On the distributions of some functions of the roots of a covariance matrix and non-central Wilks' $\Lambda^*$

by

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1. Introduction and Summary. Let $\sim \chi(p \times n)$ be a matrix variate with columns independently distributed as $N(0, \Sigma)$. Then the distribution of the latent roots, $0 \leq w_1 \leq \ldots \leq w_p < \infty$, of $\sim \chi \sim' \Sigma$ are first considered in this paper for deriving the distributions of the ratios of individual roots $w_i/w_j$ ($i < j = 2, \ldots, p$). In particular, the distributions of such ratios are derived for $p = 2, 3$ and $4$. The use of these ratios in testing the hypothesis $\delta \Sigma_1 = \Sigma_2$, $\delta > 0$ unknown, has been pointed out where $\Sigma_1$ and $\Sigma_2$ are the covariance matrices of two $p$-variate normal populations. Further, when $\Sigma = I_p$, the distribution of the sum of the two smallest roots is studied for $p = 3, 4$ and $5$. This latter criterion is useful for various tests of hypotheses, for example, those regarding the number of independent linear equations satisfied by the means, $\mu_i$, $i = 1, \ldots, p$, $t = 1, \ldots, N$ in $N$ $p$-variate normal populations with a common covariance matrix. ([1],[10]).

Further, the non-central distribution of Wilks' $\Lambda$ criterion has been obtained for $p = 2, 3$ and $4$. In this connection a lemma has been proved using some results on Mellin transform.

2. Distribution of ratios of the roots of a covariance matrix. The distribution of the latent roots, $0 < w_1 \leq w_2 \leq \ldots \leq w_p < \infty$, of $\sim \chi \sim'$ depends only upon the latent roots of $\Sigma$ and can be given in the form (James [6])

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\[(2.1) \quad K(p,n) |\Sigma|^{-\frac{1}{2}n} W^m \prod_{i>j} (w_i - w_j) \int_{\mathcal{O}(p)} \exp(-\frac{1}{2} \text{tr} \Sigma^{-1} \tilde{W} \tilde{W}^t) \, d(\tilde{\Sigma}), \]

where the integral is taken over the orthogonal group of \((p \times p)\) orthogonal matrices \(\tilde{\Sigma}; \quad m = \frac{1}{2}(n-p-1)\) and \(K(p,n) = \frac{\pi^{\frac{p^2}{2}}}{2^{\frac{3}{4}pn}} \Gamma_p(\frac{1}{2n}) \Gamma_p(\frac{1}{2p})\) and \(\tilde{W} = \text{diag}(v_1, \ldots, v_p)\).

It may be shown that (2.1) can be written in the form James [6]

\[(2.2) \quad K(p,n) |\Sigma|^{-\frac{1}{2}n} W^m \{\exp(-\frac{1}{2} \text{tr} \tilde{W}) \prod_{i>j} (v_i - v_j) \}_o \mathcal{F}_o(\frac{1}{2}(I_p - \Sigma^{-1}), \tilde{W}), \]

where

\[
\mathcal{F}_p^{a_1, \ldots, a_p; \ b_1, \ldots, b_q; \ S, T} = \sum_{k=0}^{\infty} \frac{\Gamma(\sum_{k} (a_{p_k}) \prod_{k=1}^{p} \Gamma_k(b_{p_k}) C_k(S) C_k(T)}{\prod_{k=1}^{p} \Gamma_k(b_{p_k}) C_k(\mathcal{F}_p)k!} \]

where \(a_1, \ldots, a_p, b_1, \ldots, b_q\) are real or complex constants and the multivariate coefficient \((a)_k\) is given by

\[
(a)_k = \prod_{i=1}^{p} (a - \frac{1}{2}(i-1))_k, \]

where

\[
(a)_k = a(a+1)\ldots(a+k-1), \]
partition \( \kappa \) of \( k \) is such that \( \kappa = (k_1, k_2, \ldots, k_p) \), \( k_1 \geq k_2 \geq \ldots \geq k_p \geq 0 \), \( k_1 + k_2 + \ldots + k_p = k \) and the zonal polynomials, \( C_\kappa(S) \), are expressible in terms of elementary symmetric functions (esf) of the latent roots of \( S \).

James [6].

It may be pointed out that the form (2.2) can also be viewed as a limiting form of the non-central distribution of the latent roots Khatri [4] associated with the test of the hypothesis: \( \Sigma_1 = \Sigma_2 \), where \( \Sigma_1 \) and \( \Sigma_2 \) are the covariance matrices of two \( p \)-variate normal populations, when \( n_2 \to \infty \), where \( n_2 \) is the size of the sample from the second population. Now, if we wish to test instead the null hypothesis \( \delta \Sigma_1 = \Sigma_2 \), \( \delta > 0 \) unknown, the ratios of the latent roots would be of interest as test criteria. In this context, in the limiting form (2.2), \( \Sigma \) should be replaced by \( \delta \Sigma_1 \Sigma_2^{-1} \).

