On an Asymptotic Representation of the Distribution
of the Characteristic Roots of $SS^{-1}$

by

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Mimeograph Series No. 165
July 1968

* This research was supported by the National Science Foundation Grant No. GP-7663.
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1. Introduction and Summary. Let $S_1 : pxp$ $(i = 1, 2)$ be independently
distributed as Wishart $(n_1', p, \Sigma_1')$. Let the characteristic roots of $S_1S_2^{-1}$
and $S_2^{-1}$ be denoted by $\lambda_i$ $(i = 1, 2, \ldots, p)$ and $\lambda_i$ $(i = 1, 2, \ldots, p)$
respectively such that $\lambda_1 > \lambda_2 > \ldots > \lambda_p > 0$ and $\lambda_1 > \lambda_2 > \ldots > \lambda_p > 0$.
Then the distribution of $\lambda_1, \ldots, \lambda_p$ can be expressed in the form (Khatri [8])

\[ (1.1) \quad C|A|^{-(\frac{3}{2}n_1 + \frac{3}{2}n_2 - p - 1)} \alpha_p (L) \int_{O(p)} |L_p + \Lambda|^{-1} \left( \frac{1}{2} \right)^{(n_1 + n_2)} (H'\bar{H}) \]

where

\[ C = 2^{-p} \pi^{p(p-1)/4} \frac{1}{p} \prod_{i=1}^{p} \Gamma \left( \frac{1}{2} \right) \Gamma_p \left( \frac{3}{2}n_1 + \frac{3}{2}n_2 \right) \left\{ \Gamma_p (\frac{1}{2}) \Gamma_p \left( \frac{3}{2}n_1 \right) \right\}^{-1} , \]

\[ \Gamma_p (t) = \pi \prod_{j=1}^{p} \Gamma \left( t - \frac{1}{2} j + \frac{1}{2} \right) , \quad \alpha_p (L) = \prod_{i < j} (\lambda_j - \lambda_i) , \]

$L = \text{diag} (\lambda_1, \ldots, \lambda_p)$, $\Lambda = \text{diag} (\lambda_1', \ldots, \lambda_p')$ and $(H'\bar{H})$ is the invariant
measure on the group $O(p)$. However, this form is not convenient for
further development. Also, since

\[ (1.2) \quad I = \int_{O(p)} |L_p + \Lambda|^{-1} \left( \frac{1}{2} \right)^{(n_1 + n_2)} (H'\bar{H}) = \sum_{k=0}^{p} \binom{p}{k} \frac{c' \prod_{i=1}^{p} \Gamma (\frac{i}{2})}{\prod_{i=1}^{p} \Gamma (\frac{i}{2})} \]

where

\[ c' = 2^{p} \pi^{p(p+1)/4} \prod_{i=1}^{p} \Gamma (\frac{i}{2}) \]

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and the zonal polynomial $C_k(T)$ of any $p 	imes p$ symmetric matrix $T$ is defined in James [7], the use of (1.2) in (1.1) gives a power series expansion, but the convergence of this series is very slow. In the one sample case G. A. Anderson [1] has obtained a gamma type asymptotic expansion for the distribution of the characteristic roots of the estimated covariance matrix. In this paper we obtain a beta type asymptotic representation of the roots distribution of $S_1 S_2^{-1}$ involving linkage factors between sample roots and corresponding population roots. A study is also made of the approximation to the distribution of $\nu_1, \ldots, \nu_p$ where 

$$\nu_i = \ell_i / (1 + \ell_i), \quad (i = 1, 2, \ldots, p).$$

If the roots are distinct the limiting distribution as $n_2$ tends to infinity has the same form as that of Anderson [1]. If, moreover, $n_1$ is assumed also large, then it agrees with Girshick's result [4].

2. The asymptotic representation of $I$. The procedure used to find the expansion of (1.2) is an extension of the method sketched below for the case $p = 2$. In the asymptotic theory it is necessary to assume $\ell_1 > \ell_2 > \ldots > \ell_p > 0$ and $\lambda_1 > \lambda_2 > \ldots > \lambda_p > 0$. For the simplification of notations we let $A = \lambda^{-1}$, i.e. $a_i = 1/\lambda_i$ (i = 1, \ldots, p), $0 < a_1 < a_2 < \ldots < a_p < \infty$, and $n = n_1 + n_2$. Thus for $p = 2$, let $0^+(2) = \{A \in 2(2); |A| = \pm 1\}$ then

$$(2.1) \quad I = 2 \int_{0^+(2)} |T|^{-2} A^{-2} H P H' \frac{n}{2} (H' dH).$$

Now let $H = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$, $-\pi < \theta \leq \pi$, so that $(H' dH) = d\theta$ and
\begin{equation}
I = 4 \left[ \prod_{i=1}^{2} (1 + a_{i} \ell_{i}) \right] \left[ 1 + \frac{1}{2} c_{12} (1 - \cos 2\theta) \right]^{-\frac{1}{2}} \theta^{\frac{n}{2}} d\theta
\end{equation}

where
\[
c_{12} = \frac{(a_{2} - a_{1})(\ell_{1} - \ell_{2})}{(1 + a_{1} \ell_{1})(1 + a_{2} \ell_{2})}.
\]

