Further contributions to some inequalities for normal distributions and their applications to simultaneous confidence bounds

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1. Introduction and summary. Let \( \mathbf{x} = (x_1, \ldots, x_p)' \) be distributed as multivariate normal with zero means and covariance matrix \( V(x) \) and this will be denoted by \( \mathbf{x} \sim N(0, V(x)) \). Dunn's conjecture [3], namely.

\[
(1) \quad P \left[ |x_i| \leq c_i, \ i = 1, 2, \ldots, p \right] \geq \prod_{i=1}^{p} P \left[ |x_i| \leq c_i \right]
\]

was established by Khatri [4], Sidak [5] and Scott [5] by using different approaches. Moreover, Khatri [4] conjectured that

\[
(2) \quad P \left[ |x_i| \geq c_i, \ i = 1, 2, \ldots, p \right] \geq \prod_{i=1}^{p} P \left[ |x_i| \geq c_i \right]
\]

and the proof of (2) given by Scott [5] is incorrect. The purpose of this paper is to generalize (1) and (2) in the case of convex and symmetric regions about the origin. The generalized results are mentioned as under:

Let \( \mathbf{x}' = (y_1', y_2', \ldots, y_q') \) where \( y_i' = (x_{p_1}+\ldots+x_{p_i-1}l, \ldots, x_{p_i}+\ldots+x_p) \), \( i = 1, 2, \ldots, q \) with \( \sum_{i=1}^{q} p_i = p \). Moreover, let \( \mathcal{S}_1(y_i) \) be a convex and symmetric region in \( y_i \) about the origin in \( p \)-dimensional space with \( -\infty < x_j < \infty \), \( j = 1, 2, \ldots, p_1+p_2+\ldots+p_i-1, p_1+p_2+\ldots+p_i+1, \ldots, p \). Let \( \mathcal{S}_i(y_i) \) be the complementary region of \( \mathcal{S}_1(y_i) \). Then, we have

\[
(3) \quad P \left( \bigcap_{i=1}^{q} \mathcal{S}_i(y_i) \right) \geq \prod_{i=1}^{q} P(\mathcal{S}_i(y_i))
\]

and

\[
(4) \quad P \left( \bigcap_{i=1}^{q} \mathcal{S}_1(y_i) \right) \geq \prod_{i=1}^{q} P(\mathcal{S}_1(y_i))
\]
Some applications of these results are given on simultaneous confidence intervals. All the results mentioned by Knatri [4] are now valid omitting the structure $L$.

2. Inequalities for multivariate normal distributions.

The following lemmas will be used in establishing (3) and (4).

**Lemma 1.** Let $x \sim N(0, V(x))$, $x^{(2)} = (x_2, x_3, \ldots, x_p)$ and let $\mathcal{D}_1(x_1)$ and $\mathcal{D}_2(x^{(2)})$ be two convex and symmetric regions in $x_1$ and $x^{(2)}$ respectively about the origin in $p$-dimensional space containing axes due to other variates. Then,

$$P(\mathcal{D}_1(x_1) \cap \mathcal{D}_2(x^{(2)})) \geq P(\mathcal{D}_1(x_1)) P(\mathcal{D}_2(x^{(2)})) .$$

For proof, refer Knatri [4].

**Lemma 2.** Let $x: p \times 1 \sim N(0, V(x))$ and $z: p \times 1 \sim N(0, V(z))$. Then, if $V(x) - V(z)$ is positive semi-definite,

$$P(\mathcal{D}(z)) \geq P(\mathcal{D}(x))$$

where $\mathcal{D}(z)$ is a convex and symmetric region in $z$ about the origin.

For proof, refer Anderson [1].