Now, let \( l_i = w_i / \omega_p \), \( i = 1, \ldots, p-1 \), then the distribution of \( \lambda_1, \ldots, \lambda_p, \omega_p \) can be written in the form

\[
(2.3) \quad K(p,n) |\Sigma|^{-\frac{3n}{2}} w_p^{\frac{3n-1}{2}} |L|^{\frac{1}{2}} |L|^{-\frac{1}{2}} \prod_{i>j} (\lambda_i - \lambda_j) \exp \left( \frac{-1}{2} \omega_p trL \right) = \frac{w_p^k}{2^kk!} \sum_{\kappa} \frac{C_\kappa(I_p - \Sigma^{-1}) C_\kappa(L_1)}{C_\kappa(I_p)}
\]

where

\[
\Sigma = \text{diag}(\lambda_1, \ldots, \lambda_{p-1}) \quad \text{and} \quad L_1 = \text{diag}(\lambda_1, \ldots, \lambda_{p-1}, 1).
\]

Integrating (2.3) with respect to \( \omega_p \), then the distribution of \( \lambda_1, \ldots, \lambda_{p-1} \) is of the form

\[
\int_{\omega_p} K(p,n) |\Sigma|^{-\frac{3n}{2}} w_p^{\frac{3n-1}{2}} |L|^{\frac{1}{2}} |L|^{-\frac{1}{2}} \prod_{i>j} (\lambda_i - \lambda_j) \exp \left( \frac{-1}{2} \omega_p trL \right) \]
(2.4) \[ K_1(p,n) \left| \Sigma \right|^{-\frac{3n}{2}} \left| \Sigma \right|^{-m} \left| I - L \right|_{i>j}^{\Pi} (\ell_1 - \ell_j) \]

\[
\left[ \sum_{k=0}^{\infty} \frac{\Gamma(3pn+k)}{k!} \sum_{k} \frac{C_k(L_p - \Sigma^{-1}) C_k(L_1)}{C_k(L_p)(\text{tr}L_1)^{3pn+k}} \right],
\]

where \( K_1(p,n) = 2^{\frac{3}{2}pn} K(p,n) \).

**Case i.** Let \( p = 2 \) in (2.4), then the distribution of \( \ell = w_1 / w_2 \) is of the form

\[
(2.5) \quad K_1(2,n) \left| \Sigma \right|^{-\frac{3n}{2}} \ell^{\frac{1}{2}(n-3)}(1-\ell) \left[ \sum_{k=0}^{\infty} \frac{\Gamma(n+k)}{k!(1+\ell)^{n+k}} \sum_{k} \frac{C_k(L_p)/\Sigma^{1}}{C_k(L_p)} \right].
\]

**Case ii.** Putting \( p = 3 \) in (2.4) and by the use of the results of Khatri and Pillai [5], the distribution of \( \ell_1, \ell_2 \) can be written in the form

\[
(2.6) \quad K_1(3,n) \left| \Sigma \right|^{-\frac{3n}{2}} (\ell_1 \ell_2)^{\frac{1}{2}(n-4)} (\ell_2 - \ell_1)(1-\ell_1)(1-\ell_2)
\]

\[
\left[ \sum_{k=0}^{\infty} \frac{\Gamma(a_k)}{k!} \sum_{k} \frac{C_k(L_p - \Sigma^{-1}) C_k(L_1)}{C_k(L_p)} \sum_{i=0}^{\infty} \sum_{\eta} b_{\eta,k} C_{\eta}(0 \ell_1) \right],
\]

\[
\left[ \sum_{r=0}^{\infty} (-a_k)^r (1+\ell_2)^{-r-a_k} \right],
\]

where \( a_k = (3n/2) + k \), \( b_{\eta,k} \) are the constants defined [7], and \( \eta \) is the partition of \( i \) into not more than \( p \) elements.
It may be noted that the distribution of $\ell_1$ and of $\ell_2$ can be found by writing $C(0, \ell_2) = \sum_{i_1 + i_2 = i} a_{i_1, i_2} \ell_1^{i_1} \ell_2^{i_2}$ and expanding $(1 + \ell_2)^{-r-a_k}$ then integrating $\ell_2$ and $\ell_1$ respectively.

Let $r_1 = \ell_1/\ell_2$ so the distribution of $r_1, \ell_2$ can be written in the form

\begin{equation}
K(3, n) |\Sigma|^{-\frac{3n}{2}} r_1^{\frac{1}{2}(n-4)} (1-r_1) \left[ \sum_{k=0}^{\infty} \frac{\Gamma(a_k)}{k!} \sum_{\kappa} C_k(I-\Sigma^{-1}) \sum_{i=0}^{\infty} \sum_{\eta} b_{\eta, \kappa} C(0, 1) \sum_{r=0}^{\infty} \sum_{h=0}^{\infty} (-a_k)^{r} r_1^{h} r_1^{n-2+i+r+h}(1-\ell_2)(1-r_1 \ell_2) \right].
\end{equation}

Integrating (2.7) with respect to $\ell_2$, the distribution of $r_1$ can be written in the form

\begin{equation}
K(3, n) |\Sigma|^{-\frac{3n}{2}} r_1^{\frac{1}{2}(n-4)} (1-r_1) \left[ \sum_{k=0}^{\infty} \frac{\Gamma(a_k)}{k!} \sum_{\kappa} C_k(I-\Sigma^{-1}) \sum_{i=0}^{\infty} \sum_{\eta} b_{\eta, \kappa} C(0, 1) \sum_{r=0}^{\infty} \sum_{h=0}^{\infty} (-a_k)^{r} r_1^{h} \{b(a_1, 2) - r_1 b(a_1+1, 2)\} \right]
\end{equation}

where $a_1 = n-1+i+r+h$.

