On the non-central distributions of Wilks' $\Lambda^*$

for tests of three hypotheses

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Mimeograph Series No. 164

July, 1968

*This research was supported in part by the National Science Foundation, Grant No. GP-7663.
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1. Introduction and Summary. In multivariate analysis we are interested in testing three hypotheses, namely

1) that of equality of the dispersion matrices of two $p$-variate normal populations,

2) that of equality of the $p$-dimensional mean vectors for $k$ $p$-variate normal populations having a common covariance matrix and

3) that of independence between a $p$-set and a $q$-set of variates in a $(p+q)$-variate normal population, with $p \leq q$. We obtain the non-central distribution of Wilks' criterion $\Lambda = W^{(p)} = \prod_{i=1}^{p} (1-c_i)$ in each of the above cases, where the $c_i$'s are functions of the characteristic roots of the appropriate matrices. The density functions for case 2 have been obtained by Pillai and Al-Ani [8] for $p = 2, 3, 4$ and here we obtain the density functions for all three cases for general $p$ in terms of Meijer's G-function [7] with special cases being explicitly evaluated. In this connection a theorem has been proved using some results on Mellin transforms [2,3,4]. Also the cumulative distribution function (c.d.f.) of $W^{(p)}$ is obtained for $p = 2$ in the above three cases. The densities in all cases may be put in a single general form given by

*This research was supported in part by the National Science Foundation, Grant No. GP-7663.
\( (1.1) \quad f(w(p)) = \frac{\Gamma_p(\delta)}{\Gamma_p(\frac{1}{2}\gamma)} \alpha [w(p)]^{\frac{1}{2}(\gamma - p - 1)} \sum_{k=0}^{\infty} \frac{(\delta)_k \beta}{k!} c_k(M) \Gamma_p^{\gamma}(w(p)|_{b_1, b_2, \ldots, b_p})^{a_1, a_2, \ldots, a_p} \)

where

\[ a_i = \frac{1}{2}(2\delta - \gamma) + k_p - i + b_i \quad \text{and} \quad b_i = (i - 1)/2 \]

and

\[
\begin{array}{ccc}
\text{Case 1} & \text{Case 2} & \text{Case 3} \\
\gamma = n_2 & t & n - q \\
\delta = \frac{1}{2}n & \nu & \frac{1}{2}n \\
\beta = (\frac{1}{2}n)_k & 1 & (\frac{1}{2}n)_k \\
\alpha = |\lambda|^\frac{1}{2}n_2 & e^{-tr\Omega} & |I_p^\lambda - \frac{1}{2}I_p^2|^{\frac{1}{2}n} \\
M = I_p^\lambda - (\lambda)^{-1} \Omega & \Omega & \frac{1}{2}I_p^2 \\
\end{array}
\]

See the following sections for definitions of the parameters as well as the G-function.

2. Preliminary Results. Some results on Mellin transforms \([2,3,4]\) and Meijer's G-function \([7]\) useful in proving the theorem below will now be given.

**Lemma 1.** If \( s \) is any complex variate and \( f(x) \) is a function of a real variable \( x \), such that

\[ (2.1) \quad F(s) = \int_0^\infty x^{s-1} f(x) \, dx \]
exists, then under certain regularity conditions

\[ (2.2) \quad f(x) = (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} x^{-s} F(s) \, ds. \]

\( F(s) \) is called the Mellin transform of \( f(x) \) and \( f(x) \) is the inverse Mellin transform of \( F(s) \).

