^m \binom{f_2+1}{m} \frac{e^{-\ell^2/2}}{\sqrt{2\pi}} \right] .\]

where

\[(3.42)\quad \beta_3 = (4\beta[f_{1,f_2}]^2[f_{1-3,f_2+2}])^{-1} ,\]

\[(3.43)\quad g_3(\ell,m) = \binom{f_2-1}{\ell} \binom{f_2+1}{m} .\]

and

\[(3.44)\quad a = f_1+\ell-1 \quad b = f_1+m-3 .\]
Substituting

\[ W_3 = -\log V \quad dW_3 = -\frac{dV}{V} \]

in (3.41), we obtain the density of \( V_{p_1,2,2; N} \) as

\[
V_{p_1,2,2; N} \sim f_1 \left[ \beta_1 \sum_{\ell=0}^{f_2-1} g_3(\ell,\ell+2)(-\log V) V \right]^{(a-2)/2} + 2 \sum_{\ell=0}^{f_2-1} \sum_{m=0 \atop m \neq \ell+2} (-1)^{m+\ell} \frac{g_3(\ell,m)}{b-a} \left[ v^{(a-2)/2} - v^{(b-2)/2} \right].
\]

Integrating (3.46) between the limits \((0,v)\), \(0 \leq v \leq 1\), the cdf of \( V_{p_1,2,2; N} \) is

\[
\Pr[V_{p_1,2,2; N} \leq v] = 2\beta_1 \left[ \sum_{\ell=0}^{f_2-1} g_3(\ell,\ell+2) \frac{v^{a/2}}{a} \right]^{2-a \log v} + 2 \sum_{\ell=0}^{f_2-1} \sum_{m=0 \atop m \neq \ell+2} (-1)^{m+\ell} \frac{g_3(\ell,m)}{b-a} \left[ \frac{v^{a/2}}{a} - \frac{v^{b/2}}{b} \right].
\]

which is a finite series for \( f_2 \) even or odd.

It can easily be shown that the expression obtained by Wilks [37] for the cdf of \( V \) in the case when \( p_1 = l, p_2 = p_3 = 2 \) is a special case of (3.47) with \( p_1 = 1 \). This follows by letting \( f_1 = N-2, f_2 = 1 \) in (3.47) and using (3.33) in Wilks' expression and simplifying. Also the expressions obtained by Wilks [37] in the cases when \( p_1 = p_2 = p_3 = 2 \) and \( p_1 = p_2 = 2 \),
\[ p_3 = 3 \text{ are special cases of (3.47) with } p_1 = 2, f_1 = N-3, f_2 = 1 \text{ and } p_1 = 3, \]
\[ f_1 = N-4, \text{ and } f_2 = 3 \text{ respectively.} \]

In as much as the mechanics of obtaining the remaining results are the same as in the previous cases, only the final results will be given.

**Distribution of \( V_{p_1,3;1} ; N \)**

The cdf of \( V_{p_1,3;1} ; N \) for all values of \( p_1 \) is

\[ (3.48) \quad \Pr[V_{p_1,3;1} ; N \leq v] = \beta_4 \left[ \sum_{m=1}^{f_2} \sum_{t=0}^{(f_2-2)/2} (-1)^{m+t+1} \frac{\mathcal{E}_4(2m-1,m,t)}{c-b} \cdot \left\{ \frac{v b/2}{b^2 [2-b \log v]} - \frac{2v b/2}{b(c-b)} + \frac{2v c/2}{c(c-b)} \right\} \right] \]

\[ + 4 \sum_{m=0}^{f_2-1} \sum_{t=1}^{(f_2-2)/2} (-1)^{m+t} \frac{\mathcal{E}_4(2t-2,m,t)}{(b-c)c^2} v^{c/2} \left[ 2-c \log v \right] \]

\[ + 8 \sum_{\ell=0}^{f_2-1} \sum_{m=0}^{\ell} \sum_{t=0}^{(f_2-1)/2} (-1)^{\ell+m+t} \frac{\mathcal{E}_4(\ell,m,t)}{(b-a)(c-a)} \left\{ \frac{v a/2}{a} - \frac{v c/2}{c} \right\} \]

\[ - 8 \sum_{\ell=0}^{f_2-1} \sum_{m=0}^{\ell} \sum_{t=0}^{(f_2-1)/2} (-1)^{\ell+m+t} \frac{\mathcal{E}_4(\ell,m,t)}{(b-a)(c-b)} \left\{ \frac{v b/2}{b} - \frac{v c/2}{c} \right\} \]

which is an infinite series for \( f_2 \) even or odd and where

\[ \beta_4 = (2\beta[f_1-1,f_2] \beta[(f_1-2)/2,f_2/2] \beta[(f_1-3)/2,(f_2+3)/2])^{-1} \]

and
\[ g_{h}(\lambda, m, t) = \left( \begin{array}{c} f_{2} - 1 \\ \lambda \\ \end{array} \right) \left( \begin{array}{c} f_{2} - 2 \\ m \\ \end{array} \right) \left( \begin{array}{c} f_{2} + 1 \\ t \\ \end{array} \right) \]

and

\[ a = f_{1} + k - 1, \quad b = f_{1} + 2m - 2, \quad c = f_{1} + 2t - 3 . \]

Distribution of \( V_{p_{1}, 3, 2; N} \)

The cdf of \( V_{p_{1}, 3, 2; N} \) for all values of \( p_{1} \) is

\[
\Pr[V_{p_{1}, 3, 2; N} \leq v] = \frac{(f_{2} - 2)/2}{b^{2}} \sum_{m=1}^{(f_{2} - 2)/2} \sum_{t=0}^{f_{2} + 2} \frac{(-1)^{m+1} g_{5}(2m-1, m, 2m+2)}{b} \cdot \left\{ (\log v)^{2} + \frac{8}{b^{2}} - \frac{4 \log v}{b} \right\}
\]

\[
+ \frac{4}{c-b} \sum_{m=1}^{(f_{2} - 2)/2} \sum_{t=0}^{f_{2} + 2} \frac{(-1)^{m+t+1} g_{5}(2m-1, m, t)}{c-b} \cdot \left\{ \frac{b^{2}}{b^{2} [2 - b \log v] - 2 b^{2}/b (c-b) + 2 c^{2}/c} \right\}
\]

\[
+ \frac{f_{2} - 1 (f_{2} - 2)/2}{b^{2}} \sum_{\lambda=0}^{f_{2} - 1} \sum_{m=0}^{f_{2} - 2/2} \frac{(-1)^{m+1} g_{5}(\lambda, m, \lambda+3)}{(b-a) a^{2}} [2 - a \log v]
\]

\[
+ \frac{4}{c-b} \sum_{\lambda=0}^{f_{2} - 1} \sum_{m=0}^{f_{2} - 2/2} \sum_{t=0}^{f_{2} + 2} \frac{(-1)^{\lambda+m+t} g_{5}(\lambda, m, t)}{(b-a)(c-a)} \left\{ \frac{v/2}{a} - \frac{v c/2}{c} \right\}
\]
\[
\sum_{l=0}^{f_2-1} \sum_{m=0}^{(f_2-2)/2} \frac{(-1)^{m+l} g_5(l, m, 2m+2) \nu^{b/2}}{(b-a) b^2} \left[ 2 - b \log \nu \right] \\
\sum_{l=0}^{f_2-1} \sum_{m=0}^{(f_2-2)/2} \sum_{t=0}^{f_2+2} \frac{(-1)^{\ell+m+t} g_5(l, m, t) \left( \nu^{b/2} \left( \frac{\nu}{b} - \frac{\nu}{c} \right) \right)}{(b-a)(c-a)} 
\]

which is a finite series for \( f_2 \) even and an infinite series for \( f_2 \) odd and where

\[
g_5 = (4^8[f_1-1, f_2] B[(f_1-2)/2, f_2/2] B[f_1-4, f_2+3])^{-1}
\]

and

\[
g_5(l, m, t) = \left( \binom{f_2-1}{l} \binom{f_2-2}{m} \binom{f_2+2}{t} \right)
\]

and

\[
a = f_1 + l - 1, \quad b = f_1 + 2m - 2, \quad c = f_1 + t - 4.
\]

It can easily be verified that the expression obtained by Wilks [37] for the cdf of \( V \) when \( p_1 = p_2 = 2, p_3 = 3 \) is a special case of (3.49) with \( p_1 = 2, f_1 = N-3 \) and \( f_2 = 2 \).
Distribution of $V_{p_1,4,1;N}$

The cdf of $V_{p_1,4,1;N}$ for all values of $p_1$ is

\begin{equation}
\Pr[V_{p_1,4,1;N} \leq v] = \beta_6 \left[ \sum_{t=2}^{f_2/2} \frac{(-1)^t g_6(2t-3,2t-1,t)}{c} \right] \{\log v\}^2 \\
+ \frac{4v c/2}{c^2} \left[ 2 - c \log v \right]
\end{equation}

\begin{align*}
&+ 4 \sum_{\ell=0}^{f_2-3} \sum_{t=0}^{(f_2+2)/2} \frac{(-1)^t g_6(\ell,\ell+2,t)}{c} \left( \frac{v a/2}{a} - \frac{2v c/2}{a(c-a)} \right) \\
&+ 4 \sum_{m=0}^{f_2-1} \sum_{t=0}^{(f_2+2)/2} \frac{(-1)^{m+t+1} g_6(2t-3,m,t) v c/2}{(b-c)^2} \left( \frac{v a/2}{a} - \frac{v c/2}{c} \right)
\end{align*}

\begin{align*}
&+ 8 \sum_{\ell=0}^{f_2-1} \sum_{m=0}^{f_2-1} \sum_{t=0}^{(f_2+2)/2} \frac{(-1)^{\ell+m+t} g_6(\ell,m,t)}{(c-a)(b-a)c^2} \left( \frac{v a/2}{a} - \frac{v c/2}{c} \right)
&- 4 \sum_{\ell=0}^{f_2-1} \sum_{t=1}^{f_2-1} \frac{(-1)^{\ell+t+1} g_6(\ell,2t-1,t) v c/2}{(c-a)^2} \left( \frac{v b/2}{b} - \frac{v c/2}{c} \right)
&- 8 \sum_{\ell=0}^{f_2-1} \sum_{m=0}^{f_2-1} \sum_{t=0}^{(f_2+2)/2} \frac{(-1)^{\ell+m+t} g_6(\ell,m,t)}{(b-a)(c-a)^2} \left( \frac{v b/2}{b} - \frac{v c/2}{c} \right)
\end{align*}
which is a finite series for $f_2$ even and an infinite series for $f_2$ odd and where

$$
\beta_6 = (4\beta[f_1-1,f_2]\beta[f_1-3,f_2]\beta[(f_1-4)/2,(f_2+4)/2])^{-1}
$$

and

$$
g_6(l,m,t) = \binom{f_2-1}{l} \binom{f_2-1}{m} \left(\frac{f_2+2}{t}\right)
$$

and

$$
a = f_1+l-1, \quad b = f_1+m-3, \quad c = f_1+2t-4 .
$$

It can be verified that the cdf of $V$ obtained by Wilks [37] in the case where $p_1=1$, $p_2=2$, $p_3=4$ is a special case of (3.50) by letting $p_1=2$, $f_1=N-3$ and $f_2=2$.

