On the distributions of Hotelling's $T^2_0$ for three latent roots and the smallest root of a covariance matrix*

by

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1. Introduction and Summary. In this paper, first, the distribution of $U^{(s)}$, the sum of the $s$ non-null characteristic roots of a matrix (which is a constant times Hotelling's $T^2_0$) is derived for $s = 3$, starting with the joint density of the $s$ roots given by Roy [10] (see Section 2). The C.D.F. of $U^{(3)}$ thus obtained is used to compute upper 5 per cent points for selected values of two sample parameters which show that the approximate percentage points given by Pillai [8] are generally accurate to the three decimals provided. The distribution of the sum of the three smallest roots of a sample covariance matrix is obtained next for $p = 4$, where $p$ is the number of variables, taking the population covariance matrix $\Sigma = I$. Further, the distribution of the smallest characteristic root of a sample covariance matrix is derived for an arbitrary $\Sigma$. For tests based on the sum of the $i$ smallest of $p$ roots and the smallest root alone of a covariance matrix, reference may be made to [1], [9], [10].

2. Exact distribution of $U^{(3)}$. The distribution of non-null characteristic roots of a matrix derived from sample observations taken from multivariate normal populations, given by Roy [10], is of the form

*The work of this author was supported by the National Science Foundation, Grant Number GP-7663.
\[ f(\lambda_1, \lambda_2, \ldots, \lambda_s; m, n) = C(s, m, n) \prod_{i=1}^{s} \frac{\lambda_i^m}{(1 + \lambda_i)^{m+n+s+1}} \prod_{i<j} (\lambda_i - \lambda_j) \]

\[ 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_s < \infty \]

where \[ C(s, m, n) = \prod_{i=1}^{s} \left\{ \Gamma\left[\frac{1}{2}(2m+2n+i+2)\right] \Gamma[\frac{1}{2}(2m+i+1)] \Gamma[\frac{1}{2}(2n+i+1)] \Gamma(\frac{1}{2}) \right\} \]

and \( m \) and \( n \) are defined differently for various situations described by Pillai [7] and [8]. In this section, we will obtain the density of \( U(3) = \lambda_1 + \lambda_2 + \lambda_3 \) with \( s = 3 \). First put \( s = 3 \) in (2.1) and let \( \ell_i = \lambda_i/\lambda_3 \), \( i = 1, 2, \) then we have

\[ C(3, m, n) \lambda_3^{3m+5} (\ell_2 - \ell_1) \prod_{i=1}^{2} \ell_i^{m}(1-\ell_i) / (1+\lambda_3)^3(m+n+4)(1-d)^{m+n+4} \]

\[ 0 < \lambda_3 < \infty, \quad 0 < \ell_1 \leq \ell_2 < 1 \]

where \( d = (\frac{\lambda_3}{1+\lambda_3})(2-\ell_1-\ell_2) - (\frac{\lambda_3}{1+\lambda_3})(1-\ell_1)(1-\ell_2) \). It can be shown that \( 0 < d < 1 \) and we expand (2.2) in the following series form:

\[ C(3, m, n) \lambda_3^{3m+5} (\ell_2 - \ell_1) \prod_{i=1}^{2} \ell_i^{m}(1-\ell_i) / (1+\lambda_3)^3(m+n+4) \sum_{k=0}^{\infty} \frac{(m+n+4)_k}{k!} \frac{d^k}{k!} \]

where \( (a)_k = a(a+1)\ldots(a+k-1) \) and \( (a)_0 = 1 \). Now transform \( M = \ell_1 + \ell_2 \) and \( G = \ell_1 \ell_2 \), then the joint density of \( M, G \) and \( \lambda_3 \) is given by

\[ C(3, m, n) \lambda_3^{3m+5} (1+\lambda_3)^3(m+n+4) \sum_{k=0}^{\infty} \frac{(m+n+4)_k}{k!} \frac{(-1)^{k-j}}{(k-j)!} j! (2-M)^j \frac{\lambda_3^{2k-j}}{(1+\lambda_3)^j} \]