**Theorem 1.** Let $x \sim N(0, V(x))$, $x' = (x_1', \ldots, x_q')$, $y_i' = (x_{i1} + \ldots + x_{i2}, \ldots, x_{ip} + \ldots + x_{i2})$ and $\mathcal{D}_i(x_{i1})$ be convex and symmetric region in $x_{i1}$ about the origin in $p$-dimensional space containing axes due to other variates, for $i = 1, 2, \ldots, q$. Then

$$P \left[ \bigcap_{i=1}^{q} \mathcal{D}_i(x_{i1}) \right] \geq P(\mathcal{D}_1(x_{i1})) P(\bigcap_{i=2}^{q} \mathcal{D}_i(x_{i1})) \geq \prod_{i=1}^{q} P(\mathcal{D}_i(x_{i1})) .$$
Proof. When any \((q-1)\) values of \(p_i, i=1,2,\ldots,q\) are at the most one, then theorem 1 is established by Knatri [14] or theorem 1 is equivalent to lemma 1. Here, we assume that \(p_i > 1, i = 1,2,\ldots,q\). First of all, we shall consider the case when \(V(x)\) is positive definite. Without loss of generality, we can write \(V(x) = AA'\) where \(A = (A_{i,j})\) is nonsingular, \(A_{i,i'} = \sim\) for \(i > i', i,i' = 1,2,\ldots,q\). Let \(A^{-1} x = \sim\) and \(\sim' = (\sim_1', \ldots, \sim_q')\) with \(\sim_i' = (\sim_{p_i+1'}\ldots+\sim_{p_{i-1}+1'}, \ldots, \sim_{p_i'}\ldots+\sim_{p_1'})\). Then, it is easy to see that
\[
\sim_i' = \sum_{j=1}^q A_{i,j} \sim_j
\]
for \(i = 1,2,\ldots,q\).

and \(\sim \sim N(0, I_p)\) or \(\sim_i \sim N(0, I_{p_i})\), \(i = 1,2,\ldots,q\). It is easy to see that theorem 1 will be established for \(V(x)\) to be positive definite if we can establish
\[
(5) \quad \left[ \bigwedge_{i=1}^q \mathcal{E}_i \left( \sum_{j=i}^q A_{i,j} \sim_j \right) \mid \sim \sim Q \right] \geq S
\]

\[
P \left[ \mathcal{E}_1(\sim_1') \right] P \left[ \bigwedge_{j=2}^q \mathcal{E}_i \left( \sum_{j=i}^q A_{i,j} \sim_j \mid \sim \sim Q \right) \right]
\]

for every \((p_1+1)\)-flat \(Q\) containing \((Z_1, Z_2', \ldots, Z_{p_1}')\)-axes.

Let us take such a \((p_1+1)\)-flat \(Q\) and let us suppose that this is determined by the set of linearly independent equations given by
\[
P_{-1} \sum_{j=1}^{p_1} k_j Z_j + p_1 = 0 \quad \text{for} \quad k = 1,2,\ldots,P_{-1} - 1
\]
where without loss of generality, take \( \sum_{j=1}^{p-p_1} l_{kj}^2 = 1 \) and \( \sum_{j=1}^{p-p_1} \sum_{j'} l_{kj}^j l_{k'j} = 0 \) for \( k \neq k' \). Let \( L_1 = (l_{kj}) \): \((p-p_1-1) \times (p-p_1)\). Then, we can complete \( L_1 \) by a vector \( \lambda \) such that \( L' = (\lambda \ L_1') \) is an orthogonal matrix. Now use the transformation

\[
L(x_1', \ldots, x_q') = (v_1, v_2, \ldots, v_{p-p_1})'.
\]