The integrand has a maximum of unity at \( \theta = 0 \) and then decreases to \( 1 + \frac{1}{2} c_{12} \) at \( \theta = \pm \frac{\pi}{2} \). Write (2.2) as

\begin{equation}
4 \left( \prod_{i=1}^{2} (1 + a_{i} \ell_{i}) \right) \left[ \frac{1}{2} \frac{\pi}{2} \right] \exp \left[ -\frac{1}{2} \frac{\pi}{2} \log \left( 1 + c_{12} (1 - \cos 2\theta) \right) \right] d\theta
\end{equation}

Since the integral is mostly concentrated in a small neighborhood of the origin, for large \( n \) we can expand the argument of the exponential function and \( \cos 2\theta \) in the usual power series and set the limit to be \( \pm \infty \) (see Erdélyi [3]). Thus for large degrees of freedom \( I \) is approximately

\begin{equation}
4 \left[ \prod_{i=1}^{2} (1 + a_{i} \ell_{i}) \right] \left[ \frac{1}{2} \frac{\pi}{2} \right] \exp \left\{ -\frac{n}{2} c_{12} \theta^{2} \right\} d\theta, \left\{ 1 + o \left( \frac{1}{n} \right) \right\}
\end{equation}

or

\begin{equation}
I \sim 4 \left[ \prod_{i=1}^{2} (1 + a_{i} \ell_{i}) \right] \left[ \frac{1}{2} \frac{\pi}{2} \right] \exp \left\{ \frac{1}{2} \theta \right\} \left\{ 1 + o \left( \frac{1}{n} \right) \right\}
\end{equation}
**Lemma 1.** Let $A$ and $L$ are defined as before then $f(H) = |I_p + A H L H'|$

$H \in O(p)$ attains its identical minimum value $|I_p + AL|$ when $H$ is of the form

\begin{equation}
H = \begin{pmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{pmatrix}.
\end{equation}

**Proof:**

$$df = d|I_p + A H L H'|$$

$$= d|I_p + \frac{1}{2} H L H' A^\frac{1}{2}|$$

$$= \text{tr} \left( I_p + \frac{1}{2} H L H' A^\frac{1}{2} \right)^{-1} \left( \frac{1}{2} d H L H' A^\frac{1}{2} + \frac{1}{2} H L d H' A^\frac{1}{2} \right)$$

$$= 2 \text{tr} L H' A^\frac{1}{2} \left( I_p + \frac{1}{2} H L H' A^\frac{1}{2} \right)^{-1} A^\frac{1}{2} H' d H.$$  

Note that $H' d H$ is a skew symmetric matrix therefore, $df = 0$ implies that $L H': A^\frac{1}{2} (I_p + \frac{1}{2} H L H' A^\frac{1}{2})^{-1} A^\frac{1}{2} H$ is a symmetric matrix. But $H' A^\frac{1}{2} \left( I_p + \frac{1}{2} H L H' A^\frac{1}{2} \right)^{-1} A^\frac{1}{2} H$ is itself a symmetric matrix and $L$ is a diagonal matrix with distinct positive roots,

so $H': A^\frac{1}{2} \left( I_p + \frac{1}{2} H L H' A^\frac{1}{2} \right)^{-1} A^\frac{1}{2} H$ has to be a diagonal matrix, say $D$.

Thus $I_p = A^\frac{1}{2} H (L - D^{-1}) H' A^\frac{1}{2}$. This can happen only if $H$ is of the form with $\pm 1$ in one position in a column or a row and zero in other positions.

After substituting those stationary values into $f(H)$ we obtain a general form

\begin{equation}
\Pi_{i=1}^{p} \left( 1 + a_i k_{i} \right),
\end{equation}

where $k_{i}$ is any permutation of $k_{i}(i = 1, \ldots, p)$. It is easy to see that (2.5) attains its minimum value when $k_{i} = k_{i}(i = 1, 2, \ldots, p)$. Or $f(H)$ attains its identical minimum value $|I_p + A L|$ when $H$ is of the form of (2.4).
The above lemma enables us to claim that, for large \( n \), the integrand of \( I \) is negligible except for small neighborhoods about each of these matrices of (2.4) and \( I \) consists of identical contributions from each of these neighborhoods so that

\[
I = e^P \int_{N(I)} \left| X_p + A \frac{X_p^{1/2} X_p^2}{2} \right|^{n/2} (X_p^{1/2} d X_p),
\]

where \( N(I) \) is a neighborhood of the identity matrix on the orthogonal manifold. Since any proper orthogonal matrix can be written as the exponential of a skew symmetric matrix we transform \( I \) under

\[
H = \exp S, \quad S \text{ a } pxp \text{ skew symmetric matrix},
\]

so that \( N(I) \rightarrow N(S = 0) \). The Jacobian of this transformation has been computed by G. A. Anderson [1],

\[
J = 1 + \frac{B - 2}{24} \text{ tr } S^2 + \frac{8 - B}{4 \times 6!} \text{ tr } S^4 + \ldots.
\]