**Distribution of $V_{p_1,4,2;N}$**

The cdf of $V_{p_1,4,2;N}$ for all values of $p_1$ is given by

$$
(3.51) \quad \text{Pr}[V_{p_1,4,2;N} \leq v] = \beta_7 \sum_{l=0}^{f_2-3} \left\{ \sum_{t=0}^{f_2-3} \frac{(-1)^l g_7(l,l+2,l+4)v^{a/2}}{a} \right\} 
$$

\[ \cdot \left\{ (\log v)^2 + \frac{8}{a} - \frac{4}{a} \log \frac{v}{a} \right\} \]

\[ + 4 \sum_{l=0}^{f_2-3} \sum_{t=0}^{f_2+3} \frac{(-1)^t g_7(l,l+2,t)}{c-a} \left\{ \frac{v^{a/2}}{a} \log v - \frac{2v^{a/2}}{a(c-a)} \right\} + \frac{2v^{c/2}}{c(c-a)} \]
\[ f_2 \sum_{k=0}^{f_2-1} \frac{(-1)^m \varepsilon_\ell(m+\ell)}{(b-a)^2} \left[ 2 - a \log v \right] \]
\[ + 4 \sum_{\ell=0}^{f_2-1} \sum_{m=0}^{f_2+3} \frac{(-1)^{\ell+m+t} \varepsilon_\ell(m+t)}{(b-a)(c-a)} \left[ \frac{v^{a/2}}{a} - \frac{v^{c/2}}{c} \right] \]
\[ - 4 \sum_{\ell=0}^{f_2-1} \sum_{m=0}^{f_2+3} \frac{(-1)^{\ell+m+t} \varepsilon_\ell(m+t)}{(b-a)(b^2)} \left[ \frac{v^{b/2}}{b} - \frac{v^{c/2}}{c} \right] \]
\[ + 8 \sum_{\ell=0}^{f_2-1} \sum_{m=0}^{f_2+3} \frac{(-1)^{\ell+m+t} \varepsilon_\ell(m+t)}{(b-a)(c-b)} \left[ \frac{v^{b/2}}{b} - \frac{v^{c/2}}{c} \right] \]

which is a finite series for \( f_2 \) even or odd and where

\[ \beta_\ell = (\beta_\ell[f_{1-1}, f_{2}]) (\beta_\ell[f_{1-3}, f_{2}]) (\beta_\ell[f_{1-5}, f_{2}])^{-1} \]

and

\[ \varepsilon_\ell(m, t) = \left( \begin{array}{c} f_2 - 1 \\ k \\ m \\ t \end{array} \right) \]

and

\[ a = f_{1+k-1}, \quad b = f_{1+m-3}, \quad c = f_{1+t-5} \]

Further it can be shown that the cdf of \( V \) obtained by Wilks [37] for \( p_1 = 1, p_2 = 4, p_3 = 2 \) is a special case of (3.51) with \( p_1 = 1, f_1 = N-2, f_2 = 1 \).
4. Exact Distributions of $V$ When $q = 4$

Distribution of $V_{p_1,2,2;1;N}$

The cdf of $V_{p_1,2,2;1;N}$ for all $p_1$ may be obtained from (3.51) by changing the limits of summation of $m$ and $l$ from $f_2-1$ and $f_2-3$ to $f_2+1$ and $f_2-1$, where appropriate and replacing $\beta_7$ and $g_7(l,m,t)$ by $\beta_8$ and $g_8(l,m,t)$ where

$$\beta_8 = (8\beta[f_2-1,f_2][f_1-3,f_2+2][f_1-5,f_2+4])^{-1}$$

and

$$g_8(l,m,t) = \binom{f_2-1}{l} \binom{f_2+1}{m} \binom{f_2+3}{t}.$$ 

Distribution of $V_{p_1,2,2;1;N}$

The cdf of $V_{p_1,2,2;1;N}$ for all $p_1$ may be obtained from (3.50) by changing the upper limits of summation of $l,m,t$ from $f_2-3$, $f_2-1$ and $f_2/2$ to $f_2-1$, $f_2+1$ and $(f_2+2)/2$ where appropriate and replacing $g_6(l,m,t)$ and $\beta_6$ by $g_9(l,m,t)$ and $\beta_9$ where

$$\beta_9 = (4\beta[f_1-1,f_2][f_1-3,f_2+2][f_1-4]/2,(f_2+4)/2])^{-1}$$

and

$$g_9(l,m,t) = \binom{f_2-1}{l} \binom{f_2+1}{m} \binom{(f_2+2)}{t}/2.$$
Distribution of $V_{p_1,2,1,1;N}$

The cdf of $V_{p_1,2,1,1;N}$ for all $p_1$ may be obtained from (3.48) by changing the upper limit of summation of $m$ from $(f_2-2)/2$ to $f_2/2$, whenever it appears, and substituting $\beta_{10}$ and $g_{10}(\ell,m,t)$ for $\beta_4$ and $g_4(\ell,m,t)$ where

$$\beta_{10} = (2\beta[f_1-1,f_2]p[(f_1-2)/2,(f_2+2)/2]p[(f_1-3)/2,(f_1+3)/2])^{-1}$$

and

$$g_{10}(\ell,m,t) = \binom{f_2-1}{\ell} \binom{f_2/2}{m} \binom{(f_2+1)/2}{t}.$$  

Distribution of $V_{p_1,1,1,1;N}$

The cdf of $V_{p_1,1,1,1;N}$ for all values of $p_1$ is

$$\Pr[V_{p_1,1,1,1;N} \leq v] = \beta_{11} \sum_{\ell=0}^{(f_2-2)/2} \sum_{m=0}^{(f_2-1)/2} \frac{(-1)^m \gamma_{11}(\ell,m,\ell+1,v \cdot a/2)}{a^2(b-a)}$$

$$\cdot [2-a \log v]$$

$$+ 8 \sum_{\ell=0}^{(f_2-2)/2} \sum_{m=0}^{(f_2-1)/2} \sum_{t=0}^{f_2/2} \frac{(-1)^{\ell+m+t} \gamma_{11}(\ell,m,t)}{(c-a)(b-a)}$$

$$\cdot \left\{ \frac{v^a/2}{a} - \frac{v^c/2}{c} \right\}$$
\[
\left( \frac{f_2-2}{2} \right) \left( \frac{f_2-1}{2} \right) \frac{f_2}{2} - 8 \sum_{t=0} \sum_{m=0} \sum_{l=0} \frac{(-1)^{l+m+t} g_{11}(l,m,t) v_{b/2}}{(c-b)(b-a)} \left( v_{c/2} - b \right)
\]

which is an infinite series for \( f_2 \) even or odd.

Results for many other combinations of \( p_1 \)'s could be obtained using this method but more than two convolutions are involved and the cdf's become quite unwieldy.

5. Computation of Percentage Points

The expressions derived in the preceding sections were used for tabulation of percentage points of \( V \). Values of \( V_{p_1, p_2, p_3; N} \) were first computed on the CDC 6500 to a minimum accuracy of four significant digits based on three arguments \( (p_1, N, \alpha) \) where \( \alpha \) is the lower probability level. For values of \( N > 24 \) Anderson's approximation ([3] page 239) was used. These values were then used to obtain correction factors for converting chi-square percentiles with \( f = \frac{1}{2}(p^2 - \Sigma p_i^2) \) degrees of freedom to exact percentiles of \(-[N - 1.5 - (p^2 - \Sigma p_i^3)/3f] \) log \( V \). Finally tabulation of the correction factors,

\[
C = \left[ \text{percentile of } -[N - 1.5 - (p^3 - \Sigma p_i^3)/3f] \right] \log V / \left( \text{percentile of } X^2_f \right)
\]

was made. These are given to three decimal places although they were generally obtained to four decimals. The correction factors are presented for \( \alpha = 0.01, 0.05, M = 1(1)10(2)20, 24, 30, 60, 120, \alpha, \ p_1 = 1(1)10 \) in Tables 5 and 6 and \( p_1 = 2(2)10 \) in Tables 7-10.
Table 5. Chi-Square Adjustments to
the $V_{p_1,2;N}$ Criterion, Factor C for
Lower 1% Points of $V$ (Upper Percentiles of $\chi^2$)

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$p_1 = \text{number of variates in the first set}; \ M = N-p_1^{-\frac{1}{2}}; N = \text{number of observations}, C = \left[\text{percentile for } \left\{ \frac{1}{2}(p^3 - \Sigma p_1^3)(p^2 - \Sigma p_1^2)^{-1}\log V \right\}/\text{percentile for } \chi^2 \text{ with } \frac{1}{2}(p^2 - \Sigma p_1^2) \text{ degrees of freedom}.\right]$
Table 6. Chi-Square Adjustments to
the $V_{p_1,2;2:N}$ Criterion, Factor C for