**Lemma 2.** If \( f_1(x) \) and \( f_2(x) \) are the inverse Mellin transforms of \( F_1(s) \) and \( F_2(s) \) respectively, then the inverse Mellin transform of \( F_1(s)F_2(s) \) is

\[ (2.3) \quad (2\pi i)^{-1} \int_{c-i\infty}^{c+i\infty} x^{-s} F_1(s)F_2(s) \, ds = \int_0^\infty f_1(u)f_2(uy) \, du/uy. \]

Meijer [7] defined the G-function by

\[ (2.4) \quad G_{m,n}^{p,q}(x|a_1,a_2,\ldots, a_p, b_1,b_2,\ldots, b_q) = (2\pi i)^{-1} \int_C x^s ds. \]

where \( C \) is a curve separating the singularities of \( \prod_{j=1}^{m} \Gamma(b_j-s) \prod_{j=1}^{n} \Gamma(1-a_j+s) \) from those of \( \prod_{j=1}^{n} \Gamma(1-a_j+s) \), \( q \geq 1, 0 \leq n \leq p \leq q, 0 \leq m \leq q; x \neq 0 \) and \(|x|<1 \) if \( q = p; x \neq 0 \) if \( q > p \). It is easily verified that
where the generalized hypergeometric function $\ _2F_1$ is given by James [5].

The $G$-function of (2.4) can be expressed as a finite number of generalized hypergeometric functions as follows.

$$G_{m,n}^{p,q}(x|b_1,\ldots,b_q) = \sum_{h=1}^{m+n+1} \prod_{j=1}^{m} \frac{\Gamma(b_j - b_h)}{\Gamma(b_j)} \prod_{j=1}^{n} \frac{\Gamma(1 + b_h - a_j)}{\Gamma(1 + b_h)} x^h$$

$$\cdot \ _pF_{q-1}(1+b_h - a_1, \ldots, 1+b_h - a_p; 1+b_h - b_1, \ldots, 1+b_h - b_q; (-1)^{p-m-n} x)$$

where the asterisk denotes that the number $1+b_h - b_h$ is omitted in the sequence $1+b_h - b_1, \ldots, 1+b_h - b_q$. Although the following theorem gives a more complicated form for expressing the $G$-function, it is useful in that expression (2,4) of Consul [4] and Lemma 1 of Pillai and Al-Ani [8] are special cases.

**Theorem 1.** If $s$ is a complex variate, $a_i, b_i, i=1,2,\ldots,p$ are reals, then for $p \geq 3$
\( g_{\beta,\alpha}(x|_{a_1,a_2,\ldots,a_p}) = \frac{b_{\beta}(1-x)^{c_{-1}}}{\Gamma(c_{1}+c_{2}+c_{3})} \sum_{i=1}^{\pi} \frac{(b_{\beta-1}+c_{\beta-1})(c_{i-1}+c_{i})}{(j_{i}!)^{j_{i}}!} \)

\[ \sum_{j=0}^{p} \frac{(c_{1},c_{2}+c_{3}+b_{1})}{(1-x)^{j_{i}}(c_{1}+c_{2}+c_{3})} \]

\[ \Gamma(g_{\ell}+j_{\ell}) \]

\[ j=1 \]

\[ \Gamma(h_{\ell}) \]

\[ \Gamma(\ell+1) \]

\[ \ell+1 \]

\[ \ell-1 \]

\[ \ell+3 \]

\[ \ell \]

where for notational convenience \( c_{1} = a_{i} - b_{i}, c = \sum_{i=1}^{p} c_{i}, c_{\ell} = \sum_{i=1}^{\ell-1} c_{i}, f_{\ell} = \sum_{i=1}^{\ell} j_{i} + j \),

\[ g_{h} = \sum_{i=1}^{\ell} c_{i} + \sum_{i=1}^{j_{i}} j_{i} + j \]

\[ h_{h} = \sum_{i=1}^{\ell} c_{i} + \sum_{i=1}^{j_{i}} j_{i} + j \]

and \( (a)_{k} = a(a+1)\ldots(a+k-1) \).