\[ (1-M+G)^{k-j+1} G^j \]
(2.4) is true only if both \( m \) and \( n \) are non-negative integers. We may integrate by parts term by term with respect to \( G \) from 0 to \( M^2/4 \) for \( 0 < M \leq 1 \) and from \( M - 1 \) to \( M^2/4 \) for \( 1 < M \leq 2 \). Further, transform \( U^{(3)} = \lambda_3(M+1) \) and integrate with respect to \( \lambda_3 \) from \( U^{(3)}/2 \) to \( U^{(3)} \) for \( 0 < M \leq 1 \) and from \( U^{(3)}/3 \) to \( U^{(3)}/2 \) for \( 1 < M < 2 \), we have finally the density of \( U^{(3)} \)

\[
(2.5) \quad C(3,m,n)m! \sum_{k=0}^{\infty} \left( \frac{m+n+4}{k} \right)_k \frac{(-1)^k}{(k-j)!} j! \left\{ \sum_{\forall \gamma} \sum_{\forall \sigma} \eta_1(m,k,j,v,p,q) \right\}
\]

\[
u^{n+j} B\left(\frac{u}{3+u}, \frac{u}{1+u}; 3m+a, 3n+b\right) + \frac{1}{\gamma} \frac{k-j+m+2}{\gamma} \eta_2(m,k,j,s,t) u^{m+d-7} B\left(\frac{u}{2+u}, \frac{u}{1+u}; 2m+c, m+3n+d\right)
\]

where

\[
\eta_1(m,k,j,v,p,q) = \frac{(2m-2v)^2(2k-j+2v+4)}{p} \frac{(-1)^k v-p-q}{q} \frac{3^q}{(k-j+2)^{4+2m}(m-v)!},
\]

\[
\eta_2(m,k,j,s,t) = \frac{j^s}{s} \frac{k-j+m+2}{t} \frac{2^t}{(k-j+2)^{m+1}} \frac{(-1)^{j-s-t}}{3^j},
\]

\[
a = 1-p-2v+q, \quad b = 2k-j+p-q+2v+11, \quad c = k-j+t+s+3, \quad d = k-t+s+9,
\]

and \( B(x_1x_2; p, q) = \int_{x_1}^{x_2} y^{p-1}(1-y)^{q-1} dy, \quad 0 \leq x_1 \leq x_2 \leq 1 \). Although

\[
(2.5) \text{is expressed in a series form, it converges for all values of } 0 < u < \infty. \text{ Further, the C.D.F. of } U^{(3)} \text{ obtained from (2.5) is of the form}
\]
\[(2.6) \quad P[U(3) \leq x] = c(3, m, n) m! \sum_{k=0}^{\infty} (m+n+k)^k \left( \sum_{j=0}^{(k-j)/2} \frac{(-1)^j (k-j)!}{j!} \right) \sum_{v=0}^{\infty} \sum_{p=0}^{v} \sum_{q=0}^{p} \frac{m-2v}{2k-j+2v+4} \]

\[
\eta_{\infty}(m, k, j, v, p, q) = \frac{b-6}{B \left( x^{b-6}, \frac{x}{3^{1+x}} ; 3m+a, 3n+b \right) + 3^{b-6} B(0, \frac{x}{3^{1+x}} ; 3m+2k-j+6, 3n+6) - B(0, \frac{x}{1+x} ; 3m+2k-j+6, 3n+6) + \sum_{s=0}^{j} \frac{k-j+2+m}{s} \left( \sum_{t=0}^{s} \right)}{m+d-6} \]

\[
\eta_{\infty}(m, k, j, v, p, q) = \left[ x^{m+d-6} B(\frac{x}{2^{1+x}}, \frac{x}{1+x} ; 2m+c, m+3n+d) + 2^{m+d-6} B(0, \frac{x}{2^{1+x}} ; 3m+2k-j+6, 3n+6) \right], \quad 0 < x < \infty.
\]

The C.D.F. of \( U(3) \) in (2.6) has been used to compute upper 5 percent points for selected values of \( n \) and \( m = 0 \) and 1. These values are given along with the approximate values obtained from the Pearson type approximation (Pillai [8]) for comparison.