**Case iii.** Let $p = q$ in (2.4), then the distribution of $\ell_1, \ell_2, \ell_3$ can be written in the form

\begin{equation}
K(4, n) |\Sigma|^{-\frac{3n}{2}} \prod_{i=1}^{3} \{\ell_i^{\frac{1}{2}(n-5)}(1-\ell_i)\} \prod_{i>j} (\ell_i - \ell_j) \left[ \sum_{k=0}^{\infty} \frac{\Gamma(2n+k)}{k! (1+\ell_1+\ell_2+\ell_3)^{2n+k}} \sum_{\kappa} C_k(I-\Sigma^{-1}) \sum_{i=0}^{\infty} \sum_{\eta} b_{\kappa, \eta} C(0, \Sigma) \right],
\end{equation}
where \( L = \text{diag}(\lambda_1, \lambda_2, \lambda_3) \).

Now, let \( r_i = \frac{\lambda_i}{\lambda_3}, \ i = 1, 2 \) and integrate \( \lambda_3 \) from 0 to 1, then the distribution of \( r_1, r_2 \) can be written in the form

\[
(2.10) \quad K_1^{(4, n)} \sum n \frac{1}{2n} (r_1^2 r_2)^{\frac{1}{2}(n-5)} (1-r_1)(1-r_2)(r_2-r_1) \left[ \sum_{k=0}^{\infty} \frac{\Gamma(2n+k)}{k!} \sum_{n} \right.
\]

\[
\frac{c_k(I_p - E^{-1})}{c_k(I_p)} \sum_{i=0}^{k} \sum_{\eta} b_{\kappa, \eta} c_{\eta}(R_1) \sum_{r=0}^{\infty} \frac{(-2n-k)(r_1+r_2)^{r}}{r} \sum_{h=0}^{\infty} \frac{(-2n-k-r)}{h}
\]

\[
\{ \beta(b, 2) - (r_1+r_2) \beta(b+1, 2) + r_1 r_2 \beta(b+2, 2) \}\n\]

where \( b = \frac{3}{2}(n-1)+i+h+r \) and \( R_1 = \text{diag}(r_1, r_2, 1) \). Now, we can find the distribution of \( r_1 \) or \( r_2 \) by expressing \( (r_1+r_2)^{r} \) in terms of zonal polynomials of \( R = \text{diag}(r_1, r_2) \) and using the method outlined in Pillai and Al-Ani [6] and integrating with respect to \( r_2 \) or \( r_1 \) such that \( 0 < r_1 \leq r_2 < 1 \).

Now, let \( r'_1 = \frac{r_1}{r_2} \), then the distribution of \( r'_1 \) can be written in the form

\[
(2.11) \quad K_1^{(4, n)} \sum n \frac{1}{2n} r'_1^{\frac{1}{2}(n-5)} (1-r'_1) \left[ \sum_{k=0}^{\infty} \frac{\Gamma(2n+k)}{k!} \sum_{n} \frac{c_k(I_p - E^{-1})}{c_k(I_p)} \sum_{i=0}^{k} \sum_{\eta} \right.
\]

\[
b_{\kappa, \eta} \sum_{t=0}^{i} \sum_{\tau} b_{\tau}^t c_{\tau} r'_1 \sum_{r=0}^{\infty} \frac{(-2n-k)(r'_1+1)^{r}}{r} \sum_{h=0}^{\infty} \frac{(-2n-k-r)}{h}
\]

\[
\{ \beta(b, 2) \beta(c, 2) + r'_1 \left[ \beta(c+2, 2) \beta(b+2, 2) - \beta(c+1, 2) \beta(b, 2) \right]
\]

\[+ (r'_1) \beta(b+1, 2) \right] (r'_1 \beta(c+2, 2) - \beta(c+1, 2)) - r'_1 \beta(b+2, 2) \beta(c+3, 2) \}
where \( c = n-2+t+r \) and the constants \( b'_{i,i+1} \) and \( \tau \) are defined in [8].

3. The distribution of the sum of the two smallest roots. Let \( \Sigma = \mathbf{I}_p \) in (2.2) and transform \( g_i = \frac{1}{2} w_i, i = 1, \ldots, p \), we get the joint density of \( g_1, \ldots, g_p \) in the form

\[
K_1(p,n) \prod_{i=1}^{p} \left( \frac{m}{g_i e^{-g_i}} \right) \prod_{i>j} (g_i - g_j), \quad 0 < g_1 \leq g_2 \leq \ldots \leq g_p < \infty.
\]

In this section we will derive the distribution of \( M_1 = g_1 + g_2 \) for \( p = 3 \) and 4.