**Proof.** Using mathematical induction starting with \( p=3 \), we see making the substitution \( (a,b,c,m,n,p) \rightarrow (b_{3},b_{2},b_{1},c_{3},c_{2},c_{1}) \) in (2.4) of Consul [3] that

\( g_{3,0}(x|_{a_{1},a_{2},b_{3}}) = \frac{b_{3}(1-x)^{c_{1}+c_{2}+c_{3}-1}}{\Gamma(c_{1}+c_{2}+c_{3})} \sum_{j=0}^{\infty} \frac{(c_{1})(b_{2}+c_{2}+b_{3})}{j!(c_{1}+c_{2}+c_{3})} (1-x)^{j} \)

\[ \binom{2}{1} \]

\[ (b_{3}+c_{3}+b_{2}, c_{1}+c_{2}+j; c_{1}+c_{2}+c_{3}+j; 1-x) \]

which is (2.6) with \( p=3 \). Now assuming (2.6) is true for \( p=n \), we show it holds for \( p=n+1 \). Applying Lemma 2 with
\[ F_1(s) = \frac{\prod_{i=1}^{n} \Gamma(s+b_i)}{\prod_{i=1}^{n} \Gamma(s+a_i)} \quad \text{and} \quad F_2(s) = \frac{\Gamma(s+b_n+1)}{\Gamma(s+a_n+1)} \]

we have \( f_1(x) \) is (2.6) with \( p=n \) and \( f_2(x) = \frac{b_{n+1}}{c_{n+1}} \frac{c_{n+1}-1}{\Gamma(c_{n+1})} \)

and it follows that

\[
(2.8) \quad g_{n+1,o}^{n+1,n+1}(x|b_1,a_2,\ldots,a_{n+1}) = \frac{x^{b_{n+1}}}{\Gamma(c_1+c_2+c_3) \Gamma(c_{n+1})} \int_0^1 u^{b_{n+1}-c_{n+1}} (1-u)^{n+1} \sum_{i=1}^{n} c_i^{i+1} \]

\[
\times \Pi \left( \sum_{i=1}^{n-3} \frac{(b_{n-i+1}+c_{n-i+1}-b_i)_{j_i}}{(j_i)!} \right) \sum_{j=0}^{j+\Sigma_{j_i}} \frac{(c_1+j_1+b_2+c_2-b_1)_j}{(c_1+c_2+c_3)_j} \frac{1}{j!} \sum_{i=1}^{n-3} \frac{\Gamma(g_{k}+j_{k})}{\Gamma(h_{k})} \]

\[
\times \frac{\Gamma^{n-2}(b_3+c_3-b_2,f_1,f_2,\ldots,f_{p-2};g_1,g_2,\ldots,g_{p-2};1-u)(u-x)^{c_{n+1}-1}}{du}. \]

Expanding \( u^{b_{n+1}-c_{n+1}} \) in powers of \( 1-u \) when \( b_{n+1}+c_{n+1} > b_n \), letting \( u = x+(1-x)t \) and integrating with respect to \( t \), the result is the same as (2.6) with \( p=n+1 \).

It is easily verified that Lemma 1 of Pillai and Al-Ani [8] is a special case of (2.6) with \( p=4 \) by making the following substitution

\( (b_1,b_2,b_3,b_4,c_1,c_2,c_3,c_4) \rightarrow (c,b,a,d,p,n,m,l) \).
It should be mentioned that this theorem doesn't apply when \( p=1,2 \). This is due to the fact that a simplification in the form of the \( G \)-function for \( p=3 \) reduces the hypergeometric function involved from \( 3F_2 \) to \( 2F_1 \). A general form for all \( p \) can be given as below, but we see it is more cumbersome to use because we have \( pF_{p-1} \) rather than \( p-1F_{p-2} \) as in (2.6)

\[
G_{p,p}^{0 \, p}(x|_{b_1}^{a_1, a_2, \ldots, a_p}^b) = x^{b_1(1-x)} \frac{(b_1+x)_c-1}{\Gamma(c)} \prod_{i=1}^{p-3} \frac{(b_1+c_i-b_i+1)x^{p-i-2}}{(p-i-2)!} \sum_{r=0}^{\infty} \frac{\prod_{i=1}^{p} (\sum_{j=1}^{p-1} c_j + \sum_{j=1}^{p} \ell_j + r)x^\ell}{r! (c)_{\ell+r}} \cdot \frac{F_{p-1}^{p-1}(c, b_1, p-1-b, f_1, \ldots, f_{p-2}; c, p-1+c_1, \ldots, c_{p-1}; 1-x)}{0 < x < 1}
\]