Then it is obvious that the \((p_1+1)\)-flat \( Q \) will have the coordinate system given by \((w_1', v_1, v_i = 0 \text{ for } i = 2, \ldots, p-p_1)\) and \( v_1 \sim N(O, I) \) and \( w_i \sim N(O_1 \sim_{p-1} ) \) and they are independently distributed. Hence, using this system of coordinates in the left side of (5), we get

\[
(5) \quad I(Q) = P \left[ \bigcap_{i=1}^{q} g_i \left( \sum_{j=1}^{p} A_{ij} w_j \bigg| Z \in Q \right) \right] = P \left[ \bigcap_{i=2}^{q} g_i (\tilde{A}_1 w_1 + \tilde{\delta}_i v_1) \bigg| \bigcap_{j=2}^{q} g_j (\tilde{A}_j v_1) \right]
\]

where if \( \lambda' = (\lambda_2', \lambda_3', \ldots, \lambda_q') \) with \( \lambda_i' : p_1 \times 1, \delta_i = \sum_{j=1}^{q} A_{ij} \lambda_j \) for \( i = 2, 3, \ldots, q \) and \( \tilde{\delta}_i = \sum_{j=2}^{q} A_{ij} \lambda_j \). Since \( \delta_i, i = 2, \ldots, q \) are fixed vector and \( g_i (\delta_i v_1), i = 2, 3, \ldots, q \) are convex and symmetric in \( \delta_i v_1 \) about the origin and hence \( g_i (v_1) = \bigcap_{i=2}^{q} g_i (\delta_i v_1) \) is convex and symmetric in \( v_1 \) about the origin. Then, using this in (6) and then lemma 1, we get

\[
(7) \quad I(Q) \geq P \left[ g_1 (\tilde{A}_1 w_1 + \tilde{\delta}_1 v_1) \right] P(g(v_1) = \bigcap_{i=2}^{q} g_i (\delta_i v_1)).
\]

Note that

\[
(8) \quad P \left( \bigcap_{i=2}^{q} g_i (\delta_i v_1) \right) = P \left( \bigcap_{i=2}^{q} g_i \left( \sum_{j=1}^{p} A_{ij} w_j \bigg| Z \in Q \right) \right)
\]

and by using lemma 2,
\[(9) \quad P\left[ \mathcal{A}_i(\mathbf{A}_1 \mathbf{W}_1 + \delta_1 \mathbf{v}_1) \right] \geq P\left[ \mathcal{A}_i(\mathbf{v}_1) \right] \]

for \( V(\mathbf{y}_1) - V(\mathbf{A}_1 \mathbf{W}_1 + \delta_1 \mathbf{v}_1) = \sum_{j=1}^{q} A_{1j} A_{1j}^t - (\mathbf{A}_1 \mathbf{A}_1^t + \delta_1 \delta_1^t) = \sum_{j=2}^{q} \frac{A_{1j}}{\sqrt{p_j}} \frac{A_{1j}^t}{\sqrt{p_j}} \frac{A_{1j}^t}{\sqrt{p_j}} \frac{A_{1j}}{\sqrt{p_j}} \) is positive semi-definite.

Using (8) and (9) in (7), we get (5). Thus, theorem 1 is established when \( V(\mathbf{x}) \) is nonsingular.

Let \( V(\mathbf{x}) \) be positive semi-definite. Let \( \mathbf{u}: \mathbb{R} \times \mathbb{R} \sim N(\mathbf{Q}, \mathbf{I}_p) \) and let \( \mathbf{u} \) and \( \mathbf{x} \) be independently distributed. Then \( \mathbf{x} + \mathbf{u} = \tilde{\mathbf{z}} \sim N(\mathbf{Q}, \mathbf{I}_p + V(\mathbf{x})) \) and \( V(\mathbf{x}) + \mathbf{I}_p \) is positive definite. Hence from the result for positive definite covariance matrix, we get

\[(10) \quad P\left[ \bigcap_{i=1}^{q} \mathcal{A}_i(\mathbf{t}_i) \right] \geq \prod_{i=1}^{q} P\left[ \mathcal{A}_i(\mathbf{t}_i) \right]. \]

Taking limits as \( n \to \infty \), we get the result for the singular case, for

\[ \lim_{n \to \infty} P\left[ \mathcal{A}(\mathbf{t}_i) \right] = P\left[ \mathcal{A}(\mathbf{y}_1) \right] \]

if \( \mathcal{A}(\mathbf{y}) \) is a convex and symmetric region in \( \mathbf{y} \) about the origin.