Case 1. Put \( p = 3 \) in (3.1) and let \( M = \ell_1 + \ell_2, \quad G = \ell_1 \ell_2 \), where \( \ell_i = g_i / g_3, i = 1,2 \). Then the joint distribution of \( M \) and \( g_3 \) can be written in the form

\[
K_1(3,n) \frac{e^{-g_3(1+M)}}{g_3^{3m+5}} \int_0^{M^2/4} e^{-G(1+G)} dG, \quad 0 < M \leq 1.
\]

Further, transform \( M_1 = g_3 M \) and we get

\[
K_2(3,n) \frac{g_3^{m+2} M_{1}^{2m+2}}{\left[ (g_3 - M_1/2)^2 - K_1^2 / (4(m+2)) \right]} e^{-g_3^{3+M_1}},
\]

where

\[
K_2(p,n) = K_1(p,n) / \left\{ (m+1)2^{2m+2} \right\}.
\]

Now integrating \( g_3 \) from \( M_1 \) to \( \infty \) we get for \( 0 < M \leq 1 \)
\[
K_2(3,n) e^{-M_1} M_1^{2m+2} \left[ a_0 I(M_1, \infty; m+3) + a_1 M_1 I(M_1, \infty; m+2) + a_2 M_1^2 I(M_1, \infty; m+1) \right],
\]

where \( a_0 = 1, \ a_1 = -1, \ a_2 = (m+1)/(4(m+2)) \) and \( I(x_1,x_2; q) = \int_{x_1}^{x_2} e^{-x} x^q dx \).

Now we consider the case when \( 1 \leq M \leq 2 \). Let \( L_i^i = 1-L_i^i, i=1,2 \) such that \( M' = 2-M, \ G' = (1-M+G) \), then the distribution of \( \xi_3 \) and \( M' \) can be written in the form

\[
(3.4) \quad K_1(3,n) e^{-\xi_3(3-M')} \xi_3^{3m+5} \left[ \frac{(1-M'/2)^{2m+2}}{(m+1)} - \frac{(1-M'/2)^2}{m+2} + \frac{(1-M')^{m+2}}{(m+1)(m+2)} \right].
\]

Integrate (3.4) with respect to \( \xi_3 \) from \( M_1/2 \) to \( M_1 \) and combine the result with (3.3), then the distribution of \( M_1 \) can be written in the form

\[
(3.5) \quad K_2(3,n) e^{-M_1} M_1^{2m+2} \sum_{i=0}^{2} a_i M_1^i I(M_1/2, \infty; m+3-i) \\
+ 2^{2m+2(m+2)-1} \int_{M_1/2}^{M_1} \xi_3^{2m+2(M_1-\xi_3)^{m+2}} e^{-\xi_3 \xi_3} d\xi_3, \\
0 < M_1 < \infty.
\]

**Case ii.** Put \( p = 4 \) in (3.1) and integrate \( \xi_k \), then the distribution of \( \xi_3 \) and \( M \) is given by
\[(3.6) \quad K_2(4,n) e^{g_3(2+M)} M^{2m+2} \sum_{r=0}^{m+2} (r+1) g_3^{4m+7-r} \]

\[\left[ (a-bM) \left\{ \frac{(1-M/2)^2-M^2}{4(m+2)} \right\} + a_2CM^2 \left\{ \frac{(1-M/2)^2-M^2}{4(m+3)} \right\} \right], \]

where \( a = \frac{(m+2)!}{(m+2-r)!}, b = \frac{(m+1)!}{(m+1-r)!} \) and \( 0 < M \leq 1, C = m! / (m-r)! \).

As before transform \( M_1 = g_3 M \), and integrate \( g_3 \), then the distribution of \( M_1 \), for \( 0 < M \leq 1 \), takes the form

\[(3.7) \quad 2^{-2m+5} K_2(4,n) e^{M_1} \sum_{r=0}^{m+2} (r+1) \left\{ M_1^{2m+2} \sum_{i=0}^{3} 2^{r+i} \right\} \frac{M_1^i}{a_1^i} I(2M_1, \infty; 2m+5-r-i) + a_2CM_1^{2m+4} \sum_{i=0}^{2} 2^{r+i+2} \frac{M_1^i}{b_1} I(2M_1, \infty; 2m+3-r-i)}, \quad 0 \leq M \leq 1, \]

where \( a_0' = a, a_1' = -(a+b), a_2' = a(m+1)/\{(m+2)4\} + b, a_3' = -b(m+1)/4(m+2), b_0 = 1, b_1 = -1 \) and \( b_2 = (m+2)/4(m+3) \). Now, when \( 1 \leq M \leq 2 \), as before, transform to \( M' \) and \( G' \) and integrate out \( G' \), and further transform to \( M = 2-M' \) and \( M_1 = g_3 M \) and integrate out \( g_3 \) between \( M_1/2 \) and \( M_1 \) and combining the result with (3.7) we get
\[ 2^{-m} K_2(4, n) e^{-M_1} \sum_{r=0}^{m+2} (r+1) \left[ M_1^{2m+2} \sum_{i=0}^{4} 2^{r+i-m-7} c_i \right. \]

\[ M_1^i I(M_1, \infty; 2m-r+i+5) + (m+2)^{-1} \left( (a-c) \sum_{i=0}^{m+2} (m+2)^i (-1)^i \right. \]

\[ g(r,i+1) + (c-b) \sum_{i=0}^{m+2} (m+2)^i (-1)^i g(r,i) - c(m+3)^{-1} \]

\[ \sum_{i=0}^{m+3} (m+3)^i (-1)^i 2g(r,i) \quad 0 < M_1 < \infty \]

where

\[ g(r,i) = 2^{r-i-2} M_1^{m+3-i} I(M_1, 2M_1; 3m+2+i-6) \]