where

\[
\ell = \sum_{i=1}^{p} \ell_i, f_i = \sum_{i=1}^{p} c_i + \sum_{j=p-1}^{p} \ell_j + r, \quad g_i = \sum_{j=p-1}^{p} c_j + \sum_{j=1}^{p} \ell_j + r, \quad c = \sum_{i=1}^{p} c_i.
\]

It follows that letting \( p=2 \) we get (2.5) and \( p=1 \) gives

\[
G_{1,1}^{0 \, 1}(x|_{b_1}^{a_1}) = x^{b_1(1-x)} \frac{c_{1-1}}{\Gamma(c_1)}.
\]
3. The Non-Central Distribution of $W(p)$ in Case 1. Let $X(p x \sim n_1)$ and $Y(p x \sim n_2)$ be independent matrix variates with the columns of $X$ independently distributed as $N(0, \Sigma_1)$ and those of $Y$ independently distributed as $N(0, \Sigma_2)$. Hence $\Sigma_1 = XX'$ and $\Sigma_2 = YY'$ are independently distributed as Wishart $(n_i, \Sigma_i)$, $i=1,2$. Let $0 < f_1 < f_2 < \ldots < f_p < \infty$ be the characteristic (ch.) roots of the determinantal equation

$$|\Sigma_1 - f \Sigma_2| = 0$$

and $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_p < \infty$ be the ch. roots of

$$|\Sigma_1 - \gamma \Sigma_2| = 0.$$ 

For testing the hypothesis $H_0: \lambda \sim \Lambda \sim I_p$, $\lambda > 0$ being given, we will use

$$W(p) = \prod_{i=1}^{p} (1-w_i)$$

where

$$\Lambda \sim \text{diag} (\lambda_1, \lambda_2, \ldots, \lambda_p), \ w_i = \lambda f_i / (1-\lambda f_i) \quad \text{for} \ i=1,2,\ldots,p.$$ 

Khatri [6] has shown that
\[(3.4) \quad f(w_1, w_2, \ldots, w_p) = c |\lambda\rangle^{\frac{3}{2}n_1 - \frac{1}{2}} |W\rangle^{\frac{3}{2}(n_1 - p - 1)} |I_p - W\rangle^{\frac{3}{2}(n_2 - p - 1)} \prod_{1 < j < n} (w_i - w_j)^{-1} \Gamma_o(\frac{3}{2}n_1; I_p - (\lambda\rangle, W)\]

where

\[W = \text{diag}(w_1, w_2, \ldots, w_p), \quad n = n_1 + n_2, \quad \Gamma_p(t) = \frac{\pi^{p-1}}{2^{p-1}} \prod_{j=1}^{p} \Gamma(t - \frac{1}{2} + \frac{1}{2}),\]

\[c = \pi^{\frac{3p}{2}} \Gamma_p(\frac{1}{2}) \Gamma_p(\frac{3}{2}n_1) \Gamma_p(\frac{1}{2}n_2)]^{-1}.

To find \(E[\langle W^p \rangle]^{\frac{1}{2}}\) we multiply (3.4) by \(|I_p - W\rangle^{\frac{1}{2}} = \prod_{i=1}^{p} (1 - w_i)\] and transform

\[W \sim H V H^\dagger, \quad \text{where } H \text{ is an orthogonal and } V \text{ is a symmetric matrix,}
\]

integrate out \(H \text{ and } V\) using (44) and (22) of Constantine [1] and we find

\[(3.5) \quad E[\langle W^p \rangle]^{\frac{1}{2}} = \frac{\Gamma_p(\frac{3}{2}n_1) \Gamma_p(\frac{3}{2}n_2 + h)}{\Gamma_p(\frac{3}{2}n_1 + h)} |\lambda\rangle^{\frac{3}{2}n_1} \frac{1}{2} \Gamma_1(\frac{3}{2}n_1, \frac{3}{2}n_2 + h; I_p - (\lambda\rangle)^{-1}.