**Table 1**

<table>
<thead>
<tr>
<th>( n )</th>
<th>( m = 0 )</th>
<th>( m = 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n )</td>
<td>Exact</td>
<td>Approximate</td>
</tr>
<tr>
<td>15</td>
<td>0.747</td>
<td>0.747</td>
</tr>
<tr>
<td>20</td>
<td>0.547</td>
<td>0.547</td>
</tr>
<tr>
<td>25</td>
<td>0.437</td>
<td>0.437</td>
</tr>
<tr>
<td>30</td>
<td>0.362</td>
<td>0.362</td>
</tr>
</tbody>
</table>

The table shows that the approximate values (Pillai [8]) are generally accurate to the three decimals provided. The exact values from (2.6) were computed on CDC 6500 and terms of the series up to \( k = 25 \) were generally used.
3. The distribution of the sum of the three smallest roots of a covariance matrix when \( p = 4, \Sigma = I \). We may start with the following density which will be discussed in detail in the next section.

\[
(3.1) \quad K_1(p, n) = \prod_{i=1}^{p} \left( g_i^m e^{-g_i} \right) \prod_{i > j} (g_i - g_j) \quad 0 < g_1 \leq g_2 \leq \ldots \leq g_p < \infty,
\]

where \( K_1(p, n) = \frac{\pi^{p/2}}{\Gamma_p(\frac{1}{2}n) \Gamma_p(\frac{1}{2}p)} \).

First put \( p = 4 \) in (3.1) and integrate with respect to \( g_4 \). Next transform \( M_1 = \ell_1 + \ell_2^2, G_1 = \ell_1 \ell_2 \) where \( \ell_1 = g_1/g_3, \ell_2 = 1/2 \), and integrate with respect to \( G_1 \). Then the joint density of \( M_1 \) and \( g_3 \) is of the form:

\[
f(M_1, g_3) = f_1(M_1, g_3) + f_2(M_1, g_3)
\]

where

\[
(3.2) \quad f_1(M_1, g_3) = K_2(4, n) e^{g_3(2+M_1)} M_1^{m+2} \sum_{r=0}^{m+2} \frac{(r+1)g_3^{4m+7-r}}{(r+1)g_3^{4m+7-r}} \{(a-bM_1)
\]

\[
\left[ (1-\frac{1}{2}) - \frac{M_1^2}{4(m+2)} \right] + dC_{(1-\frac{1}{2})} \left[ (1-\frac{1}{2}) - \frac{M_1^2}{4(m+3)} \right],
\]

\[
0 < g_3 < \infty, \quad 0 < M_1 < 1
\]

and where

\[
a = (m+2)!/(m+2-r)!, \quad b = (m+1)!/(m+1-r)!, \quad c = m!/(m-r)!, \quad d = (m+1)/4(m+2),
\]

\[
K_2(4, n) = K_1(4, n) \left[ (m+1) \cdot 2^{2m+2} \right].
\]
\[ (3.3) \quad r_2(M_1, g_3) = K_1(4, n) e^{-g_3(2+M_1)} \sum_{r=0}^{\infty} \left( \frac{M_1}{r+1} \right) g_3^{4m+7-r} \sum_{r=0}^{\infty} \frac{1}{(m+1)^{1/2}} \left( \frac{1}{2} \right) \]

\[ [a-bM_1+c(-\frac{1}{2})] \frac{M_1}{(m+2)} \frac{2}{(2-M_1)^2} \left[ \frac{M_1}{2} \right]^{-2} \frac{C(\frac{1}{2})^{2m+4}}{4(m+2)} \left[ (2-M_1)^2 \right] - \frac{M_1}{2} \frac{2m+6}{(m+2)(m+3)} \]

\[ \frac{M_1}{2} \frac{2C(M_1-1)}{(m+2)(m+3)} \}

\[ 0 < g_3 < \infty, \quad 1 < M_1 < 2. \]

Now we make the following transformation \( T = g_3(M_1+1) \) in (3.2) and (3.3)
and integrate with respect to \( g_3 \) from \( \frac{1}{2}T \) to \( T \) and \( \frac{1}{3}T \) to \( \frac{1}{2}T \) respectively.

Finally the density of \( T \) is given by

\[ (3.4) \quad K_2(4, n) e^{-T} \sum_{r=0}^{\infty} \left( \frac{1}{r+1} \right) \sum_{j=0}^{2m+2} \left( \frac{2m+2}{1} \right)^{-1} \left[ \frac{C}{4} \right] \cdot I(\frac{1}{2}; T; 4m+6-r-j-i) + \]

\[ K_1 \cdot I(\frac{T}{2}; T; 4m+6-r-j-i) \left[ \frac{1}{m+1} \right] \]

\[ \sum_{j=0}^{m+2} \left( \frac{m+2}{j} \right)^{-1} \frac{T^{m+2}}{(m+2)\left(4m^2+6j-i\right)} + K_5 \cdot I(\frac{T}{3}; 2; 4m+5-r-i) \]