\[ c_0 = 4a, \quad c_1 = -4(a+b), \quad c_2 = (c+a)(m+1)(m+2)^{-1} + 4b \]

\[ c_3 = -(c+b)(m+1)(m+2)^{-1}, \text{ and } c_4 = c(m+1)/\{4(m+3)\} \]
Case iii. Put \( p = 5 \) in (3.1) and integrate \( g_5 \) and \( g_4 \), then the distribution of \( g_3 \) and \( M \) is given by

\[
(3.9) \quad K_2(5,n) e^{-g_3(3+M)} g_3^{3m+5} M^{2m+2} \sum_{r=0}^{6} \eta_r M^r g_3^{2m+7-i-j}
\]

where \( \eta_0 = K_{0,i,j}/(m+1) \), \( \eta_1 = (K_{1,i,j} - K_{0,i,j})/(m+1) \),

\[
\eta_2 = (K_{0,i,j} + K_{3,i,j})/4(m+2) + (K_{2,i,j} - K_{1,i,j})/(m+1),
\]

\[
\eta_3 = (K_{1,i,j} - K_{3,i,j} + K_{4,i,j})/4(m+2) - K_{2,i,j}/(m+1),
\]

\[
\eta_4 = (K_{2,i,j} - K_{4,i,j})/4(m+2) + (K_{3,i,j} + K_{5,i,j})/2^4(m+3),
\]

\[
\eta_5 = (K_{4,i,j} - K_{5,i,j})2^4(m+3), \text{ and } \eta_6 = K_{5,i,j}/2^6(m+4)
\]

and the \( K_{\ell,i,j} \) are defined by

\[
(3.10) \quad K_{\ell,i,j} = \sum_{j=0}^{\ell} \sum_{i=0}^{m+k} \frac{1}{(2+j+1)^{l-1-j} \ell_{6} \ell_{5}} \left[ a_{\ell}^{(1)}(2m+7-i-\ell_{6} \ell_{5} - j) - a_{\ell}^{(2)}(2m+6-i-\ell_{6} \ell_{5} - j) + a_{\ell}^{(3)}(2m+5-i-\ell_{6} \ell_{5} - j) \right],
\]

where

\[
\ell_{6} = \begin{cases} \ell, & \text{for } \ell = 0, 1, \text{ and } 2, \\ \ell_{1}, & \text{for } \ell = 3, 4, \text{ and } 5, 
\end{cases}
\]

and

\[
K = \begin{cases} 4 & \text{for } \ell = 0, 1, 3 \\ 3 & \text{for } \ell = 2, 4 \\ 2 & \text{for } \ell = 5
\end{cases}
\]

and
\[
a_0^{(1)} = (m+3)_{-i+1}, \quad a_0^{(2)} = -a_1^{(m+2)}_{-i+2}, \quad a_0^{(3)} = (m+2)_{-i+1} \\
a_1^{(1)} = a_0^{(2)}, \quad a_1^{(2)} = b_i^{(m+1)}_{-i+3}, \quad a_1^{(3)} = -c_i^{(m+1)}_{-i+1} \\
a_2^{(1)} = a_0^{(3)}, \quad a_2^{(2)} = -c_i^{(m+1)}_{-i+2}, \quad a_2^{(3)} = (m+1)_{-i+1} \\
a_3^{(1)} = d_i^{(m+1)}_{-i+3}, \quad a_3^{(2)} = -c_i^{(m)}_{-i+4}, \quad a_3^{(3)} = g_i^{(m)}_{-i+3} \\
a_4^{(1)} = a_2^{(2)}, \quad a_4^{(2)} = k_i^{(m)}_{-i+3}, \quad a_4^{(3)} = -l_i^{(m)}_{-i+2} \\
a_5^{(1)} = a_2^{(3)}, \quad a_5^{(2)} = a_4^{(3)}, \quad a_5^{(3)} = (m)_{-i+1} \\
\] (3.11)

and (a)_{-i+b} = a(a-1) \ldots (a-i+b+1); \ a_1 = 2, \ a_i = 2m+i-1, \ i \geq 2;
\[
b_1 = 4, \ b_2 = 4m+8 \quad \text{and} \quad b_i = (2m+7-i)(2m+5-i) + i-1 \quad \text{for} \quad i \geq 3;
\]
c_1 = 2, \ c_i = 2m+5-i \quad \text{for} \quad i \geq 2; \ d_1 = 2, \ d_2 = 2m+4 \quad \text{and} \quad d_i = (m+2)_{-i+2} +
\[
(m+3-i)_{-i+2} \quad \text{for} \quad i \geq 3; \ e_1 = 4, \ e_2 = 4m+6, \ e_3 = \sum_{i=0}^{3} (m+i)_{-i+2} \quad \text{and}
\]
e_i = \sum_{K=0}^{3} (m+2-i+K)(m+1)_K \quad \text{for} \quad i \geq 4; \ g_1 = 2, \ g_2 = 2m+2, \ g_i = (m+1)_{-i+2} +
\[
(m+2-i)_{-i+2} \quad \text{for} \quad i \geq 3; \ l_1 = 2, \ l_i = 2m-i+3, \ i \geq 2, \ k_1 = 4, \ k_2 = 4m + 4,
\]
k_i = 4m^2 + 16m - 4im + i^2 - 7i + 14 \quad \text{for} \quad i \geq 3.