Lower 5% Points of $V$ (Upper Percentiles of $\chi^2$)

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$p_1 = \text{number of variates in the first set}; M = N-p_1+1; N = \text{number of observations}, C = \left[\text{percentile for } -\left\{-N-15-\frac{1}{3}(p^2-\Sigma p_i^2)(p^2-\Sigma p_i^2)^{-1}\log V\right\}\text{percentile for } \chi^2 \text{ with } \frac{1}{2}(p^2-\Sigma p_i^2) \text{ degrees of freedom}.\right]$
Table 7. Chi-Square Adjustments to the $\chi^2_{p_1,2,1;N}$ Criterion, Factor C for Lower 1% Points of $\chi^2$ (Upper Percentiles of $\chi^2$)

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$p_1 =$ number of variates in the first set; $M = N - p_1 - 3$; $N =$ number of observations, $C =$ percentile for $-\{N - 1.5 - \frac{1}{3}(p^3 - E p_1^3)(p^2 - E p_1^2)^{-1}\}$ log $V)/(\text{percentile for } \chi^2 \text{ with } \frac{1}{2}(p - E p_1^2) \text{ degrees of freedom})$. 
Table 8. Chi-Square Adjustments to the $V_{p_1,2,1;N}$ Criterion, Factor C for Lower 5% Points of $V$ (Upper Percentiles of $\chi^2$)

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$\chi^2_{p_1}$ $15.5073$ $23.6848$ $31.4140$ $38.8851$ $46.1943$

$p_1$ = number of variates in the first set; $M=N-p_1-3$, $N$ = number of observations, $C$ = percentile for $-\{N-1.5-\frac{1}{3}(p^3-\Sigma p_i^3)(p^2-\Sigma p_i^2)^{-1}\} \log V$/(percentile for $\chi^2$ with $\frac{1}{2}(p^2-\Sigma p_i^2)$ degrees of freedom).
Table 9. Chi-Square Adjustments to the $V_{p_1,3,2;N}$ Criterion,
Factor C for Lower 2% Points of $V$ (Upper Percentiles of $\chi^2$)

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<td>58.6192</td>
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$p_1$ = number of variates in the first set; $M=N-p_1-5$, $N$=number of observations, $C$=[percentile for $\{N-1.5-1/(3(p^3-\Sigma p^3)(p^2-\Sigma p^2)^{-1})\log V\}$/(percentile for $\chi^2$ with $\frac{1}{2}(p^2-\Sigma p^2)$ degrees of freedom).
Table 10. Chi-Square Adjustments to the $V_{p_1,3;2;N}$ Criterion, Factor C for Lower 5% Points of $V$ (Upper Percentiles of $\chi^2$)

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$X^2_F$ | 26.2962 | 38.8851 | 50.9985 | 62.8296 | 74.4683

$p_1$ = number of variates in the first set; $M=N-p_1-5$, $N=$number of observations, $C=\left[N-1.5-\frac{1}{3}(p^3-\Sigma p^3)(p^2-\Sigma p^2)^{-1}\right] \log V/(percentile for $X^2$ with $\frac{1}{2}(p^2-\Sigma p^2)$ degrees of freedom).
CHAPTER IV
SOME DISTRIBUTION PROBLEMS IN
THE MULTIVARIATE COMPLEX GAUSSIAN CASE

1. Introduction and Summary

Let $X_1: p \times n$ and $X_2: p \times n$ be real random variables having the joint density function

\[(1.1) \quad (2\pi)^{-\frac{p(n+1)}{2}} \exp\left[-\frac{1}{2} \operatorname{tr} \Sigma^{-1}(X_1 - \mu_1)(X_1 - \mu_1)'\right], \quad -\infty < X < \infty\]

where

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_1 & -\Sigma_1 \Sigma_2 \\ -\Sigma_1 \Sigma_2 & \Sigma_2 \end{pmatrix}, \quad \mu = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad M = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix},$$

$\Sigma_1: p \times p$ is a real symmetric positive definite (p.d.) matrix,

$\Sigma_2: p \times p$ is a real skew-symmetric matrix,

$\mu_j: p \times q$ and $M_j: q \times n$ $(j = 1, 2)$, are given matrices or their joint density does not contain $\Sigma_1, \Sigma_2, \mu_1, \mu_2$ as parameters. Then it has been shown by Goodman [10] that the distribution of the complex matrix $Z = X_1 + iX_2, \ (i = (-1)^{\frac{1}{2}})$, is complex Gaussian and its density function is given by

\[(1.2) \quad N_c(\nu, \Sigma) = \pi^{-\frac{p(n+1)}{2}} \exp\left[-\operatorname{tr} \Sigma^{-1}(Z-\mu)(Z-\mu)'\right] \]
where \( \Sigma = \Sigma_1 + i\Sigma_2 \) is Hermitian p.d., i.e. \( \Sigma^* = \Sigma \), \( \mu = \mu_1 + i\mu_2 \) and 
\( \bar{\Sigma} = \Sigma_1 + i\Sigma_2 \). Goodman [10], Wodin [38], James [13], Al-Ani [1], and
Khatri [16], [17], [19], [20] have studied distributions derived from a
sample of a complex p-variate normal distribution.

Some concepts which are important and necessary notation are given
below.

\[
\bar{\Gamma}_m(a) = \pi^{\frac{1}{2}m(m-1)} \prod_{i=1}^{m} \Gamma(a-i+1)
\]

\[
[a]_k = \prod_{i=1}^{m} (a-i+1)_{k_i} = \bar{\Gamma}_m(a,k)/\bar{\Gamma}_m(a)
\]

where \( k = (k_1,k_2,\ldots,k_p) \) is a partition of the integer \( k \) and

\[
\bar{\Gamma}_m(a,k) = \pi^{\frac{1}{2}m(m-1)} \prod_{i=1}^{m} \Gamma(a+k_i-i+1)
\]

The hypergeometric functions are defined as

\[
\bar{F}_p^q(a_1,\ldots,a_p; b_1,\ldots,b_q; A,B) = \sum_{k=0}^{\infty} \sum_{\kappa} \frac{\prod_{i=1}^{p} [a_i]_k \bar{C}_k(A) \bar{C}_k(B)}{\prod_{i=1}^{q} [b_i]_k \bar{C}_k(\bar{\Sigma}_m) k!}
\]

or when \( B = \Sigma_m \) we denote it by

\[
\bar{F}_p^q(a_1,\ldots,a_p; b_1,\ldots,b_q; A)
\]

and \( \bar{C}_k(A) \) is a zonal polynomial of a Hermitian matrix \( A \) and is given
as a symmetric function of the characteristic roots of \( A \). (See Section 6)
The non-central distributions of the characteristic roots concerning the classical problems of the covariance model, MANOVA model, and canonical correlation coefficients have been found by James [13] and Khatri [16], [19]. Here for the three cases mentioned, we give the general moment and the density which is expressed in terms of Meijer's G-function [25],[26], for \( W^{(p)} = \prod (1 - w_i) \), where the \( w_i \), \( i = 1, 2, \ldots, p \) are the characteristic roots in the above cases. The moments and densities are analogous to those given in the real case in Chapter II. Further the density functions of \( \mathbf{U} \) and Pillai's \( \mathbf{V} \) criteria in the complex central case are obtained for \( p = 2 \) and from the non-central complex multivariate \( F \) distribution various independence relationships are shown and independent beta variables are obtained. The last section is devoted to complex zonal polynomials. A method for computing them in terms of elementary symmetric function (esf's) is given and they are tabulated through degree 8 in Tables 11-14.

2. **Density Functions of \( W^{(p)} \) in the Non-Central Case**

Testing the Equality of Two Covariance Matrices

Let \( \mathbf{X}: (p \times n_1) \sim N_c(\mathbf{Q}_1, \mathbf{\Sigma}_1) \) and \( \mathbf{Y}: (p \times n_2) \sim N_c(\mathbf{Q}_2, \mathbf{\Sigma}_2) \) be independent and \( n_1 \geq p \). Then Khatri [19] has shown the density function of the characteristic roots, \( 0 < f_1 < \ldots < f_p \) of \( (\mathbf{X} \mathbf{X}' - \mathbf{Y} \mathbf{Y}')^{-1} \) can be written as

\[
\frac{n_1}{(p_1 - p)^{n_1}} F_0(n_1, \mathbf{I} - \mathbf{A}^{-1}, \mathbf{F}^{-1} (\mathbf{I} + \mathbf{F})^{-1}) \prod f_i^{n_1 - p} \prod_{i>j} (f_i - f_j)^2
\]

where
\[ C(p) = \frac{\pi^p(p-1) \sim \Gamma_0(n)}{\sim \Gamma_p(n_1) \sim \Gamma_p(n_2) \sim \Gamma_p(p)}, \quad n = n_1 + n_2, \quad F = \text{diag}(f_1, \ldots, f_p) \]

and \( \Lambda \) is a diagonal matrix whose diagonal elements are the characteristic roots of \( (\Sigma \Sigma^{-1}) \). Transforming

\[ w_i = \frac{f_i}{(1 + f_i)} \]

we find the density of \( 0 < w_1 < \ldots < w_p \) is

\[ C(p)|\Lambda|^{-n_1} \sim |\Gamma_0(n_1; \Lambda^{-1})| |\Lambda|^{-n_2-p} \prod_{i,j} (w_i - w_j)^2 \]

where

\[ \sim = \text{diag} (w_1, w_2, \ldots, w_p). \]

To find \( E[W(p)]^h \) where \( W(p) = \prod_{i=1}^{p} (1 - w_i) \) we multiply (2.4) by \( |\Gamma_\sim - W|^h \)

and transform \( \sim \to \sim \to \sim \) where \( \sim \) is unitary, i.e. \( \sim \sim' = \sim \), and \( \sim \)

is Hermitian p.d. Using the Jacobian of transformation given by Khatri [16]

\[ J(\sim; U, W) = \prod_{i>j} (w_i - w_j)^2 h_2(\sim) \]

and integrating out \( \sim \) and \( \sim \) using

\[ \int_{\sim \sim'} h_2(U) = \frac{\pi^p(p-1)}{\Gamma_p(p)} \int_{\sim \sim'} h_2(U) = \frac{\pi^p(p-1)}{\Gamma_p(p)} \]

and
\begin{equation}
\int_{S=S_{>0}} |S|^{-p} |I_{-} - S|^{n + h - q - \Gamma_{-} - \Gamma_{+}} \gamma_s = \frac{\Gamma_{+}^{(q, \kappa)} \gamma_{+}^{(n + h - q)} \gamma_{-}^{(I_{-})}}{\gamma_{+}^{(n + h, \kappa)}}
\end{equation}

we get after simplifying

\begin{equation}
E[W^{(p)}] = \left| \Gamma \right|^{-n} \frac{\gamma_{+}^{(n)} \gamma_{+}^{(n + h)}}{\gamma_{+}^{(n + h)}} \gamma_{+}^{(I_{-} - 1)} .
\end{equation}

Before finding the density of \( W^{(p)} \), below are stated some needed results on Mellins transforms [7], [8], [9], and Meijer's G-function [25], [26].