**Theorem 2.** Under the notations of theorem 1, we have

\[ P\left( \bigcap_{i=1}^{q} \mathcal{A}_i(\mathbf{y}_1) \right) \geq P(\mathcal{A}_1(\mathbf{y}_1)) P\left( \bigcap_{i=2}^{q} \mathcal{A}_i(\mathbf{y}_1) \right) \geq \prod_{i=1}^{q} P(\mathcal{A}_i(\mathbf{y}_1)), \]

where \( \mathcal{A}(\mathbf{y}) \) is the complement of \( \mathcal{A}(\mathbf{y}) \).
Proof. Let us consider the case when \( V(x) \) is positive definite and we proceed in the same manner as in theorem 1 in considering

\[
P(S_1(y_1) \cap \bigcup_{i=2}^{q} S_i(y_i))
\]

Using the same arguments as those in theorem 1, we get

\[
(11) \quad P(S_1(\sum_{j=1}^{q} A_{1j} \tilde{w}_j) \cap \bigcup_{i=2}^{q} S_i(\sum_{j=1}^{q} A_{ij} \tilde{w}_j))|Z \in Q
\]

\[
= P(S_1(A_{11} \tilde{w}_1 + \delta_{1} v_1) \cap \bigcup_{i=2}^{q} S_i(v_i))
\]

Now \( \bigcup_{i=2}^{q} S_i(v_i) \iff |v_1| \leq \alpha \) for some \( \alpha > 0 \). Hence, using this in (11) and using theorem 1, we get

\[
(12) \quad P(S_1(\sum_{j=1}^{q} A_{1j} \tilde{w}_j) \cap \bigcup_{i=2}^{q} S_i(\sum_{j=1}^{q} A_{ij} \tilde{w}_j))|Z \in Q
\]

\[
\geq P(S_1(A_{11} \tilde{w}_1 + \delta_{1} v_1)) P(|v_1| \leq \alpha)
\]

and using \( P(|v_1| \leq \alpha) = P(S_1(v_1)) = P(\bigcup_{i=2}^{q} S_i(v_i)) = P(\bigcup_{i=2}^{q} S_i(\sum_{j=1}^{q} A_{ij} \tilde{w}_j))|Z \in Q \) and \( P(S_1(A_{11} \tilde{w}_1 + \delta_{1} v_1)) \geq P(S_1(y_1)) \), we get

\[
(13) \quad P(S_1(\sum_{j=1}^{q} A_{1j} \tilde{w}_j) \cap \bigcup_{i=2}^{q} S_i(\sum_{j=1}^{q} A_{ij} \tilde{w}_j))|Z \in Q
\]

\[
\geq P(S_1(y_1)) P(\bigcup_{i=2}^{q} S_i(\sum_{j=1}^{q} A_{ij} \tilde{w}_j)|Z \in Q)
\]

Then (13) gives us

\[
(14) \quad P(S_1(y_1) \cap \bigcup_{i=2}^{q} S_i(y_i)) \geq P(S_1(y_1)) P(\bigcup_{i=2}^{q} S_i(y_i))
\]
We note that if $R_1$ and $R_2$ be two regions, then

$$P(R_1) = P(R_1 \cap R_2) + P(R_1 \cap \overline{R_2}) .$$

Moreover, we have

$$\left\{ \bigcup_{i=2}^q R_i(x_i) \right\} = \bigcap_{i=2}^q \overline{R_i(y_i)} .$$

Using these in (14), we get

$$P\left( \bigcup_{i=2}^q R_i(x_i) \cap \bigcap_{i=2}^q \overline{R_i(y_i)} \right) \leq P(R_1(y_1)) P\left( \bigcap_{i=2}^q \overline{R_i(y_i)} \right) .$$

(15)

and this implies

$$P\left( \bigcap_{i=1}^q \overline{R_i(y_i)} \right) \geq P(R_1(y_1)) P\left( \bigcap_{i=2}^q \overline{R_i(y_i)} \right) .$$

(16)

Thus, theorem 2 is proved when $V(x)$ is positive definite. When $V(x)$ is singular, we can argue in the same manner as in theorem 1. This completes the proof of theorem 2.