\[ 0 < T < \infty, \]

where

\[ I(x_1, x_2; n) = \int_{x_1}^{x_2} e^{-y} y^n dy, \quad 0 \leq x_1 \leq x_2 < \infty, \]

and constant coefficients are:
\[ C_0 = \left(9 - \frac{1}{m+2}\right)(a+b) + ac\left(9 - \frac{1}{m+3}\right), \quad C_1 = -3\left(2a+5b+8dc\right) + \frac{2a+3b}{m+2} + \frac{4dc}{m+3}, \]

\[ C_2 = (a+b+22dc) - \left(\frac{a+3b}{m+2} + \frac{6dc}{m+3}\right), \quad C_3 = \frac{b}{m+2} + \frac{4dc}{m+3} - (b+8dc), \quad C_4 = ac\left(1 - \frac{1}{m+3}\right), \]

\[ K_0 = 9(a+b)^2 - \frac{9C}{16(m+2)} - \frac{(a+b+C)}{4(m+2)} + \frac{C}{8(m+2)(m+3)}, \]

\[ K_1 = [6a+15b+6c] + \frac{3C}{2(m+2)} + \frac{2a+3b+c}{4(m+2)} - \frac{C}{2(m+2)(m+3)}, \]

\[ K_2 = (a+b+11c) - \frac{11C}{8(m+2)} - \frac{a+3b+3c}{4(m+2)} + \frac{3C}{4(m+2)(m+3)}, \]

\[ K_3 = -(b+2c) + \frac{C}{2(m+2)} + \frac{b+c}{4(m+2)} - \frac{C}{2(m+2)(m+3)}, \]

\[ K_4 = \frac{C}{4} - \frac{C}{8(m+2)} + \frac{C}{8(m+2)(m+3)}, \]

\[ K_5 = \frac{[2^{m+2}(a+b-2c)(m+3) + 2^{m+4}C]}{(m+2)(m+3)}, \]

and

\[ K_6 = \frac{[2^{m+2}(c-b)(m+3) - 2^{m+3}C]}{(m+2)(m+3)}. \]

4. The distribution of the smallest characteristic root of the sample covariance matrix. Let \( X(p \times n) \) be a matrix variate with columns independently distributed as \( N(0, \Sigma) \), then the distribution of the characteristic roots, \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_p < \infty \) of \( XX' \) depends only upon the characteristic roots of \( \Sigma \) and can be given in the form (James [4])

\[(4.1) \quad K(p,n) = \frac{n}{\lambda_1^{m}} \prod_{i>j} (\lambda_i - \lambda_j) \int_{\Sigma} \exp\left(-\frac{1}{2} tr \Sigma^{-1} H W H'\right) d(H) \]

\[ 0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_p < \infty , \]
where the integral is taken over the orthogonal group of \((p \times p)\) orthogonal matrices \(H; m = \frac{1}{2}(n-p-1)\) and \(K(p,n) = \frac{1}{2}p^{2}/2^{2p} \pi^{\frac{1}{2}p^{2}} \Gamma_{p}^{\frac{1}{2}p} \Gamma_{p}^{\frac{1}{2}n} \Gamma_{p}^{\frac{1}{2}p^{2}}\) and 

\(W = \text{diag}(w_{p}, \ldots, w_{p}).\) (4.1) can also be written in the form (James \([4]\))

\[K(p,n) \sim \frac{n}{\sqrt{2}} |W|^{m} \exp(-\frac{1}{2}tr \ W) \prod_{i \geq j} (w_{i}-w_{j})^{\frac{1}{2}(I - \Sigma^{-1}), W},\]

\[0 < w_{1} \leq \cdots \leq w_{p} < \infty,\]

where \(F_{q}(a_{1}, \ldots, a_{p}; b_{1}, \ldots, b_{q}; \varphi, T) = \Sigma_{k=0}^{\infty} \frac{(a_{1})_{k}, \cdots, (a_{p})_{k}}{(b_{1})_{k}, \cdots, (b_{q})_{k}} \frac{C_{k}(\varphi)C_{k}(T)}{C_{k}(I)k!}\)