Using Lemma 1, the density of \(f(W(p))\) has the form

\[(3.6) \quad f(W(p)) = c_p \sum_{k=0}^{\infty} \sum_{k} \frac{(\frac{3}{2}n_1)_{(\frac{3}{2}n_1)_k}}{k!} c_k (I_p - (\lambda\rangle)^{-1} \{W(p)\}^{\frac{3}{2}(n_2 - p - 1)}
\]

\[c + i\omega \quad \int_{c - i\infty}^{c + i\infty} \frac{\prod_{i=1}^{p} \Gamma(r + b_i)}{\prod_{i=1}^{p} \Gamma(r + a_i)} dr\]
where
\[ r = \frac{3n_x}{2} + h - \frac{1}{2}(p-1), \quad b_i = \frac{3}{2}(i-1), \quad a_i = \frac{3n_x}{2} + k_{p-i+1} + b_i, \]

\[ c_p = \frac{\Gamma_p(\frac{3n_x}{2})}{\Gamma_p(\frac{3n_x}{2})} \left| \lambda \right|^{\frac{3n_x}{2}} \left( a \right)_{\frac{p}{k}} = \prod_{i=1}^{p} \Gamma(a-i), \quad (a)_k = a(a+1) \cdots (a+k-1), \]

\[ \sum_{\lambda} \text{ is the sum over all partitions } \lambda \text{ of the integer } k \text{ where} \]

\[ \lambda = (k_1, k_2, \ldots, k_p), \quad k_1 \geq k_2 \geq \ldots \geq k_p > 0, \quad \sum_{i=1}^{p} k_i = k, \text{ and} \]

\[ c_k(\lambda) \text{ is a zonal polynomial; see James [5].} \]

Noting that the integral in (3.6) is in the form of Meijer's G-function we can write the density of \( W(p) \) as

\[ f(W(p)) = c_p(W(p)) \frac{\Gamma(a_1, a_2, \ldots, a_p)}{\Gamma(a_1 + a_2 + \cdots + a_p)} \sum_{k=0}^{\infty} \sum_{\lambda} \frac{(\frac{3n_x}{2})_k (\frac{3n_x}{2})_{\lambda-k}}{k!} c_k(\lambda, -1) \frac{\Gamma(\frac{3n_x}{2} - (\lambda)_k)}{\Gamma(\frac{3n_x}{2})} \frac{\Gamma((a_1)_k, (a_2)_k, \ldots, (a_p)_k)}{\Gamma((a_1 + a_2 + \cdots + a_p)_k)} \cdot 2F1(a_1-b_2, a_1-b_2, a_1+a_2-b_1-b_2; 1-W(2)). \]

Letting \( p=2 \) in (3.7) and using (2.5) we obtain

\[ f(W(2)) = c_2(W(2)) \frac{\Gamma(a_1, a_2, b_1, b_2)}{\Gamma(a_1 + a_2 + b_1 + b_2)} \sum_{k=0}^{\infty} \sum_{\lambda} \frac{(\frac{3n_x}{2})_k (\frac{3n_x}{2})_{\lambda-k}}{k!} c_k(\lambda, -1) \frac{\Gamma(\frac{3n_x}{2} - (\lambda)_k)}{\Gamma(\frac{3n_x}{2})} \frac{\Gamma((a_1)_k, (a_2)_k, b_1, b_2)}{\Gamma((a_1 + a_2 + b_1 + b_2)_k)} 2F1(a_1-b_2, a_1-b_2, a_1+a_2-b_1-b_2; 1-W(2)). \]
The probability that \( W^{(2)} \leq w(\leq 1) \) can be obtained by integrating \( (3.8) \) by parts \( n_1 \) times when \( n_1 \) is even. Using the relation \([3]\)