**Corollary 1.** Let $x_j \sim N(0, V(x_j))$, $j = 1, 2, \ldots, n$ and let them be independent.

Let $\mathcal{D}_i = \mathcal{D}_i(y_{1i}, y_{2i}, \ldots, y_{ni})$ be convex and separately symmetric in $y_{1i}, y_{2i}, \ldots, y_{ni}$ about the origin for $i = 1, 2, \ldots, q$ and let $\overline{\mathcal{D}_i}$ be the complement of $\mathcal{D}_i$. Then

$$P\left( \bigcap_{i=1}^q \mathcal{D}_i \right) \geq \prod_{i=1}^q P(\mathcal{D}_i) \quad \text{and} \quad P\left( \bigcap_{i=1}^q \overline{\mathcal{D}_i} \right) \geq \prod_{i=1}^q P(\overline{\mathcal{D}_i}) .$$

(For the definition of separately symmetric, see Knatri [4].)

**Proof.** We shall only indicate the proof for one case as under:

Let $w_{ij} \sim N(0, V(w_{ij}))$, $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, q$ and let them be independent and independent of $x_1, x_2, \ldots, x_n$. By theorem 1, it is easy to see that
(17) \[ P\left[ \bigcap_{i=1}^{q} \mathcal{D}_i (\tilde{y}_{i1}, \ldots, \tilde{y}_{in}) \mid \tilde{x}_1, \ldots, \tilde{x}_{n-1} \right] \]
\[ \geq P\left[ \bigcap_{i=1}^{q} \mathcal{D}_i (\tilde{y}_{i1}, \ldots, \tilde{y}_{in-1}) \mid \tilde{x}_1, \ldots, \tilde{x}_{n-1} \right] \]

because \( \mathcal{D}_i (y_{i1}, \ldots, y_{in}) \) is convex and symmetric in \( y_{in} \) about the origin when \( y_{i1}, \ldots, y_{in-1} \) are fixed. From (17), we get

(18) \[ P\left[ \bigcap_{i=1}^{q} \mathcal{D}_i (\tilde{y}_{i1}, \ldots, \tilde{y}_{in}) \right] \geq P\left[ \bigcap_{i=1}^{q} \mathcal{D}_i (\tilde{w}_{i1}, \ldots, \tilde{w}_{in}) \right]. \]

Proceeding in the same manner for \( x_{n-1}, x_{n-2}, \ldots, x_1 \), we get the final result as

(19) \[ P\left[ \bigcap_{i=1}^{q} \mathcal{D}_i \right] \geq P\left[ \bigcap_{i=1}^{q} \mathcal{D}_i (w_{i1}, \ldots, w_{in}) \right] = \prod_{i=1}^{q} P(\mathcal{D}_i) \]

This proves the first part of corollary 1. The second part can be proved in the same manner.