If \( s \) is any complex variate and \( f(x) \) is a function of a real variable \( x \), such that

\begin{equation}
\Gamma(s) = \int_{0}^{\infty} x^{s-1} f(x) \, dx
\end{equation}

exists, then under certain regularity conditions

\begin{equation}
f(x) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} x^{-s} F(s) \, ds.
\end{equation}

\( F(s) \) is called the Mellin transform of \( f(x) \) and \( f(x) \) is the inverse Mellin transform of \( F(s) \). Meijer [25], [26] defined the G-function by

\begin{equation}
G_{p, q}^{m, n} (a_{1}, \ldots, a_{p}) = (2\pi i)^{-1} \int_{c}^{q} \frac{\prod_{j=1}^{m} \Gamma(b_{j} - s) \prod_{j=1}^{n} \Gamma(1 - a_{j} + s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_{j} + s) \prod_{j=n+1}^{p} \Gamma(a_{j} - s)} x^{s} \, ds
\end{equation}
where $C$ is a curve separating the singularities of $\prod_{j=1}^{m} \Gamma(b_j - s)$ from those of $\prod_{j=1}^{n} \Gamma(1 - a_j + s)$, $q > 1$, $0 \leq n \leq p \leq q$, $0 \leq m \leq q$; $x \neq 0$ and $|x| < 1$ if $q = p$; $x \neq 0$ if $q > p$. Using (2.9) and (2.10) we see from (2.8) that the density of $f(W(p))$ has the form

$$f(W(p)) = c_p \sum_{k=0}^{\infty} \sum_{k} \frac{[n]_k [n_i]_k}{k!} \sim c_k (I_p - \Lambda^{-1}) [W(p)]^{n_2 - p}$$

$$\cdot (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} \frac{\prod_{i=1}^{p} \Gamma(r+b_i)}{\prod_{i=1}^{p} \Gamma(r+a_i)} \frac{p}{r} \frac{1}{\prod_{i=1}^{p} \Gamma(r+b_i)} dr$$

where

$$c_p = \frac{\Gamma_p(n)}{\Gamma_p(n_2)} \left| A \right|^{-n_1}, \quad b_i = i-1, \quad a_i = n_i + k_p - i + b_i.$$

Noting that the integral in (2.12) is in the form of Meijer G-function we can write the density of $W(p)$ as

$$f(W(p)) = c_p \sum_{k=0}^{\infty} \sum_{k} \frac{[n]_k [n_i]_k}{k!} \sim c_k (I_p - \Lambda^{-1}) [W(p)]^{n_2 - p}$$

$$\cdot g_p, p, W(p) |_{a_i, \cdots, a_p} \left| b_1, \cdots, b_p \right) .$$
Using the fact that

\begin{equation}
\binom{a_1}{b_1, b_2} x^{a_1} (1-x)^{-1} = \frac{x^{a_1} (1-x)^{-1}}{\Gamma(a_1 - b_1 - b_2)}
\end{equation}

\begin{itemize}
\item $_2F_1(a_2-b_2, a_1-b_2; a_1+a_2-b_1-b_2; 1-x) \quad 0 < x < 1
\end{itemize}

we find the density of $W(2)$ to be

\begin{equation}
f(W(2)) = c_2 \sum_{k=0}^{\infty} \sum_{n_1} \frac{[n_1]_k [n_2]_k}{k!} \sum_{n_2} \gamma_1^{-1} \{W(2)\}^{n_2-2} \times \frac{[1-W(2)]^{2n_1+k-1}}{\Gamma(2n_1+k)} \times \frac{2F_1(n_1+k_1, n_1+k_2-1; 2n_1+k; 1-W(2))}{2F_1(n_1+k_1, n_1+k_2-1; 2n_1+k; 1-W(2))}
\end{equation}

where $\kappa = (k_1, k_2)$. Using the results of Consul [9] for $p = 3$ and Al-Ani [1] for $p = 4$ we could also write out the densities of $W(3)$ and $W(4)$.

**MANOVA Model**

Suppose $X: p \times m \sim N_c(\mu, \Sigma)$ and $Y: p \times n \sim N_c(0, \Sigma)$ are independent with $m \geq p$. Then the joint density of the characteristic roots

\begin{equation}
0 < f_1 < \ldots < f_p \quad \text{of} \quad (X \bar{X}^t)(Y \bar{Y}^t)^{-1}
\end{equation}

is given by Khatri [19] as

\begin{equation}
c(p)e^{-tr\Omega} \frac{1}{\Gamma(\text{rank}(X \bar{X}^t)(Y \bar{Y}^t)^{-1}))} \prod_{i,j} (f_i - f_j)^2
\end{equation}

where
\[ C'(p) = \frac{\tilde{\Gamma}_p(m+n)\,\Gamma_p(p-1)}{\tilde{\Gamma}_p(m)\Gamma_p(n)\,\Gamma_p(p)} , \quad F = \text{diag} \left( f_1, \ldots, f_p \right) \]

and \( \Omega = \text{diag}(\omega_1, \ldots, \omega_p) \) where \( \omega_i \) are the characteristic roots of \( \tilde{\mu}' \Sigma^{-1} \tilde{\mu} \). Now proceeding as in the previous case we obtain \( E[W(p)]^h \),

\[ W(p) = \prod_{i=1}^{p} (1-w_i) \quad \text{where} \]

\[ w_i = f_i/(1+f_i) \]

(2.16) \[ E[W(p)]^h = e^{-\text{tr} \Omega \, \frac{\tilde{\Gamma}_p(m+n)\,\tilde{\Gamma}_p(n+h)}{\tilde{\Gamma}_p(n)\,\tilde{\Gamma}_p(m+n+h)}} \frac{\Gamma_1(m+n;m+n+h;\Omega)}{\Gamma_1(m+n;\Omega)} \]

Using Mellin's transform and Meijer's G-function as in the previous case we get the density of \( W(p) \) as

(2.17) \[ f(W(p)) = e^{-\text{tr} \Omega \, \frac{\tilde{\Gamma}_p(m+n)}{\Gamma_p(n)}} \sum_{k=0}^{\infty} \frac{[m+n]^{k} \, C_k(\Omega)}{k!} \, \{ W(p) \}^{n-p} \]

\[ \cdot \, \text{G}^{p,0}_{p,p}(W(p)|a_1, \ldots, a_p) \]

\[ \cdot \text{G}_{p,p}^{p,0}(a_1, \ldots, a_p | b_1, \ldots, b_p) \]

where \( a_i = m+k_i, b_i = i-1 \).

As in the covariance model case, we could also obtain the density explicitly for \( p = 2, 3, 4 \).
Canonical Correlation

Let

\[(2.18) \quad \begin{bmatrix} X: p \times n \\ Y: q \times n \end{bmatrix} \sim N_c \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \]

n ≥ p+q and q ≥ p. Then the joint density of the characteristic roots

\[0 < r_1^2 < \ldots < r_p^2 \text{ of } \begin{bmatrix} \tilde{X}\tilde{Y}' \tilde{Y}\tilde{X}' \end{bmatrix}^{-1} (\tilde{X}\tilde{X}')^{-1} (\tilde{Y}\tilde{Y}')^{-1} \]

is given by Khatri [19] as

\[(2.19) \quad c(p) \sim \begin{bmatrix} I_p - R^2 \end{bmatrix}^{n} F_{2p}^{1} \begin{bmatrix} n, n-q; R^2, \Sigma \end{bmatrix} \begin{bmatrix} R^2 | q-p; I_p - R^2 \end{bmatrix}^{n-q-p} \prod_{i>j} (r_i^2 - r_j^2)^2 \]

where

\[(2.20) \quad c(p) = \frac{\Gamma_p(n)p^{p-1}}{\Gamma_p(n-q)\Gamma_q(p)\Gamma_p(p)} , \quad R^2 = \text{diag}(r_1^2, \ldots, r_p^2) \]

and \(R^2 = \text{diag}(\rho_1, \ldots, \rho_q)\) where \(\rho_i\) are the characteristic roots of \(\Sigma_{12} \Sigma_{21}^{-1} \Sigma_{22}^{-1} \Sigma_{11}^{-1} \). Proceeding as in the previous cases we find \(\mathbb{E}[W(p)]^h\),

\[W(p) = \prod_{i=1}^{p} (1-r_i^2)\], by substituting in (2.8) as follows

\[(2.21) \quad (n_1, n_2, \Lambda) \rightarrow (n, n-q, (I_p - R^2)^{-1}) \]

Further the density of \(W(p)\) is obtained from (2.14) by making the above substitution and letting
\[ a_i = q^{i+k}p^{-i+1}b_{i}, \quad b_i = i-1. \]

As in the other cases the densities could be written out explicitly for \( p = 2, 3, 4 \).