As before transform \( M_1 = g_3 M \), and integrate \( g_3 \), then the dis-

\[
(3.12) \quad K_2(5,n) M_1^{2m+2} e^{-M_1} \sum_{r=0}^{6} \eta_r M_1^{3}(3M_1,\infty;3m+10-i-j-r)_{3m+10-i-j-r}.
\]
Now, when \( 1 \leq M \leq 2 \), proceeding as before, and combining the result with (3.12) we get

\[
K_3(5,n)M_1^{m+2} e^{-M_1} \left[ (3M_1)^m \sum_{r=0}^{6} 3^{i+j+r} \eta_{r}^r M_1^r \cdot I(3M_1/2,\infty;3m+10-i-j-r) 
+ 2^{2m+2} \sum_{s=0}^{m+2} (-1)^s \sum_{r=0}^{2} p_{r}^r M_1^{r-s} 3^{s+i+j+r} I(3M_1/2,3M_1;4m+10+s-i-j-r) \right]
\]

where \( K_3(5,n) = K_2(5,n)/3^{4m+10} \),

\[
p_0 = K_{0,i,j}/(m+1)(m+2) - K_{3,i,j}/(m+2)(m+3) + K_{5,i,j}/(m+3)(m+4),
\]

\[
p_1 = K_{1,i,j}/(m+1)(m+2) + (K_{3,i,j} - K_{4,i,j})/(m+2)(m+3) - 2K_{5,i,j}/(m+3)(m+4)
\]

\[
p_2 = K_{2,i,j}/(m+1)(m+2) + K_{4,i,j}/(m+2)(m+3) + K_{5,i,j}/(m+3)(m+4).
\]

4. The Non-Central distribution of Wilks' Criterion. In this section we shall derive the non-central distribution of Wilks' criterion, namely

\[
\Lambda = W(p) = \prod_{i=1}^{p} (1-r_i) \text{ where } r_1, \ldots, r_p \text{ are the characteristic roots of}
\]

\[
|S_1 - r(S_1 + S_2)| = 0
\]

where \( S_1 \) is a \((p \times p)\) matrix distributed non-central Wishart with \( s \) degrees of freedom and a matrix of non-centrality parameters \( \Omega \) and \( S_2 \) has the Wishart distribution with \( t \) degrees of freedom, the covariance matrix in each case being \( \Sigma \). For this, first we state below a few results on Mellin transform and then prove a lemma.
Theorem 1. If \( s \) is any complex variate and \( f(x) \) is a function of a real variable \( x \), such that

\[
(4.1) \quad F(s) = \int_0^\infty x^{s-1} f(x) dx
\]

exists. Then, under certain conditions [3]

\[
(4.2) \quad f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} F(s) ds.
\]

\( F(s) \) in (4.1) is called the Mellin transform of \( f(x) \) and \( f(x) \) in (4.2) is called the inverse Mellin transform of \( F(s) \). Now we state another theorem [3].

Theorem 2. If \( f_1(x) \) and \( f_2(x) \) are the inverse Mellin transform of \( F_1(s) \) and \( F_2(s) \) respectively, then the inverse Mellin transform of \( F_1(s) F_2(s) \) is given by

\[
(4.3) \quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} F_1(s) F_2(s) ds = \int_0^\infty f_1(u)f_2(x/u) \cdot \frac{du}{u}.
\]

Further we use theorem 2 to prove the following lemma.

Lemma 1. If \( s \) is a complex variate, \( a, b, c, d, m, n, p \) and \( \ell \) are reals then
\[ I = \frac{1}{2\pi i} \int_{c-\infty}^{c+\infty} x^{-s} \frac{\Gamma(s+a) \Gamma(s+b) \Gamma(s+c) \Gamma(s+d)}{\Gamma(s+a+m) \Gamma(s+b+n) \Gamma(s+c+p) \Gamma(s+d+\ell)} \, ds \]

\[ (4.4) = \frac{x^d (1-x)^{m+n+p+\ell-1}}{\Gamma(m+n+p)} \sum_{k=0}^{\infty} \frac{(d+\ell-a)_k}{k!} \sum_{r=0}^{\infty} \frac{(p)_r (b+n-c)_r}{r! (m+n+p)_r} (1-x)^{k+r} \]

\[ \frac{\Gamma(m+n+p+k+r)}{\Gamma(m+n+p+\ell+k+r)} \]

\[ _3F_2 \left( \begin{array}{c} a+m-b, n+p+r, m+n+p+k+r; m+n+p+r, m+n+p+\ell+k+r \end{array}; 1-x \right) \]

**Proof:** Let \( F_1(s) = \{ \Gamma(s+a) \Gamma(s+b) \Gamma(s+c)/\Gamma(s+a+m) \Gamma(s+b+n) \Gamma(s+c+p) \} \),
\( F_2(s) = \Gamma(s+d)/\Gamma(s+d+\ell) \), then

\[ f_1(x) = x^a (1-x)^{m+n+p-1} \left[ \Gamma(m+n+p) \right]^{-1} \sum_{r=0}^{\infty} \frac{(p)_r (b+n-c)_r}{r! (m+n+p)_r} (1-x)^r \]

(4.5)

\[ _2F_1(a+m-b, n+p+r; m+n+p+r; 1-x) \]

\[ x^d (1-x)^{\ell-1} \]

and \( f_2(x) = \frac{\Gamma(\ell)}{\Gamma(k)} \), \( 0 < X < 1 \), [4].