**Corollary 2.** Let \( x_{j} \sim N(0, V(x_{j})), j=1,2,\ldots,n \) and be independent. Let us suppose that \( V(x_{j}) = (A_{ii}, j), A_{ii}, j = \sigma^2_{i,j} I_{\alpha_i} \) for \( \alpha_i = 1, 2, \ldots, r \)

and \( \sum_{\alpha_i} x_{i,j}^2 \) for \( i = 1, 2, \ldots, r \)

\( y_{j, t, j} = (Z_{1,t} + \ldots + t_{t, t-1} + 1, \ldots, Z_{r_{t, t-1} + 2} + \ldots + r_t), t = 1, 2, \ldots, q, \sum_{t=1}^{q} r_t = r \) and

\( \mathcal{D}_t = \mathcal{D}_t (y_{1,t}, \ldots, y_{n,t}) \) about the origin for \( t = 1, 2, \ldots, q \). Then,
\[ P_\text{\(t=1\)} = x_1 \geq \prod_{t=1}^{q} P(x_t) \text{ and } P_\text{\(t=1\)} = x_1 \geq \prod_{t=1}^{q} P(x_t) \]

This follows from corollary 1.

**Note:** In corollaries 1 and 2, if some observations are missing on \(y_1\) or \(y_2\), \ldots or \(y_q\), we have to omit these from the convex and symmetric regions \(A_i\), \(i = 1, 2, \ldots, q\). E.g. Suppose on \(y_1\), the only observations available are \(y_{1,j}\), \(j = 1, 2, \ldots, n_1\) \((n_1 < n)\). Then, \(A_1 = (y_{1,1}, \ldots, y_{1,n_1})\) is convex and symmetric region in \(y_{1,1}, \ldots, y_{1,n_1}\) about the origin.

3. **Direct applications.**

**(3.1) Confidence bounds for means.**

Let us suppose that \(x_j \sim N(\xi_j, \sigma^2)\) for \(j = 1, 2, \ldots, n\) and let them be independent. Let us assume that \(V(x) = (A_{ii})\), \(A_{ii} = \sigma_i^2 \sim \sigma_i^2\), \(i = 1, 2, \ldots, r\) and \(\xi' = (\xi_1', \xi_2', \ldots, \xi_r')\), with \(\xi_i: x_i \sim 1\).

Let \(X_j = (y_{1,j}, \ldots, y_{r,j})\), \(y_i: x_i \sim 1\), \(\overline{y}_i = \frac{1}{n} \sum_{j=1}^{n} y_{i,j}\) and

\[ S_{ii} = \sum_{j=1}^{n} y_{i,j} - n \overline{y}_i \overline{y}_i \]. Then, by corollary 2, it is easy to see that

\[ (20) \quad \prod_{i=1}^{r} \left[ (\overline{y}_i - \xi_i') (\overline{y}_i - \xi_i) \right] \leq C_i S_{ii}, \quad i = 1, 2, \ldots, r \]

because \(\overline{y}_i\) and \(S_{ii}\) are independently distributed and \(S_{ii} = \sum_{j=1}^{n} z_{i,j} z_{i,j}\) with \((\overline{y}_i, \ldots, \overline{y}_i)' \sim N(\xi, \sigma^2)\) and \(Z_{ij} = (z_{i,j}, \ldots, z_{i,j})' \sim N(0, \sigma^2)\).
Now it is easy to see that $n(n-1)(\bar{\xi}_i - \xi_i)'(\bar{\xi}_i - \xi_i)/s_{ii}'$ is distributed as $F_{\alpha_i, (n-1)\alpha_i}$ with $\alpha_i$ and $(n-1)\alpha_i$ degrees of freedom for $i = 1, 2, \ldots, r$. Hence, we can find $c_1, c_2, \ldots, c_r$ such that

$$F_{\alpha_i, (n-1)\alpha_i} \left[ (\bar{\xi}_i - \xi_i)'(\bar{\xi}_i - \xi_i)/s_{ii}' \leq c_i s_{ii}' \right] = 1 - \alpha .$$  

One choice of choosing $c_1, c_2, \ldots, c_r$ is to take

$$F_{\alpha_i, (n-1)\alpha_i} \left[ (\bar{\xi}_i - \xi_i)'(\bar{\xi}_i - \xi_i)/s_{ii}' \leq c_i s_{ii}' \right] = (1 - \alpha)^{1/r} .$$