\[
(3.9) \quad (d^n/dz^n)[z^{c-1} 2F_1(a,b;c;z)] = (c-n)_n z^{c-n-1} 2F_1(a,b;c-n;z),
\]

and recalling that \( \kappa=(k_1,k_2) \), we obtain the c.d.f. of \( W^{(2)} \) in terms of \( a_i \)'s and \( b_i \)'s as

\[
(3.10) \quad \Pr[W^{(2)} \leq w] = |\lambda|^{-\frac{1}{2}n_1} \sum_{k=0}^{\infty} \sum_{\kappa} \frac{(\frac{1}{2}n_1)\gamma_k}{\Gamma_k} \binom{\frac{1}{2}(n_2-1)}{k} w^{\frac{1}{2}(n_2-1)}
\]

\[
\cdot \left\{ \frac{\Gamma_2(\frac{1}{2}n_2)\Gamma_2(n_1)}{\Gamma_2(\frac{1}{2}n_1)\Gamma_2(n_1+\frac{1}{2}a_2-b_1-b_2-r)} \sum_{r=0}^{a} \frac{(a_1+a_2-b_1-b_2-r)_r}{\{\frac{1}{2}(n_2-1)\}_{r+1}} w^{(1-w)^{a_1+a_2-b_1-b_2-r-1}} \right\}
\]

\[
\cdot 2F_1(a_2-b_2,a_1-b_2,a_1+a_2-b_1-b_2-r;1-w) + I_w(\frac{1}{2}n_2, b) \}
\]

where \( a_i, b_i \) are defined in \( (3.6) \), \( a=a_1-1 \) and \( b=a_2-b_2 \). When \( n_1 \) is odd, after integrating \( (3.8) \) by parts \( n_2 \) times, the c.d.f. of \( W^{(2)} \) is \( (3.10) \) with \( a=a_2-1 \) and \( b=a_1-b_2 \).

Letting \( p=3 \) in \( (3.7) \) we have
\[ f(W(3)) = \frac{\Gamma_3(\frac{3n}{2})}{\Gamma_3(\frac{1}{2}n_2)} |\lambda\lambda|^{-\frac{3}{2}n_1} [W(3)]^{\frac{1}{2}(n_2-4)} \]

\[ \sum_{k=0}^{\infty} \sum_{\frac{3n}{2}K, \frac{1}{2}n_1K} \frac{(\frac{3n}{2}K, \frac{1}{2}n_1K)}{k!} C_k(I_3-(\lambda\lambda)^{-1}) G_3,0(W(3)|_{b_1, b_2, b_3}^{a_1, a_2, a_3}) \]

where \( a_i \) and \( b_i \) are defined in (3.6).

It is clear \( G_3,0(W(3)|_{b_1, b_2, b_3}^{a_1, a_2, a_3}) \) could be written out in terms of the hypergeometric function using Theorem 1, for computation purposes.

Also letting \( p = 4 \) in (3.7) yields

\[ f(W(4)) = \frac{\Gamma_4(\frac{1}{2}n)}{\Gamma_4(\frac{1}{2}n_2)} |\lambda\lambda|^{-\frac{1}{2}n_1} [W(4)]^{\frac{1}{2}(n_2-5)} \]

\[ \sum_{k=0}^{\infty} \sum_{\frac{1}{2}nK} \frac{(\frac{1}{2}nK)}{k!} C_k(I_4-(\lambda\lambda)^{-1}) G_4,0(W(4)|_{b_1, b_2, b_3, b_4}^{a_1, a_2, a_3, a_4}) \]

where \( a_i \)'s and \( b_i \)'s are defined in (3.6).