3. **The Density Function of Pillai's V-Statistic in the Central Case For Two Roots**

If \( p^2 = 0 \) in (2.19) we have the density function of the characteristic roots \( r_1^2, r_2^2, \ldots, r_p^2 \) in the central case. Letting \( p = 2 \) we have

\[ f_1(r_1^2, r_2^2) = c^{i}(2) |r_1^2|^{q-2} |r_2^2|^{q-2} (r_1^2 - r_2^2)^2. \]

Let \( V = r_1^2 + r_2^2 \) and \( G = \frac{r_1^2 r_2^2}{V} \), \( 0 < V < 1 \). To find the density function of \( V \) we make the above transformation and find

\[ f_2(V, G) = c^{i}(2) G^{q-2} (1-V+G)^{n-q-2} (V^2 - 4G)^{\frac{1}{2}}. \]

Integrating \( G \) between the limits \( 0 \) to \( V^2/4 \), [31] and writing \( (1-V+G)^{n-q-2} \) as a finite series we have

\[ f(V) = c^{i}(2) \sum_{r=0}^{n-q-2} \binom{n-q-2}{r} (1-V)^r (V^2)^{n-q-r-2} \]

\[ \times \left\{ \int_{0}^{V^2/4} G^{q+r-2} (V^2 - 4G)^{\frac{1}{2}} dG \right\}. \]

Integrating the expression in the brackets by parts we find the density function of \( V \) to be
\[(3.4) \quad f_3(V) = C''(2) \sum_{r=0}^{n-q-2} \frac{(n-q-2)(1-V)^{n-q-r-2}}{r!} \cdot \frac{(q+r-2)!V^{q+r-1}}{2^{q+r-1}3\cdot5\cdot\ldots\cdot(2(q+r)-1)} , \quad 0 < V < 1 .\]

To obtain the density function of \( V \) in the range \( 1 \leq V \leq 2 \) we change \( r_1^2 - 1 - r_1^2 \) in (3.1) and transform as before to get

\[(3.5) \quad f_4(V, G) = C''(2)(1-V+G)^{q-2}q^{n-q-2}(V^2-4G)^{\frac{1}{2}} .\]

Writing \((1-V+G)^{q-2}\) as a series and integrating \( G \) between the limits \( 0 \) to \( V^2/4 \) we have

\[(3.6) \quad f_5(V) = C''(2) \sum_{r=0}^{q-2} \frac{(q-2)(1-V)^{q-r-2}}{r!} \cdot \int_0^{V^2/4} G^{n+r-q-2}(V^2-4G)^{\frac{1}{2}} dG .\]

Evaluating the integral by parts yields

\[(3.7) \quad f_5(V) = C''(2) \sum_{r=0}^{q-2} \frac{(q-2)(1-V)^{q-r-2}}{r!} \frac{(n+r-q-2)!V^{2(n+r-q)-1}}{2^{n+r-q-1}3\cdot5\cdot\ldots\cdot2(n+r-q)-1} .\]

Transforming \( V' = 2-V, \ 1 \leq V \leq 2 \) we find

\[(3.8) \quad f_6(V) = C''(2) \sum_{r=0}^{q-2} \frac{(q-2)(1-V)^{q-r-2}}{r!} \frac{(n+r-q-2)!V^{2(n+r-q)-1}}{2^{n+r-q-1}3\cdot5\cdot\ldots\cdot2(n+r-q)-1} , \quad 1 \leq V \leq 2 .\]
By making the following changes in the parameters in (3.1)

\[(q,n-q,r_i^2) \rightarrow (m,n,w_i)\]

or

\[(q,n-q,r_i^2) \rightarrow (n_1,n_2,w_i)\]

we obtain the central density of the characteristic roots in the MANOVA or equality of two matrices cases, respectively. Thus the results of this section and the next aren't restricted to the canonical correlation case, but extend to the two cases mentioned above as well.

4. The Density Function of the U-Statistic in the Central Case For Two Roots

To obtain the density function of \( U \) we make the transformation in (3.1)

\[r_i^2 = \lambda_i (1+\lambda_i)^{-1}\]

and find

\[(4.1) \quad g_1(\lambda_1,\lambda_2) = c''(2) [Q]^q [I + Q]^{-n} (\lambda_1 - \lambda_2)^2\]

where \( Q = \text{diag}(\lambda_1,\lambda_2) \). Letting \( U = \lambda_1 + \lambda_2 \) and \( G = \lambda_1 \lambda_2 \) we see the joint density of \( U \) and \( G \) can be put in the form

\[(4.2) \quad g_2(U,G) = c''(2) U^{q-2} (1 + \frac{U}{2})^{-2n} (U^2 - 4G)^{\frac{1}{2}} \]

\[\cdot \left[ 1 - \frac{U^2 - 4G}{4(1 + \frac{U}{2})} \right]^{-n} \]
Writing the part in brackets as a series and integrating $G$ between the limits 0 to $U^2/4$ yields

\[(4.3) \quad g_3(U) = c^{''}(2)(1 + \frac{U}{2})^{-2n} \sum_{r=0}^{\infty} \frac{(-1)^r (-n)}{r} \frac{u^{r+2}}{u^r (1 + \frac{U}{2})^{2r}} \cdot \int_0^{U^2/4} (u^2 - 4\xi)^{r+\frac{1}{2}} \xi^{q-2} \, d\xi \] .

Integrating the expression in the brackets by parts, we find the density of $U$ for $p = 2$ is

\[(4.4) \quad g_3(U) = c^{''}(2) \sum_{r=0}^{\infty} \frac{(-1)^r (-n) (q-2)! u^{2r+2(q-3)+5}}{4^{r+q-1} (1 + \frac{U}{2})^{2r+2n}} \cdot \frac{1}{(r+\frac{3}{2})(r+\frac{5}{2}) \ldots (r+\frac{2(q-3)+5}{2})} \] .

5. Complex Multivariate Beta Distribution and Independent Beta Variables

If $X: p \times m$ and $Y: p \times n$ are independent complex matrix variates $m \geq p$, whose columns are independent complex $p$-variate with covariance matrix $\Sigma$, and if $E(X) = \mu$ and $E(Y) = \nu$, then the distribution of

\[(5.1) \quad F = \overline{\overline{Y}}' (\overline{\overline{Y}}')^{-1} \overline{X} \]

depends on parameters

\[(5.2) \quad \Omega = \overline{\overline{\mu}}', \Sigma^{-1} \overline{\overline{\mu}} \]
and is \([13]\)

\[ f(\tilde{F}) = k_{\perp} e^{-\text{tr} Q \tilde{F}} \prod_{j=1} I_{\tilde{F} j} (m+n; m \tilde{Q} (\tilde{F} + F^{-1})^{-1}) |F|^{m-P} |\tilde{F} + F|^{-n}(d\tilde{F}) \]

where

\[ k_{\perp} = \frac{\Gamma_p(m+n)}{\Gamma_p(m) \Gamma_p(n)} . \]

Since the density of \( \tilde{F} \) for \( p \geq m \) can be obtained from (5.3) by making the changes

\[ (p,m,n) \to (m,p,m+n-p) \]

it suffices to work only with (5.3). Making the transformation

\[ \tilde{L} = (\tilde{I}_{\tilde{F}} + \tilde{F}^{-1})^{-1} \]

in (5.3) and noting \( J(\tilde{L}; \tilde{F}) = |\tilde{I}_{\tilde{F}} - \tilde{L}|^{-2\tilde{P}} \) \([16]\) we have,

\[ f(\tilde{L}) = k_{\perp} \tilde{\lambda}^2 \prod_{j=1} I_{\tilde{F} j} (m+n; m \tilde{Q} 2 \tilde{L} \tilde{F} j |\tilde{L}|^{m-P} |\tilde{I}_{\tilde{F}} - \tilde{L}|^{-n-P}(d\tilde{L}) . \]

Proceeding in a manner similar to Khatri and Pillai \([21]\) let

\[ \tilde{L} = \begin{pmatrix} \tilde{L}_{11} & \tilde{L}_{12} \\ \tilde{L}_{21} & \tilde{L}_{22} \end{pmatrix} \]

\[ \prod_{j=1} I_{\tilde{L} j} \]

and note that \( |\tilde{L}| = \tilde{L}_{11} |\tilde{L}_{22}| \) and
\[ (5.9) \quad |I_{\sim p-L}| = (1-\lambda_{11})^{|I_{\sim p-1-L_{22}}-\lambda}\sim/[\lambda_{11}(1-\lambda_{11})] \quad . \]

Now it can be shown that \( \lambda_{11} \) and \( \{L_{22}, \gamma = \lambda/[\lambda_{11}(1-\lambda_{11})^{\frac{1}{2}}] \} \) are independently distributed and their respective distributions are

\[ (5.10) \quad f_1(\lambda_{11}) = \beta(m,n)\lambda^2 \sim f_1(m+n;m;\lambda_{11})\lambda_{11}^{m-1}(1-\lambda_{11})^{n-1} \]

and

\[ (5.11) \quad f_2(L_{22},\gamma) = k_2|L_{22}|^{m-p} \sim |I_{\sim p-1-L_{22}}-\gamma|^{n-p} \quad , \]

where

\[ (5.12) \quad k_2 = k_{11} \beta(m,n) \quad . \]

For further independence, we can use the transformation

\[ u = (I_{\sim p-1} - L_{22})^{\frac{1}{2}} \gamma \quad . \]

With Jacobian of transformation \( |I_{\sim p-1-L_{22}}|^{-1} \) it can be shown that \( u \sim \) and \( L_{22} \) are independently distributed and their respective distributions are

\[ (5.13) \quad f_3(u) = u^{-(p-1)}[\Gamma(n)/\Gamma(n-p+1)](1-u)^{n-p} \quad \]

and

\[ (5.14) \quad f_4(L_{22}) = k_3|L_{22}|^{m-(p-1)-1} |I_{\sim p-1-L_{22}}|^{n+1-(p-1)-1} \quad . \]
where

\[ k_3 = \pi^{(p-1)}[\Gamma(n-p+1)/\Gamma(n)]k_2. \]