and \(a_{1}, \ldots, a_{p}; b_{1}, \ldots, b_{q}\) are real or complex constants and the multivariate coefficient \((a)_{k}\) is given by \((a)_{k} = \frac{p}{i=1} (a-\frac{1}{2}(i-1))k_{i}\) and where partition \(K\) of \(k\) is such that \(k = (k_{1}, \ldots, k_{p}), k_{1} \geq k_{2} \geq \cdots \geq k_{p} \geq 0, k_{1}+\cdots+k_{p} = k\) and the zonal polynomials, \(C_{k}(\varphi),\) are expressible in terms of elementary symmetric functions of the characteristic roots of \(\varphi, (James [5]).\) If we let \(\Sigma = I \sim I_{p}\) in (4.2) and transform \(g_{i} = \frac{1}{2}w_{i}, i = 1, \ldots, p,\) we obtain the joint density of \(g_{1}, \ldots, g_{p}\) in the form [see (3.1)]

\[K_{p}(p,n) \prod_{i=1}^{p} (g_{i}^{m} e^{-g_{i}}) \prod_{i \geq j} (g_{i}-g_{j})^{\frac{1}{2}(I - \Sigma^{-1}), g_{1}}, 0 < g_{1} \leq g_{2} \leq \cdots \leq g_{p} < \infty.\]

Expanding (4.2) as a power series, we have

\[K(p,n) \sim \frac{n}{\sqrt{2}} |W|^{m} \exp(-\frac{1}{2}tr \ W) \prod_{i \geq j} (w_{i}-w_{j})^{\frac{1}{2}(I - \Sigma^{-1}), W},\]

Let \(u_{i} = \frac{1}{w_{p}}, i = 1, 2, \ldots, p-1,\) and make use of the known equality (Khatri and Pillai \([6]\)), \(C_{k}(U) = \Sigma_{n=0}^{\infty} E_{k, n} C_{k}(U)\) where \(U = \text{diag}(u_{1}, \ldots, u_{p})\) and \(E_{k, n}\) are constants depending on \(k\) and \(n,\) then we have
(4.5) \[ K(p,n) \sim (-\frac{n}{2})^m \exp \left( -\frac{p}{2} \right) \sum_{s=0}^{\infty} \frac{(\frac{1}{2})^s C_2(U')}{s!} \prod_{p} \frac{p^{m+s+\frac{1}{2}(p+2)}(p-1)}{p} \]

\[ |I_{\pi-1} - U'| \prod_{1>j}(u_{1-j} - u_{1-j}) \sim \sum_{k=0}^{\infty} \sum_{k=0}^{n=0} \frac{C_k(I_{\pi-1})}{k!} \frac{1}{\pi} \frac{\delta}{\delta \eta} C(U') \]

\[ 0 < w_p < \infty, \quad 1 > u_{p-1} > u_{p-2} > \ldots > u_1 > 0 \]

we need only to consider

(4.6) \[ |U'|^m \frac{C(U')}{\pi} \frac{1}{\pi} \frac{\delta}{\delta \eta} C(U') \]

Now apply the result (Khatri and Pillai [6], Hayakawa [3]),

\[ C(U') \frac{C(U')}{\eta} = \sum_{\theta} \frac{\delta}{\delta \eta} C(U') \] where the summation is over all partition \( \theta \) of \( q \) satisfying \( n+s=q \) and \( \frac{\delta}{\delta \eta} \) are constant depending on \( \theta, \eta \) and \( \eta \). In (4.5) transform \( u_i = 1-u_{i-1} \), \( i = 1,2,\ldots,p-1 \), i.e.,

\[ U = I - U' \] where \( U = \text{diag}(u_1, \ldots, u_{p-1}) \), then (4.6) becomes

(4.7) \[ |I_{\pi-1} - U'|^m |U'| \prod_{1>j}(u_{1-j} - u_{1-j}) \sum_{\theta} \frac{\delta}{\delta \eta} C(I_{\pi-1}) \quad 1 > u_1 > u_2 > \ldots > u_{p-1} > 0 \]