4. The Non-Central Distribution of \( W(p) \) in Case 2. Let \( \Lambda = W(p) = \prod_{i=1}^{p} (1-\ell_i) \)

where \( \ell_1, \ell_2, \ldots, \ell_p \) are the ch. roots of the determinantal equation
\[(4.1) \quad |S_1 - \ell(S_1 + S_2)| = 0\]

where \(S_1\) is a \((p \times p)\) matrix distributed as non-central Wishart with \(s\) degrees of freedom, \(\tilde{\Omega}\) is a matrix of non-centrality parameters and \(S_2\) has the Wishart distribution with \(t\) degrees of freedom, the covariance matrix in each case being \(\Sigma\). Pillai and Al-Ani [8] obtained the density of \(\tilde{W}(p)\) for \(p=2,3,4\). Here we obtain the density of \(\tilde{W}(p)\) in general in terms of Meijer's G-functions. As in section 3, applying Lemma 1 to the expression for \(E[\tilde{W}(p)]^h\) given by Constantine [1] and using \(2.4\) we find

\[(4.2) \quad f(\tilde{W}(p)) = C_p[\tilde{W}(p)]^{(t-p-1)/2} \sum_{k=0}^{\infty} \frac{(v)_k c_k(\tilde{\Omega})}{k!} g_p, p \tilde{W}(p)^{\frac{a_1}{2}, \frac{a_2}{2}, \ldots, \frac{a_p}{2}}_{b_1, b_2, \ldots, b_p}\]

where

\[v = \frac{1}{2}(s+t), \quad C_p = \frac{\Gamma_p(v)}{\Gamma_p(s+t) e^{-tr\tilde{\Omega}}}, \quad b_i = \frac{1}{2}(i-1), \quad a_i = \frac{1}{2}s + k, \quad p-i+1 + b_i\]

The probability that \(\tilde{W}(2) \leq w(\leq 1)\) can be obtained by using \(2.5\) in \((4.2)\), integrating by parts \(a_1\) times when \(s\) is even, then using \(3.9\) we get the c.d.f. of \(\tilde{W}(2)\) as
\[(4.3) \quad \Pr\{W^{(2)} \leq w\} = e^{-\text{tr}\Omega} \sum_{k=0}^{\infty} \frac{C_k(\Omega)}{k!} \left(\frac{1}{\sqrt{\pi (t-1)}} \frac{\Gamma_2(v)(v)}{\Gamma_2(\frac{1}{2}t)\Gamma(a_1+a_2-b_1-b_2)} \right) \]

\[
\times \sum_{r=0}^{a} \frac{(a_1+a_2-b_1-b_2-r)}{[\frac{1}{2}(t-1)]_{r+1}} \Gamma(a_1+a_2-b_1-b_2-r-1) \Gamma(a_2-b_2, a_1-b_2; a_1+a_2-b_1-b_2-r; 1-W^{(2)}) \]

\[
+ I_w(\frac{1}{2}t, b)) \]

where

\[a = a_1 - 1, b = a_2 - b_2\] and the \(a_i\)'s and \(b_i\)'s are defined in \(4.2\). When \(s\) is odd, we integrate \(4.2\) by parts \(a_2\) times and find the c.d.f. is \(4.3\) with \(a=a_2-1, b=a_1-b_2\).

The densities of \(W^{(3)}\) and \(W^{(4)}\) obtained by Pillai and Al-Ani [8] are special cases of \(4.2\) as can be verified by letting \(p=3,4\) in \(4.2\), applying Theorem 1 and making the substitution

\[(a_1, a_3, b_1, b_3) \rightarrow (a_3, a_1, b_3, b_1).\]

5. **The Non-Central Distribution of \(W^{(p)}\) in Case 3.** Let the columns of

\[
\begin{pmatrix}
X_1 \\
X_2
\end{pmatrix}
\]

be independent normal \((p+q)\) variates \((p \leq q, p+q \leq n, n\) is the sample size) with zero means and covariance matrix
\begin{equation}
\Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\
\Sigma_{21} & \Sigma_{22} \end{pmatrix}.
\end{equation}