Direct substitution of (2.7) into (2.6) yields

\[
\left| X_p + A \frac{X_p^{1/2} X_p^2}{2} \right|^{-n/2}
\]

\[
= \left| X_p + A \frac{X_p^{1/2} X_p^2}{2} \right|^{-n/2} \left| X_p + A \frac{X_p^{1/2} X_p^2}{2} \right|^{-1} \left( X_p + A \frac{X_p^{1/2} X_p^2}{2} \right) \left( X_p + A \frac{X_p^{1/2} X_p^2}{2} \right)^{-1}
\]

\[
= \left| X_p + A \frac{X_p^{1/2} X_p^2}{2} \right|^{-n/2} \left( X_p + A \frac{X_p^{1/2} X_p^2}{2} \right) \left( X_p + A \frac{X_p^{1/2} X_p^2}{2} \right)^{-1}
\]

\[
= \left| X_p + A \frac{X_p^{1/2} X_p^2}{2} \right|^{-n/2} \left( X_p + A \frac{X_p^{1/2} X_p^2}{2} \right) \left( X_p + A \frac{X_p^{1/2} X_p^2}{2} \right)^{-1}
\]

\[
\left| X_p + A \frac{X_p^{1/2} X_p^2}{2} \right|^{-n/2}.
\]
Lemma 2. For any p x p matrix $\mathcal{B}$ and its characteristic roots $b_i (i = 1 \ldots p)$, if
\[
\max_{1 \leq i \leq p} |b_i| < 1 \quad \text{then}
\]
\[
|\mathcal{I}_p + \mathcal{B}|^{-\frac{n}{2}} = \exp \left\{ -\frac{n}{2} \text{tr} \left( \mathcal{B} - \frac{\mathcal{B}^2}{2} + \frac{\mathcal{B}^3}{3} \ldots \right) \right\}.
\]

Proof:

\[
|\mathcal{I}_p + \mathcal{B}|^{-\frac{n}{2}} = \exp \left\{ -\frac{n}{2} \log \prod_{i=1}^{p} (1 + b_i) \right\}.
\]

If
\[
\max_{1 \leq i \leq p} |b_i| < 1 \quad \text{then}
\]
\[
|\mathcal{I}_p + \mathcal{B}|^{-\frac{n}{2}} = \exp \left\{ -\frac{n}{2} \text{tr} \left( \mathcal{B} - \frac{\mathcal{B}^2}{2} + \frac{\mathcal{B}^3}{3} \ldots \right) \right\}.
\]

Apply lemma 2 to (2.9) and the maximum characteristic roots of
\[
(I_p + A L)^{-1}(A S L - A L S + \ldots)
\]
can be assumed to be less than unity.

Since we are only interested in the first term we need to investigate the group of terms up to order of $S^2$ which is denoted by $\{S^2\}$. Let
\[
\mathcal{B} = (I + A L)^{-1},
\]
then
\[
\text{tr} \left\{ S^2 \right\} = \text{tr} \left[ \mathcal{B} (A LS^2 - A S L S) \right]
\]
\[
- \frac{1}{2} \begin{pmatrix}
SALSRLALS + RASLRLALS - RASLRLALS \\
- RALSRLALS
\end{pmatrix}.
\]

After simplification (2.12) reduces to

\[
\text{tr} \left[ \mathcal{B} (A LS^2 - A S L S) - (LS - SL)RLSRA \right]
\]

or
\[
\text{tr} \left\{ S^2 \right\} = \sum_{i < j}^{p} C_{ij} s_i^2 s_j^2
\]
\[(2.14) \quad c_{ij} = (a_j - \bar{a}_j)(l_i - \bar{l}_j)\left(1 + a_j l_i\right)\left(1 + a_j l_j\right).\]

Direct substitution into (2.1) yields
\[(2.15) \quad I = 2^p p \prod_{i=1}^{p} (1 + a_i l_i)^{-\frac{n_i}{2}} \sqrt{\frac{2\pi}{\Sigma C_{ij} s_{ij}}} \exp\left\{-\frac{n_i}{2} \Sigma C_{ij} s_{ij}^2\right\} \prod_{i<j} s_{ij} \left\{1 + 0\left(\frac{i}{n}\right)\right\}.\]

For large \(n\) the limits for each \(s_{ij}\) can be put to \(\pm \infty\). We finally have the following theorem.