Using (21) in (20), we can find simultaneous confidence bounds on $\xi_i$, $i = 1, 2, \ldots, r$ with confidence greater than $(1 - \alpha)$ as

$$s_{ii}' \bar{\xi}_i - \left( \sum_{i=1}^{2} a_i a_i' \right)^{1/2} \leq \xi_i \leq s_{ii}' \bar{\xi}_i + \left( \sum_{i=1}^{2} a_i a_i' \right)^{1/2}$$

for all $i = 1, 2, \ldots, r$ and for all non-null vectors $a_i : a_i x 1, i = 1, 2, \ldots, r$.

(3.2) One sided confidence bounds on variances.

Let us suppose that $\bar{x} = (y_1', \ldots, y_q')' \sim N(0, V(x))$ and let us have $n$ independent observations on $\bar{x}$. Out of these $n$ observations, it is found that $n_i$ observations are missing on $y_i$, $i = 1, 2, \ldots, q$. Let $V(x) = (\sum_{i=1}^q S_i)$, and $\sum_{i=1}^q S_i$, the sample sum of squares matrix due to available observations on $y_i$, $i = 1, 2, \ldots, q$.

If $\sum_{i=1}^q \max_{\sum_{i=1}^q S_i} (A_i^{-1}) \leq c_i$, then $\sum_{i=1}^q S_i$ is section-wise convex and separately symmetric in available observations about the origin (see DasGupta, Mudholkar and Anderson [2]). Hence, by corollary 1, we get

$$F_{\sum_{i=1}^q \sum_{i=1}^q S_i} \left[ \bigcap_{i=1}^q P(S_i) \right] \geq \prod_{i=1}^q P(S_i) \text{ and } F_{\sum_{i=1}^q \sum_{i=1}^q S_i} \left[ \bigcap_{i=1}^q \bar{S}_i \right] \geq \prod_{i=1}^q P(\bar{S}_i) .$$
In order to obtain the lower bounds on the parameters $A_{ii}$, $i = 1,2,\ldots,q$, we use the first part of (22). Let us choose $c_1, c_2, \ldots, c_q$ such that

$$\prod_{i=1}^{q} \mathbb{P}[\max_{\tilde{A}_{ii}} (A_{ii}^{-1} S_i) \leq c_i] = 1 - \alpha.$$  \hspace{1cm} (24)

Using (24) in the first part of (23), we get simultaneous lower bounds on $\tilde{A}_{ii}$, $i = 1,2,\ldots,q$ with confidence greater than or equal to $(1 - \alpha)$ as

$$\tilde{a}_i' \tilde{A}_{ii} \tilde{a}_i \geq \tilde{a}_i' S_i \tilde{a}_i / c_i, \quad i = 1,2,\ldots,q$$  \hspace{1cm} (25)

for all non-null vectors $\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_q$.

Similarly, by choosing $c'_i, i = 1,2,\ldots,q$ from

$$\prod_{i=1}^{q} \mathbb{P}[\max_{\tilde{A}_{ii}} (A_{ii}^{-1} S_i) > c'_i] = 1 - \alpha,$$  \hspace{1cm} (26)

we find the simultaneous upper bounds on $\tilde{A}_{ii}$, $i = 1,2,\ldots,q$ with confidence greater than or equal to $(1 - \alpha)$ as

$$\tilde{a}_i' \tilde{A}_{ii} \tilde{a}_i \leq c'_i (\tilde{a}_i' S_i \tilde{a}_i), \quad i = 1,2,\ldots,q$$  \hspace{1cm} (27)

for all non-null vectors $\tilde{a}_i$, $i = 1,2,\ldots,q$.

By combining (25) and (27), we get the simultaneous confidence bounds on $\tilde{A}_{ii}$, $i = 1,2,\ldots,q$ with confidence greater than or equal to $(1 - \alpha)^2$. 

- References -