Let $R^2 = \text{diag}(r_1^2, r_2^2, \ldots, r_p^2)$ where $r_i^2$ are the ch. roots of

\begin{equation}
|x_{i\neq j} x_{i\neq j}^{-1} x_{i\neq j} - r^2 x_{i\neq j}^2| = 0
\end{equation}

and $P^2 = \text{diag}(\rho_1^2, \rho_2^2, \ldots, \rho_p^2)$ where $\rho_i^2$ are the ch. roots of

\begin{equation}
|\Sigma_{22} \Sigma_{22}^{-1} \Sigma_{12} - P^2 \Sigma_{11}| = 0.
\end{equation}

Constantine [1] obtained the density of $r_1^2, r_2^2, \ldots, r_p^2$ as

\begin{equation}
f(r_1^2, r_2^2, \ldots, r_p^2) = C \left| I_{p-2} - R^2 \right|^{\frac{1}{2} n} \left| R^2 \right|^{\frac{1}{2} (q-p-1)} \left| I_{p-2} - R^2 \right|^{\frac{1}{2} (n-q-p-1)}
\end{equation}

\begin{equation}
\cdot \prod_{i<j} \sum_{k=0}^\infty \frac{(\frac{1}{2}n)_k (\frac{1}{2}n)_k}{(\frac{1}{2}q)_k} C_k\begin{pmatrix} R^2 \\ \Sigma \end{pmatrix} C_k(R^2)
\end{equation}

where

\begin{equation}
C = n^{\frac{1}{2}p^2} \Gamma_p(\frac{1}{2}n) \left[ \Gamma_p(\frac{1}{2}q) \Gamma_p(\frac{1}{2}(n-q)) \Gamma_p(\frac{1}{2}p) \right]^{-1}.
\end{equation}
To find $E[W(p)^h]$, $W(p) = \prod_{i=1}^{\mathbf{P}} (1-r_i^2)$, we multiply (5.4) by $|I_p-R^2|^h$, proceed as in section 3 for case 1 and we find

$$E[W(p)^h] = \frac{\Gamma_p(\frac{3n}{2})\Gamma_p(\frac{1}{2}(n-q)+h)}{\Gamma_p(\frac{1}{2}(n-q))\Gamma_p(\frac{1}{2}n+h)} |I_p-R^2|^{\frac{1}{2}n} \gamma_p^2(\frac{\frac{1}{2}n}{2},\frac{\frac{1}{2}n}{2};\frac{1}{2}n+h;\frac{1}{2}n)^2.$$  

(5.5)

Noting that (5.5) can be obtained from (3.5) by substituting

$$(n_2, n_1, (\lambda_1)^{-1}) \to (n-q, n, I_p-R^2)$$

(5.6)

it can be verified that the density of $W(p)$ in this case is

$$f(W(p)) = c_p[W(p)]^{\frac{1}{2}(n-q-p-1)} \sum_{k=0}^{\infty} \frac{(\frac{1}{2}n)_k(\frac{1}{2}n)_kC_k(P^2)}{k!} g_p,\gamma(W(p)) |_{b_1, b_2, \ldots, b_p}^{a_1, a_2, \ldots, a_p}$$

(5.7)

where

$$c_p = \frac{\Gamma_p(\frac{1}{2}n)}{\Gamma_p(\frac{1}{2}(n-q))} |I_p-R^2|^{\frac{1}{2}n}, a_i = \frac{1}{2}q+k_p-i+1+b_i, b_i = \frac{1}{2}(i-1).$$

The c.d.f. of $W^{(2)}$ is obtained from (3.10) when $q$ is even by substituting as in (5.6) and using the $a_i$'s as just defined.
For $q$ odd the c.d.f. of $W^{(2)}$ follows from that of case 1 for $n_1$ odd by making the substitution (5.6) and using the $a_i$'s just defined. The densities of $W^{(p)}$ for $p=2,3,4$ follow from (3.8), (3.11), (3.12) respectively making substitution (5.6).

Acknowledgement

The author would like to express his sincere thanks to Professor K.C.S. Pillai of Purdue University for suggesting this area of study and for his discussion and guidance in the preparation of this paper.