Now by the use of Theorem 2 we get

\[ I = \frac{x^d}{\Gamma(\ell) \Gamma(m+n+p)} \int_X^1 u^{a-d-\ell} (1-u)^{m+n+p-1} \sum_{r=0}^{\infty} \frac{(p)_r (b+n-c)_r}{r! (m+n+p)_r} (1-u)^{k+r} \]

(4.6)

\[ (1-u)^r _2F_1(a+m-b, n+p+r; m+n+p+r; 1-u) (U-X)^{\ell-1} \, du \]

Further, put \( u = 1 - (1-X)t \) in the above and by simplifying we have
\[
I = \frac{X^d (1 - X)^{m+n+p+\ell-1}}{\Gamma(\ell) \Gamma(m+p+n)} \int_0^1 \left( \sum_{k=0}^{\infty} \frac{(d+\ell-a)_k}{k!} \sum_{r=0}^{\infty} \frac{(p)_r(b+n-c)_r}{r!(m+n+p)_r} \right) \\
\sum_{i=0}^{\infty} \frac{(a+m-b)(m+p+r)_i}{i!(m+n+p+r)_i} (1 - X)^{k+i+r} t^{m+n+p+k+i+r-1} (1-t)^{\ell-1} dt .
\]

(4.7)

Now integrate (4.7) with respect to \( t \), then the lemma follows immediately.

The moments of the Wilks' Criterion has been given [2] in the following form.

\[
E[W^{(h)}] = \left[ \Gamma_p(h+\frac{1}{2}t) \Gamma_p(v)/\Gamma_p(t/2) \Gamma_p(h+v) \right] \chi_1^2(h; h+v; -\Omega) ,
\]

where \( v = \frac{1}{2}(s+t) \), and \( \Gamma_p(u) = \prod_{i=1}^{p} (u-\frac{1}{2}(i-1)) \).

By using Kummar transformation then (4.8) can be written in the following form

\[
E[W^{(h)}] = \left[ \Gamma_p(h+\frac{1}{2}t) \Gamma_p(v)/\Gamma_p(t/2) \Gamma_p(h+v) \right] e^{-tr\Omega} \chi_1^2(v; h+v; \Omega) .
\]

(4.9)

Case i. Put \( p = 2 \) in (4.9), then

\[
E[W^{(h)}] = \frac{\Gamma(2v-1)}{2^v \Gamma(t-1)} e^{-tr\Omega} \sum_{k=0}^{\infty} \sum_{K} \frac{(\nu)_k \chi_k(\Omega)}{k!} . \frac{\Gamma(r) \Gamma(r+\frac{1}{2})}{\Gamma(r+\frac{3}{2}+k_1+\frac{1}{2}) \Gamma(r+\frac{3}{2}+k_2)} ,
\]

(4.10)

where \( r = h+\frac{3}{2}t-\frac{3}{2} \) and \( k_1 > k_2 > 0, k_1 + k_2 = k \),

then
\[(4.11) \quad f(W^{(2)}) = \frac{\Gamma(2v-1)}{2^s \Gamma(t-1)} \exp(\text{tr} \cdot \Omega) \sum_{k=0}^{\infty} \sum_{K} \frac{(v)_K C_k(\Omega)}{K!} \cdot \]

\[\frac{1}{2\pi i} \oint_{c-i\infty}^{c+i\infty} \left[ W^{(2)} \right]^{-h-1} \left[ \Gamma(r) \Gamma(r+\frac{1}{2})/\Gamma(r+\frac{1}{2}s+k_2) \Gamma(r+\frac{1}{2}s+\frac{1}{2}+k_1) \right] dr . \]

Now, by the use of the results of Consul [4], we get the density function of \( W^{(2)} \) in the following form

\[(4.12) \quad f(W^{(2)}) = \frac{\Gamma(2v-1)}{2^s \Gamma(t-1)} [W^{(2)}]^{\frac{1}{2}(t-3)} \exp(\text{tr} \cdot \Omega) \cdot \sum_{k=0}^{\infty} \sum_{K} \frac{(v)_K C_k(\Omega)}{K! \Gamma(s+k)} \]

\[(1-W^{(2)})^{s+k-1} \quad _2F_1\left(\frac{3s+k_1}{2s+k_2}, \frac{1}{2}; s+k, 1-W^{(2)}\right). \]

Putting \( \Omega = Q \), then the central case can be written in the following form

\[(4.13) \quad f(W^{(2)}) = \frac{\Gamma(2v-1)}{2^s \Gamma(t-1) \Gamma(s)} [W^{(2)}]^{\frac{1}{2}} (1-W^{(2)})^{s-1} \quad _2F_1\left(s/2, (s-1)/2; s; 1-W^{(2)}\right). \]