Notice that \( \sim L_{22} \): \((p-1) \times (p-1)\) is the central complex multivariate beta distribution with \(m\) and \(n+1\) degrees of freedom. Making the transformation

\[(5.15) \quad x_i = u_i/(1-u_1u_2 \cdots - u_{p-1}u_{p-1}), \quad i = 1,2,\ldots,p-1, \quad u_0 = 0\]

in (5.13) with Jacobian of transformation \( \prod_{i=1}^{p-1} (1-x_i x_i)^{p-i-1} \), we obtain the density of \( \sim X = (x_1, x_2, \ldots, x_{p-1})' \) as

\[(5.16) \quad f(X) = \pi^{-(p-1)} \prod_{i=1}^{p-1} \frac{\Gamma(n-i+1)}{\Gamma(n-i)} (1-x_i x_i)^{n-i-1}. \]

After making the transformation of \( x_j = a_j + ib_j \) to polar coordinates \((r_j, \theta_j)\), we find with \( \sim r = (r_1, \ldots, r_{p-1})' \)

\[(5.17) \quad f(r) = \prod_{i=1}^{p-1} \frac{\Gamma(n-i+1)}{\Gamma(n-i)} (1-r_i^2)^{n-i-1} \cdot 2r_i \, dr_i. \]

Finally, the transformation \( v_i = r_i^2 \) yields independent real beta variates and their respective densities are given by

\[(5.18) \quad f_v(v) = [\beta(l,n-i)]^{-1} (1-v_i)^{n-i-1}. \]
6. **Complex Zonal Polynomials**

The zonal polynomials of a Hermitian matrix \( A \) \([13]\), are given by

\[
\tilde{c}_k(A) = \chi_{[k]}(1) \chi_{[k]}(A),
\]

where \( k = (k_1, k_2, \ldots, k_m) \) is a partition of the integer \( k \) and \( \chi_{[k]}(1) \) is the dimension of the representation \([k]\) of the symmetric group and is given by

\[
\chi_{[k]}(1) = k! \prod_{i < j}^{m} (k_i - k_j - i + j) / \prod_{i = 1}^{m} (k_i + m - 1)!,
\]

\( \chi_{[k]}(A) \) is the character of the representation \([k]\) of the linear group and is given as a symmetric function of the characteristic roots \( e_1, e_2, \ldots, e_m \) of \( \sim A \) by

\[
\chi_{[k]}(A) = |(e^1_j)^{k_j + m - j} / |(e^1_j)^{m - j}|
\]

where the determinants are Vandermonde type. Further the following equality is satisfied

\[
\sum_{k} \tilde{c}_k(A) = (a_1)^k
\]

where \( a_1 \) is the ith esf of the \( e_1 \)'s. Using the following lemma obtained by Pillai [27] we can get the zonal polynomials as a linear combination of the esf's. Tables 11-14 give \( \chi_{[k]}(A) \) and \( \chi_{[k]}(1) \) through degree 8.
Lemma: Let $D(g_s, g_{s-1}, \ldots, g_1), (g_j \geq 0, j = 1, 2, \ldots, s)$, denote the determinant

$$D(g_s, g_{s-1}, \ldots, g_1) = \begin{vmatrix} g_s & g_{s-1} & \ldots & g_1 \\ e_s & e_s & & e_s \\ & & \vdots & \\ g_1 & g_{s-1} & \ldots & g_1 \\ e_1 & e_1 & & e_1 \end{vmatrix}.$$  

If $a_r (r \leq s)$ denotes the $r$th esf in $s$ es's, then

$$(6.6) \quad \text{i)} \quad a_r D(g_s, g_{s-1}, \ldots, g_1) = \Sigma' \ D(g_s^\prime, g_{s-1}^\prime, \ldots, g_1^\prime)$$

where $g_j^\prime = g_j + \delta, j = 1, 2, \ldots, s, \delta = 0, 1$ and $\Sigma'$ denotes the sum over the $(s)\choose r$ combinations of $s$ $g$'s taken $r$ at a time for which $r$ indices $g_j^\prime = g_j + 1$ such that $\delta = 1$ while for other indices $g_j^\prime = g_j$ such that $\delta = 0$.

$$(6.6) \quad \text{ii)} \quad (a_r)^k (a_h)^\ell D(g_s, g_{s-1}, \ldots, g_1), k, \ell \geq 0,$$ can be expressed as a sum of $(s)\choose r$ \( (s)\choose h \) determinants obtained by performing on $D(g_s, g_{s-1}, \ldots, g_1)$ in any order (i) $k$ times and (ii) $\ell$ times with $r=h$. However if at least two of the indices in any determinant are equal, the corresponding term in the summation vanishes.

An example will suffice to show how $\chi_{(k)}(A)$ for any degree can be obtained from those of lower degree. Here we obtain $\chi_{(k)}(A)$ for $k=3$. Let

$$(6.7) \quad D = \mid (e_i^{m-j}) \mid$$

and
(6.8) \[ D(k_1^{m-1}, k_2^{m-2}, \ldots, l) = \{ (e_1^{j_1^{m-1}}, \ldots, e_j^{k_1}) \}. \]

When \( k = 2 \) we have

(6.9) \[ (a_1^{2} - a_2)D = D(m+1, m-2, m-3, \ldots, l) \text{ for } \kappa = (2) \]

and

(6.10) \[ a_2^D = D(m, m-1, m-3, \ldots, l) \text{ for } \kappa = (1^2) \].

Multiplying (6.9) and (6.10) by \( a_1 \), using Pillai's lemma, gives

(6.11) \[ (a_1^{3} - a_1 a_2)D = D(m+2, m-2, \ldots, l) + D(m+1, m-1, m-3, \ldots, l) \]

and

(6.12) \[ a_1 a_2 D = D(m+1, m-1, m-3, \ldots, l) + D(m, m-1, m-2, m-4, \ldots, l). \]

But since

(6.13) \[ a_3^D = D(m, m-1, m-2, m-4, \ldots, l) \]

we have substituting in (6.12)

(6.14) \[ (a_1 a_2 - a_3)D = D(m+1, m-1, m-3, \ldots, l) \].

When \( \kappa = (1^3) \) and \( \kappa = (21) \) in (6.8), we obtain (6.13) and (6.14) respectively. Thus
\[ x_{\{13\}}(a) = a_3 \quad \text{and} \quad x_{\{21\}}(a) = a_1 a_2 - a_3 \quad . \]

Substituting (6.14) in (6.11) we find

\[ (6.15) \quad (a_1^3 - 2a_1 a_2 + a_3)D = D(m+2, m-2, \ldots, 1) \]

and thus

\[ x_{\{3\}}(a) = a_1^3 - 2a_1 a_2 + a_3 \quad . \]
Table 11. Complex Zonal Polynomials of 1st - 5th Degree

<table>
<thead>
<tr>
<th></th>
<th>In terms of elementary symmetric functions of the latent roots of $A$</th>
<th>$X_{[k]}^{(1)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X[1]$</td>
<td>1st Degree</td>
<td>1</td>
</tr>
<tr>
<td>$X[2]$</td>
<td>$a_1$</td>
<td></td>
</tr>
<tr>
<td>$X[1^2]$</td>
<td>$a_2^2 - a_2$</td>
<td>1</td>
</tr>
<tr>
<td>$X[3]$</td>
<td>2nd Degree</td>
<td>1</td>
</tr>
<tr>
<td>$X[21]$</td>
<td>$a_1 - 2a_1a_2 + a_3$</td>
<td>1</td>
</tr>
<tr>
<td>$X[1^3]$</td>
<td>$a_2a_3 - a_3$</td>
<td>2</td>
</tr>
<tr>
<td>$X[4]$</td>
<td>3rd Degree</td>
<td>1</td>
</tr>
<tr>
<td>$X[31]$</td>
<td>$a_4^4 - 4a_1a_2 - 2a_2^2 + a_3$</td>
<td>1</td>
</tr>
<tr>
<td>$X[2^2]$</td>
<td>4th Degree</td>
<td>3</td>
</tr>
<tr>
<td>$X[21^2]$</td>
<td>$a_2^4 - 2a_1a_2 - a_2^2 + a_3$</td>
<td>2</td>
</tr>
<tr>
<td>$X[1^4]$</td>
<td>5th Degree</td>
<td>3</td>
</tr>
<tr>
<td>$X[5]$</td>
<td>$a_5^5 - 5a_1a_2 - a_2^2 + a_2a_3 - a_3^2 + a_3a_4 + a_4$</td>
<td>4</td>
</tr>
<tr>
<td>$X[41]$</td>
<td>$a_5^4 - 4a_1a_2 - a_2^2 + 2a_2a_3 - a_3^2 + a_3a_4 + a_4$</td>
<td>5</td>
</tr>
<tr>
<td>$X[32]$</td>
<td>$a_5^3 - 3a_1a_2 - a_2^2 + 3a_2a_3 - a_3^2 + a_3a_4 + a_4$</td>
<td>6</td>
</tr>
<tr>
<td>$X[31^2]$</td>
<td>$a_5^2 - 2a_1a_2 - a_2^2 + 2a_2a_3 - a_3^2 + a_3a_4 + a_4$</td>
<td>5</td>
</tr>
<tr>
<td>$X[221]$</td>
<td>$a_5 - a_2^2 + a_2a_3 - a_3^2 + a_3a_4 + a_4$</td>
<td>4</td>
</tr>
<tr>
<td>$X[213]$</td>
<td>5th Degree</td>
<td>4</td>
</tr>
<tr>
<td>$X[1^5]$</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>$x[k]$ &amp; $a_1^6 a_1^4 a_2^2 a_3^2 a_4 a_5 a_6$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>--------</td>
<td>-------------------------------</td>
<td></td>
</tr>
<tr>
<td>$x[6]$</td>
<td>1 -5 6 4 -1 -6 -3 1 2 2 -1</td>
<td></td>
</tr>
<tr>
<td>$x[51]$</td>
<td>1 -3 -1 1 4 1 -1 -2 -1 1</td>
<td></td>
</tr>
<tr>
<td>$x[42]$</td>
<td>1 -1 -1 0 1 0 1 -1 0</td>
<td></td>
</tr>
<tr>
<td>$x[41^2]$</td>
<td>1 0 -2 -1 1 1 1 -1</td>
<td></td>
</tr>
<tr>
<td>$x[3^2]$</td>
<td>1 -2 1 1 -1 0 0</td>
<td></td>
</tr>
<tr>
<td>$x[321]$</td>
<td>1 -1 -1 0 1 0</td>
<td></td>
</tr>
<tr>
<td>$x[31^3]$</td>
<td>1 0 -1 -1 1 1</td>
<td></td>
</tr>
<tr>
<td>$x[2^3]$</td>
<td>1 -1 0 0</td>
<td></td>
</tr>
<tr>
<td>$x[21^2]$</td>
<td>1 -1 0</td>
<td></td>
</tr>
<tr>
<td>$x[21^4]$</td>
<td>1 -1</td>
<td></td>
</tr>
<tr>
<td>$x[16]$</td>
<td>1</td>
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</tbody>
</table>