**Theorem**: The asymptotic distribution of the roots, \(l_1 > l_2 \ldots > l_p > 0\), of \(S_{12}^{-1}\) for large degrees of freedom \(n = n_1 + n_2\) when the roots of \(\Sigma_{12}^{-1}\) are \(\lambda_1 > \lambda_2 \ldots > \lambda_p > 0\) and \(a_i = 1/\lambda_i (i = 1, \ldots p)\), is given by
\[(2.16) \quad c_{ij} = (\lambda_i) \prod_{i=1}^{p} \left[\left(l_i \frac{n_i}{2} (a_i)^{\frac{n_i}{2}} (1 + a_i l_i)^{-\frac{n_i}{2}}\right) \right] \prod_{i<j} s_{ij} \left\{1 + 0\left(\frac{i}{n}\right)\right\}^\frac{1}{2}.\]

The asymptotic formula shows that the distribution function of a group of adjacent roots is sensitive only to those other roots which are close to them.

3. **A Dual Expansion** cf. \(I\) and Some Remarks. If we let
\[\mathcal{I} = I_0(I_0 + I_1)^{-1}\]

in (1.1) i.e. \(w_i = \frac{\mathcal{I}_i}{(1 + \mathcal{I}_1)} (i = 1, 2, \ldots p)\) where \(\mathcal{I} = \text{diag}(w_1, \ldots w_p)\), then the joint distribution of \(w_i\)'s is given by
\[(3.1) \quad c_{ij} = \frac{\alpha_p(\mathcal{W})}{|\mathcal{W}|^{\frac{1}{2}}} \int_{\mathcal{W}} \left|I_{\mathcal{W}} - \mathcal{W}\right| \left|I_{\mathcal{W}}^+ - \mathcal{W}\right| \alpha_p(\mathcal{W})^2 |\mathcal{I}_0|^{\frac{1}{2}} |\mathcal{I}_0 + \mathcal{H}^+ L H^+|^{\frac{1}{2}} - \frac{n}{2} \quad \mathcal{W} = w_1 > w_2 > \ldots > w_p > 0.\]
Application of lemmas 1 and 2 to (3.1) yields its asymptotic representation

\[ C[A] \frac{n_1}{2} |W| \left( I_p - W \right) \frac{1}{2(p-1)\frac{n_1}{2} - \frac{n_2}{2} - \frac{1}{2}(n_1 + n_2)} \]

\[ \frac{p}{\prod_{i=1}^{p} (\sigma_i W_i)} \]

\[ \frac{p}{\prod_{i<j} \frac{1}{2} (a_i - a_j) \left( W_i - W_j \right)} \]

where \( C_{ij} = \frac{(a_i - a_j) (W_i - W_j)}{[1 + (a_i - 1) W_i][1 + (a_j - 1) W_j]} \).

Now let us proceed to look at (2.16) once again. The asymptotic distribution of characteristic roots of \( S_1 S_2^{-1} \) given there can be rewritten as

\[ F_1(A) \prod_{i<j} (\ell_i - \ell_j)^{\frac{1}{2}} \prod_{i=1}^{p} \left[ \ell_i^{2(p-1)} \left( 1 + a_i \ell_i \right) - \frac{n_2}{2} + \frac{n_i}{2} \right] \prod_{i=1}^{p} \text{d} \ell_i \]

where \( F_1(A) \) \((i = 1, 2, 3)\) depends on \( a_i \) but not on \( \ell_i \). If we make \( \xi_i = \ell_i/n_2 \) \((i = 1, 2, \ldots, p)\) and let \( n_2 \) tends to infinity then (3.3) reduces to the limiting form

\[ F_2(A) \prod_{i=1}^{p} \xi_i^{\frac{1}{2}} \text{e}^{\frac{1}{2} \sum_{i=1}^{p} a_i \xi_i} \prod_{i<j} (\xi_i - \xi_j) \]

Moreover, let \( \ell_i^{*} = n_1 \xi_i \) \((i = 1, 2, \ldots, p)\), then (3.4) becomes

\[ F_3(A) \prod_{i=1}^{p} \ell_i^{*}^{\frac{1}{2}(n_1 - p - 1)} \text{e}^{\frac{1}{2} \sum_{i=1}^{p} a_i \ell_i^{*}} \prod_{i<j} (\ell_i^{*} - \ell_j^{*}) \]

Note that \( \ell_i^{*} \)'s here are, in limiting sense, the characteristic roots of \( S_1^{*} S_2^{-1} \) where \( S_1^{*} \) is the covariance matrix of the first sample.
References