It may be pointed out that (4.13) can be reduced to

\[(4.14) \quad \frac{\Gamma(2v-1)}{2\Gamma(t-1) \Gamma(s)} [W^{(2)}]^{\frac{1}{2}(t-3)} (1-W^{(2)})^{s-1} \quad _2F_1\left(\frac{3s+k_1}{2s+k_2}, \frac{1}{2}; s+k, 1-W^{(2)}\right), \]

by observing that

\[(4.15) \quad _2F_1\left(s/2, (s-1)/2; s; 1-U\right) = 2^{s-1}/(1+U) \quad _{s-1} \quad ([11]). \]

Also the density function of \( W^{(2)} \) can be written in the following form

by the use of the results in [3].
\begin{equation}
\begin{aligned}
\text{Case ii.} \text{ Put } p = 3 \text{ in } (4.9), \text{ and by the use of } (4.2) \text{ the density function of } W^{(3)} \text{ can be written in the following form}

f(W^{(3)}) &= \frac{\Gamma_3(v)}{\Gamma_3(t/2)} \exp(\text{tr} - Q) \{W^{(2)}\}^{\frac{1}{2}(t-3)} \sum_{k=0}^{\infty} \sum_{k} \left( \frac{v_k}{k!} \frac{C_k(Q)}{\Gamma(s+2k)} \right) \sum_{r=0}^{s+2k-1} \binom{s+2k-1}{r} (-1)^r \left\{ W^{(2)} \right\}^{r/2}.

\int_{c-i\infty}^{c+i\infty} \frac{W^{(3)}}{\Gamma(r + \frac{3}{2}s + k)} \frac{\Gamma(r) \Gamma\left(r + \frac{1}{2}k_1\right) \Gamma\left(r + \frac{1}{2}k_2\right) \Gamma\left(r + \frac{1}{2}s + k + 1\right)}{r! \Gamma(3s/2 + k)} \,dr

\text{where } k_1 \geq k_2 \geq k_3 \geq 0, \ k_1 + k_2 + k_3 = k.

By (4.5), the density function of \( W^{(3)} \) can be written in the form

\begin{aligned}
f(W^{(3)}) &= \frac{\Gamma_3(v)}{\Gamma_3(t/2)} \exp(\text{tr} - Q) \left\{ W^{(3)} \right\}^{\frac{1}{2}(t-1)} (1 - W^{(3)})^{\frac{3}{2}s-1}.

\sum_{k=0}^{\infty} \sum_{k} \left( \frac{v_k}{k!} \frac{C_k(Q)}{\Gamma(3s/2 + k)} \right) \sum_{r=0}^{\infty} \frac{\left( \frac{1}{2}s + k_1 \right)_r \left( \frac{1}{2}(s-1) + k_2 \right)_r}{r! \Gamma\left(3s/2 + k + r\right)} (1 - W^{(3)})^{r+k}

2F_1\left(\frac{1}{2}(s-1) + k_3, s + k_1 + k_2 + r; 3s/2 + k + r; 1 - W^{(3)}\right).
\end{aligned}
\end{aligned}
\end{equation}
Case iii. Put \( p = 4 \) in (4.9) and by the use of (4.2) the density function of \( W^{(4)} \) can be written in the form

\[
(4.19) \quad f(w^{(4)}) = \frac{\Gamma_h(v)}{\Gamma_h(2t)} \exp(-\Omega) \{W^{(4)}\}_{\frac{1}{2}(t-5)} - \sum_{k=0}^{\infty} \sum_{\forall k} \frac{(v)_k \cdot C_k(\Omega)}{k!}.
\]

\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(r) \Gamma(r+\frac{1}{2}) \Gamma(r+1) \Gamma(r+2) \{W^{(4)}\}_{r-1} \Omega}{\Gamma(r+\frac{3}{2}s+k_4) \Gamma(r+\frac{3}{2}s+1+k_3) \Gamma(r+\frac{3}{2}s+1+k_2) \Gamma(r+\frac{3}{2}s+\frac{3}{2}+k_1)} dr
\]

where \( k_4 \geq k_3 \geq k_2 \geq k_1 \geq 0 \), and \( \sum_{i=1}^{k} = k \).

By using lemma 1, the density function of \( W^{(4)} \) can be written in the form

\[
(4.20) \quad f(w^{(4)}) = \frac{\Gamma_h(v)}{\Gamma_h(2t)} \exp(-\Omega) \{w^{(4)}\}_{\frac{1}{2}(t-2)} (1 - W^{(4)})^{2s-1}
\]

\[
\sum_{k=0}^{\infty} \sum_{\forall k} \frac{(v)_k \cdot C_k(\Omega)}{k!} \sum_{j=0}^{\infty} \frac{(\frac{3}{2}(s+3)+k_1)_j}{j!} \sum_{r=0}^{\infty} \frac{(\frac{3}{2}(s+2)+k_1)_r}{r!} \Gamma(3s/2+k+j+k_1+r) \Gamma(3s/2+k+j+k_1+r)
\]

\[
\frac{1}{\Gamma(3s/2+k_1+r)} (1 - W^{(4)})^{k+j+r}
\]

\[
F_{\frac{3}{2}}(\frac{3}{2}(s-1)+k_4, s+k_2+k_3+r, 3s/2+k_1+j+r; 3s/2+k_1+r, 2s+j+k+r; 1 - W^{(4)}).
\]

It may be pointed out that the non-central distribution of Wilks' criterion could be found for more than \( p = 3 \) by extending lemma 1. However the distribution would be complicated.
References