Applying Constantine's result [2], \( C_\theta(I_{\pi-1}) = C_\theta(I_{\pi-1}) \sum_{q=0}^{\infty} \sum_{\alpha} (-1)^{\alpha} \]

\[ \frac{A_\theta \psi_{v}(U)}{C_\theta(I_{\pi-1})} \] and making use of the following equality (Khatri and Pillai [6]),

\[ |I_{\pi-1} - U'|^m C(U) = \sum_{t=0}^{\infty} \frac{\delta_{v}}{t!} \frac{C(U)}{t!} \] where \( A_\theta \) 's are constants depending on \( \theta \) and \( v \) and \( \delta_{v,\sigma} \) is the coefficient of \( C(U) \),
\[(4.10) \quad K(p,n)\left|\Sigma\right|^{\frac{n}{2}} \exp\left(-\frac{p}{2}\right) \sum_{s=0}^{\infty} \sum_{k=0}^{s} \left(\frac{s}{2}\right)^k \frac{k!}{\prod_{\eta=0}^{\infty} \eta^{b_{k,\eta} \sum_{\theta}^{\frac{\eta}{\theta}} c_{\theta\left(l_{p-1}\right)}}}

\sum_{z=0}^{q} \sum_{v} \frac{(-1)^z A_{\theta,v} \sum_{\sigma}^{\infty} \theta_{\sigma}}{C_{\theta\left(l_{p-1}\right)}} \sum_{t=0}^{\infty} \sum_{\sigma}^{\delta} \frac{\delta}{F(p,\delta)(1-\frac{1}{w_p})^{l_{p+1}}+h+2},

\infty > w_p > w_l > 0.

Note that the series in \( t \) is actually only a finite summation and \((4.10)\) converges for all values of \( \infty > w_p > w_l > 0 \). So if we integrate \((4.10)\) with respect \( w_p \), we have the density of the smallest characteristic root

\[(4.11) \quad K(p,n)\left|\Sigma\right|^{\frac{n}{2}} \sum_{s=0}^{\infty} \sum_{k=0}^{s} \left(\frac{s}{2}\right)^k \frac{k!}{\prod_{\eta=0}^{\infty} \eta^{b_{k,\eta} \sum_{\theta}^{\frac{\eta}{\theta}} c_{\theta\left(l_{p-1}\right)}}}

\sum_{z=0}^{q} \sum_{v} \frac{(-1)^z A_{\theta,v} \sum_{\sigma}^{\infty} \theta_{\sigma}}{C_{\theta\left(l_{p-1}\right)}} \sum_{t=0}^{\infty} \sum_{\sigma}^{\delta} \frac{\delta}{F(p,\delta)(1-\frac{1}{w_p})^{l_{p+1}}+h+2},

\infty > w_p > w_l > 0.

Also note that if we expand \((4.1\) as a power series and proceed as above, we will obtain the joint density of the largest and smallest characteristic roots.
$$K(p,n) = \frac{1}{n!} \sum_{k=0}^{\infty} \frac{C_k(-\frac{1}{2})}{C_k(I_p)} w^{pm+\frac{1}{2}(p+2)(p-1)+k} \sum_{n=0}^{\infty} \sum_{\eta} \delta_{v,\sigma} \frac{\xi^{\frac{1}{2}}}{t!}$$

$$F(p,\delta)(1-w_1/w_p)^{\frac{1}{2}p(p+1)+h+2} \approx w_p > w_1 > 0,$$

where $A_{\eta,v}$ is defined similar to $A_{\Theta,v}$. We may also set $u_1 = 1 - w_1/w_p$ in (4.10), then $u_1$ and $w_p$ are independently distributed. If we integrate with respect to $w_p$, then the density of $u_1$, i.e. the ratio of the smallest root to the largest root is given by

$$K(p,n) = \frac{1}{\sqrt{\pi}} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \sum_{k=0}^{\infty} \frac{C_k(-\frac{1}{2})}{C_k(I_p)} k! \sum_{n=0}^{\infty} \sum_{\eta} \delta_{v,\sigma} \frac{\xi^{\frac{1}{2}}}{t!}$$

$$b_{k,\eta} \sum_{\theta} \delta_{k,\theta} C_{\eta}(I_{p-1}) \sum_{z=0}^{\infty} \frac{(-1)^z A_{\Theta,v}}{C_v(I_{p-1})} \sum_{t=0}^{\infty} \sum_{\sigma \delta} \frac{\xi^{\frac{1}{2}}}{t!} F_1(p,s,k;\delta) u_1^{\frac{1}{2}p(p+1)+h+2} \quad 0 < u_1 < 1$$

where $F_1(p,s,k;\delta) = \frac{1}{\Gamma(p+2)} \frac{p^m+s+k+1}{\Gamma(p+2)(p-1)+pm+s+k+1} \cdot F(p,\delta).$
References