Table 12. Complex Zonal Polynomials for 6th Degree.
Table 13. Complex Zonal Polynomials for 7th Degree

<table>
<thead>
<tr>
<th></th>
<th>$a_1^7 a_2^5 a_3 a_2^2 a_4^3 a_3^2 a_2 a_4 a_3 a_3^2 a_4^2 a_5 a_4 a_4 a_5 a_4 a_6 a_7$</th>
<th>$x_{[K]}(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{[7]}$</td>
<td>1 -6 10 5 -4 -12 -4 3 3 6 3 -2 -2 -2 1</td>
<td>1</td>
</tr>
<tr>
<td>$x_{[6]}$</td>
<td>1 -4 -1 3 6 1 -3 -2 -4 -1 2 2 1 -1</td>
<td>6</td>
</tr>
<tr>
<td>$x_{[5]}$</td>
<td>1 -1 -2 1 1 2 -1 0 -1 0 -1 1 0</td>
<td>14</td>
</tr>
<tr>
<td>$x_{[5]}^2$</td>
<td>1 0 -3 -1 1 2 2 1 -2 -1 -1 1</td>
<td>15</td>
</tr>
<tr>
<td>$x_{[43]}$</td>
<td>1 -2 1 -1 2 0 -1 -1 1 0 0</td>
<td>14</td>
</tr>
<tr>
<td>$x_{[421]}$</td>
<td>1 -1 -1 -1 1 1 1 0 -1 0</td>
<td>35</td>
</tr>
<tr>
<td>$x_{[413]}$</td>
<td>1 0 0 -2 -1 1 1 1 -1</td>
<td>20</td>
</tr>
<tr>
<td>$x_{[321]}$</td>
<td>1 -1 -1 1 1 -1 0 0</td>
<td>21</td>
</tr>
<tr>
<td>$x_{[32]}$</td>
<td>1 -1 0 -1 1 0 0</td>
<td>21</td>
</tr>
<tr>
<td>$x_{[321]}$</td>
<td>1 -1 -1 0 1 0</td>
<td>35</td>
</tr>
<tr>
<td>$x_{[31]}^4$</td>
<td>1 0 -1 -1 1</td>
<td>15</td>
</tr>
<tr>
<td>$x_{[23]}$</td>
<td>1 -1 0 0</td>
<td>14</td>
</tr>
<tr>
<td>$x_{[221]}$</td>
<td>1 -1 0 1</td>
<td>14</td>
</tr>
<tr>
<td>$x_{[21]}^5$</td>
<td>1 -1 6</td>
<td>1</td>
</tr>
<tr>
<td>$x_{[1]}^7$</td>
<td>1 1</td>
<td>1</td>
</tr>
</tbody>
</table>
Table 14. Complex Zonal Polynomials for 8th Degree

<p>| x[8] | 1  | -7 | 15 | 6 | -10 | -20 | -5 | 12 | 12 | 4 |
| x[71] | 1  | -5 | -1 | 6 | 8 | -1 | -9 | -3 | -6 | -1 |
| x[62] | 1  | -1 | -3 | 2 | 1 | 1 | 3 | -2 | -1 | -1 |
| x[61^2] | 1  | -2 | 1  | -1 | 1 | 2 | -1 | 1 |
| x[53] | 1  | -1 | 0 | -2 | 1 | 2 | 1 |
| x[521] | 1  | 0 | 0 | 0 | 0 | -3 | -1 |
| x[51^3] | 1  | -3 | 1 | 2 | -1 | 1 |
| x[4^2] | 1  | -1 | -1 | 1 | 1 | -1 |
| x[431] | 1  | -1 | 0 | 0 | 1 | -1 |
| x[42^2] | 1  | -1 | 0 | 0 | 1 | -1 |
| x[421^2] | 1  | -1 | 0 | 0 | 1 | -1 |
| x[41^4] | 1  | -1 | 0 | 0 | 1 | -1 |
| x[3^2] | 1  | -1 | 0 | 0 | 1 | -1 |
| x[3^21^2] | 1  | -1 | 0 | 0 | 1 | -1 |
| x[32^2] | 1  | -1 | 0 | 0 | 1 | -1 |
| x[321^3] | 1  | -1 | 0 | 0 | 1 | -1 |
| x[31^5] | 1  | -1 | 0 | 0 | 1 | -1 |
| x[2^4] | 1  | -1 | 0 | 0 | 1 | -1 |
| x[2^31^2] | 1  | -1 | 0 | 0 | 1 | -1 |
| x[2^21^4] | 1  | -1 | 0 | 0 | 1 | -1 |
| x[21^6] | 1  | -1 | 0 | 0 | 1 | -1 |
| x[1^8] | 1  | -1 | 0 | 0 | 1 | -1 |</p>
<table>
<thead>
<tr>
<th>Table 14. (Cont'd.)</th>
<th>$a_2a_3^2a_2a_4^2a_1a_3a_4^2a_1a_2a_5^2a_1a_6^2a_4a_3a_5^2a_2a_6a_1a_7a_8$</th>
<th>$x_{[K]}^{(1)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_{[8]}$</td>
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</tr>
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<td>-2 -1 -4 -2 -1 1 2 1 1 -1 21</td>
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</tr>
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CHAPTER V

AN APPROXIMATION TO THE DISTRIBUTION OF THE
LARGEST ROOT OF A MATRIX AND PERCENTAGE POINTS

1. Introduction and Summary

Khatri [16] has pointed out that one can handle all the classical problems of point estimation and testing hypotheses concerning the parameters of complex multivariate normal populations much as one handles those for multivariate normal populations in real variates. Further, he suggested the maximum latent root statistic for testing the reality of a covariance matrix [17]. The joint distribution of the latent roots \( w_1, w_2, \ldots, w_q \) under certain null hypotheses can be written as [15], [16]

\[
(1.1) \quad C_1 \{ \prod_{j=1}^{q} w_j^m (1 - w_j)^n \} \{ \prod_{i>j} (w_i - w_j)^2 \}
\]

where

\[
(1.2) \quad C_1 = \prod_{j=1}^{q} \frac{\Gamma(m+n+q+j)}{\Gamma(n+j)\Gamma(m+j)\Gamma(j)}
\]

and

\[ 0 \leq w_1 \leq w_2 \leq \ldots \leq w_q \leq 1. \]
Khatri [15] has derived the distribution of \( w_q \) (or \( w_1 \)) in a determinant form as follows

\[
(1.3) \quad \Pr\{w_q \leq x; m, n\} = C_1 \left| \begin{array}{cccc}
\beta_0 & \beta_1 & \cdots & \beta_{q-1} \\
\beta_1 & \beta_2 & \cdots & \beta_q \\
\vdots & \vdots & \ddots & \vdots \\
\beta_{q-1} & \beta_q & \cdots & \beta_{2q-2} \\
\end{array} \right| = |(\beta_{i+j-2})|
\]

where \( C_1 \) is defined in (1.2),

\[
(1.4) \quad \beta_{i+j-2} = \int_0^x w^{m+i+j-2} (1-w)^n \, dw
\]

for \( i, j = 1, 2, \ldots, q \), and \((\beta_{i+j-2})\) is a \( q \times q \) matrix. \( \Pr\{w_1 \leq x; m, n\} \) can be obtained from (1.3) using

\[
(1.5) \quad \Pr\{w_1 \leq x; m, n\} = 1 - \Pr\{w_q \leq 1-x; n, m\}.
\]

In this paper an approximation to the cdf of \( w_q \) at the upper end is obtained and upper 1% and 5% points are given for \( q=2, 3, 4, 5, 6 \) in Tables 16-25. A general form of the approximation is given with the results for \( q=2 \) and 3 written out explicitly.

2. **Case For Two Roots**

It is easily seen from (1.3) that letting \( q = 2 \) and expanding the determinant
(2.1) \[ \Pr\{w_2 \leq x;m,n\} = C_1 \left\{ \int_0^x w^m (1-w)^n \, dw \int_0^x w^{m+2} (1-w)^n \, dw \right\} - \left( \int_0^x w^{m+1} (1-w)^n \, dw \right)^2 \].

This can be written as a finite sum for integral values of \( m \) and \( n \)
integrating by parts and we obtain

(2.2) \[ \Pr\{w_2 \leq x;m,n\} = C_1 \left\{ \beta_0 \beta_2 - \beta_1^2 \right\} \]

where \( \beta_\ell \) is defined in (1.4) or a form which is more convenient for our
purposes is

(2.3) \[ \beta_\ell = \beta(m+\ell+1,n+1) - \sum_{j=0}^{m+\ell} \frac{(m+\ell)!}{j!} \frac{x^j(1-x)^{m+n+\ell-j+1}}{\prod_{k=0}^{m+\ell-j} (n+k+1)} \]

where \( \beta(a,b) \) is the usual beta function. For small values of \( m \), the
approximation is obtained by neglecting all terms involving \((1-x)^{2n}\) and
higher powers. Expanding (2.2) and neglecting terms involving \((1-x)^{2n}\)
and higher powers we have

(2.4) \[ \Pr\{w_2 \leq x;m,n\} \approx C_1 \left[ \beta(m+1,n+1) \beta_2 - \beta(m+3,n+1) \sum_{j=0}^{m} \frac{m!}{j!} \frac{x^j(1-x)^{m+n-j+1}}{\prod_{k=0}^{m-j} (n+k+1)} \right. \]
\[ - \beta(m+2,n+1) \beta_1 + \beta(m+2,n+1) \sum_{j=0}^{m+1} \frac{(m+1)!}{j!} \frac{x^j(1-x)^{m+n-j+2}}{\prod_{k=0}^{m-j+1} (n+k+1)} \right]. \]
Adding and subtracting $\beta(m+3,n+1)\beta(m+1,n+1)$ and $\beta(m+2,n+1)^2$ in (2.4) we find

\begin{equation}
\Pr\{w_2 \leq x; m,n\} = c_1 \beta(m+1,n+1) + \beta(m+3,n+3)\beta_0 - 2\beta(m+2,n+1)\beta_1 - 1.
\end{equation}

Noting from (2.2) with $x=1$ that

\begin{equation}
c_1 = \frac{\beta(m+3,n+1)\beta(m+1,n+1) - \beta(m+2,n+1)^2}{-1},
\end{equation}

and simplifying in (2.5) we find

\begin{equation}
\Pr\{w_2 \leq x; m,n\} = \frac{m+n+2}{\beta(m+2,n+2)} \left[ \beta_2 + \frac{(m+1)\beta_2}{(m+n+2)\beta_0} - \frac{2m}{m+n+2} \beta_1 \right] - 1
\end{equation}

where

\begin{equation}
(a)_k = a(a+1) \cdots (a+k-1).
\end{equation}

This approximation is very simple for computational use and no products of incomplete beta functions are involved. Upper 1% and 5% points using (2.7) are given in Tables 16, 17. The error involved in using this approximation has been computed and the difference between the exact and approximate percentage points occurs in the seventh place. (See Table 15).
3. Case For Three Roots

When there are 3 roots, we have

\[(3.1) \quad \Pr\{w_3 \leq x; m, n\} = C_1\{\beta_o \beta_2 \beta_4 + 2\beta_1 \beta_2 \beta_3 - \beta_2^3 - \beta_o \beta_3^2 - \beta_1 \beta_4}\]

where \(\beta_k\) is defined in (2.3) and \(C_1\) in (1.2). As in the two-roots case we write the \(\beta_k\)'s as finite sums, expand, neglect terms involving \((1-x)^2n\) and higher powers and find

\[(3.2) \quad \Pr\{w_3 \leq x; m, n\} = C_1\{[\beta(3)\beta(5) - \beta(4)^2]\beta_o + 2[\beta(3)\beta(4) - \beta(2)\beta(5)]\beta_1\]

\[+[2\beta(2)\beta(4) + \beta(1)\beta(5) - 3\beta(3)^2]\beta_2 + 2[\beta(2)\beta(3) - \beta(1)\beta(4)]\beta_3\]

\[+ [\beta(1)\beta(3) - \beta(2)^2] \beta_4\} - 2\]

where \(\beta(j) = \beta(m+j, n+1)\). Simplifying in (3.2) we find

\[(3.3) \quad \Pr\{w_3 \leq x; m, n\} = [\beta(m+1, n+3)]^{-1}\{ \frac{(m+2)}{2} \beta_o - 2(m+3)(m+n+4)\beta_1\]

\[+ \frac{3m(m+n+6)+2n+7}{m+1}(m+n+4)\beta_2 - \frac{2(m+n+3)}{m+1}\beta_3 + \frac{(m+n+4)(m+n+3)}{2(m+1)2}\} - 2\]

As in the two-roots case, no products of incomplete beta functions are involved. Upper 1% and 5% points using (3.3) are given in Tables 18 and 19. Comparison of some exact and approximate values is made in Table 15.
4. Case For q Roots

Denote the product \( \beta_{i_1 i_2} \cdots \beta_{i_q} \) occuring in \( |(\beta_{i_q})| \) by \( \beta'_{i_1 i_2} \cdots \beta_{i_q} \)

where \( i_j \) are any of the integers \( 0,1,\ldots,2q-2 \). It can be seen from cases

with \( q=2 \) and \( 3 \) that \( \beta_{i_1 i_2} \cdots \beta_{i_q} \) is approximated by \( \beta'_{i_1 i_2} \cdots \beta_{i_q} \)

\[
(4.1) \quad \beta'_{i_1 i_2} \cdots \beta_{i_q} = \sum_{j=1}^{q} \prod_{k=1}^{q} \beta_{i_k + j} \beta_{i_j + k} \beta_{i_i + k + j} \beta_{i_j + k + j}
\]

where \( \beta_{i_j} \) is defined in (1.4). Thus the distribution of \( w_{q/2} \) can be

approximated by

\[
(4.2) \quad \Pr\{w_{q/2} \leq x;m,n\} \approx C_x \left| (\beta_{i_q}) \right|' - (q-1)
\]

where \( \left| (\beta_{i_q}) \right|' \) is obtained by replacing \( \beta_{i_1 i_2} \cdots \beta_{i_q} \) in \( |(\beta_{i_q})| \) by \( \beta'_{i_1 i_2} \cdots \beta_{i_q} \). By collecting the coefficients of incomplete beta functions

we get the following form which is simpler for computer computations

\[
(4.3) \quad \Pr\{w_{q/2} \leq x;m,n\} = C_x \sum_{k=0}^{2q-2} D_k \beta_{k-} - (q-1),
\]

where \( D_k \) is the sum of the cofactors of \( \beta(k+1) \) in the qxq matrix

\[
\begin{pmatrix}
\beta(1) & \beta(2) & \cdots & \beta(q) \\
\beta(2) & \beta(3) & \cdots & \beta(q+1) \\
\vdots & \vdots & \ddots & \vdots \\
\beta(q) & \beta(q+1) & \cdots & \beta(2q-1)
\end{pmatrix}
\]
where $\beta(j)$ is defined as $\beta(m+j,n+1)$, the usual beta function. Letting $q=2$ and $3$ in (4.3) we get (2.5) and (3.2) respectively.

5. Computation of Percentage Points

Based on the results of the previous sections upper 1% and 5% points were computed for $q=2,3,4,5,6$ on the CDC 6500 computer. Results are given to five significant figures for the arguments $m=0(1)5,7,10,15$ and $n=5(5)$ $30(10)40(20)120(40)200,300,500,1000$. As can be seen from the comparisons below the percentage points from the exact and approximate cdf’s agree through five figures and generally six.

Table 15. Comparison of Percentage Points From the Exact and Approximate CDF’s

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<th>$n$</th>
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<th>approximate</th>
<th>exact</th>
<th>approximate</th>
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In addition to the above, the approximate expression is attractive for two reasons; first, computation time is less for the approximation because we don’t evaluate a determinant at each step in the iteration scheme, as we do for the exact case; second, round off error is less troublesome in the approximate expression.
<table>
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<th>4</th>
<th>5</th>
<th>7</th>
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Table 17. Upper 5% Points of the Largest Root for q=2

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<th>10</th>
<th>15</th>
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CHAPTER VI
SUMMARY AND CONCLUSION

The study of some central and non-central distribution problems in real and complex multivariate analysis has been carried out in this work. The main objective has been to investigate the distributions of characteristic roots, and functions of characteristic roots, of certain matrices, with emphasis on various likelihood ratio criteria, in the central and non-central cases. Although in the first chapter a general form was found for the first three moments of \( U_i^{(p)} \) and the first two moments of \( V_i^{(p)} \), both in the non-central (linear) MANOVA case, it would be useful to obtain expressions for the general moments of both statistics. Further it is hoped that a general expression can be obtained for \( a_{k,\tau} \) coefficients given in Chapter I, although such expressions were given in some cases there.

In Chapter II we make use of Mellin's transform and Meijers G-function to find the non-central distributions of Wilks' \( \Lambda \)-criterion in the cases of MANOVA, canonical correlation and the equality of two covariance matrices. The densities were found for any \( p \) and the cdf of \( \Lambda \) was explicitly found for \( p = 2 \) but the cdf remains to be found for higher \( p \) values. Also power computations could be carried out. The exact distributions of the likelihood ratio criterion \( V_{p_1,p_2,\ldots,p_q;N} \) for testing
the independence of $q$-sets of variates under the null hypothesis are found in Chapter III for various combinations of $p_1$ values and general $N$ using the convolution operation. Due to the fact that the expressions become unwieldy, results were obtained when no more than two convolutions were necessary, but the work could be extended to more than two convolutions. $\chi^2$ correction factors, from which lower 1% and 5% points can be obtained, were tabulated in special cases but could be obtained for other cases using the results of Chapter III.

In Chapter IV the density functions of Wilks' $\Lambda$-criterion in the complex case for the three situations mentioned in Chapter II, are obtained using Mellin's transform and expressed in terms of Meijer's G-function. Results are obtained concerning the complex multivariate beta distribution and independent beta variates. While the density functions of the $U$-statistic and of Pillai's $V$-criterion were found for $p = 2$ in the complex central case, results haven't been obtained for larger $p$ values. Using Pillai's lemma on the product of an esf and a Vandermonde determinant, zonal polynomials of a Hermitian matrix were found through degree 8 and results for higher degrees could be found in the same manner. In Chapter V, an approximation to the distribution of the maximum characteristic root in the complex case is obtained. Although a computational form, using determinants, was found, it is felt that further research might yield a general term for the coefficient of each incomplete beta function involved. Tables were given for upper 1% and 5% points for values of $q = 2(1)6$. These tables could be easily extended using the results of Chapter V.
REFERENCES


