A Martingale Version of a Theorem of Marcinkiewicz and Zygmund*

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1. Introduction. Suppose that \((x_n, n \geq 1)\) is an orthonormal sequence of independent random variables and \((a_n, n \geq 1)\) is a sequence of real numbers. Marcinkiewicz and Zygmund [5] proved that if \(P[\sum a_k x_k \text{ converges}] = 1\), \(\sum a_k^2 < \infty\). Recently, Gundy [3] extends their theorem to martingales as follows:

Let \((d_n, n \geq 1)\) be a sequence of martingale differences with \(E(d_n^2 | \mathcal{F}_{n-1}) = 1\) a.e. and \(P(|d_n| > \lambda |\mathcal{F}_{n-1}) \geq \gamma \text{ a.e. for some positive constants } \lambda \text{ and } \gamma\), and let \((v_n, \mathcal{F}_{n-1}, n \geq 1)\) be a stochastic sequence, i.e., \(v_n\) is an \(\mathcal{F}_n\)-measurable random variable for each \(n\). Then \(\sum v_k^2 < \infty\) a.e. on the set \([\sum v_k d_k \text{ converges}].\)

Let \(x_1, x_2, \ldots\) be independent, identically distributed random variables and \((a_m, n \geq 1, n \geq 1)\) be a double sequence of real numbers such that \(\lim_n a_{m,n} = a_m\) for each \(n\). In [6], Zygmund proved that if \(\sum_{k=1}^{\infty} a_{m,k} x_k = T_m\) a.e. and \(P[\sup_m |T_m| < \infty] = 1\), then \(\sum a_k^2 < \infty\).

In this note, by stopping rules, we will extend Marcinkiewicz and Zygmund's theorem in a different direction and at the same time generalize Zygmund's theorem.

2. Main theorem. In this section, as well as in the following one, we will assume that \((d_k, \mathcal{F}_k, k \geq 1)\) is a sequence of martingale differences with \(E(d_k^2 | \mathcal{F}_{k-1}) = 1\) (\(\mathcal{F}_0 = \{\phi, \Omega\}\)), \((a_{m,n}, m \geq 1, n \geq 1)\) is a double sequence.

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of real numbers with $\lim_{m} a_{m,n} = a_n$ for each $n \geq 1$, and that $s_{m,n} = \sum_{k=1}^{n} a_{m,k} d_k$.

Theorem 1. Let

(1) $\inf_n E|d_n| \geq \delta > 0$,

(2) $\lim_{K \to \infty} P[\sup_n |s_{m,n}| \geq K] = 0$ uniformly in $m$,

(3) $\sup_{m,n} |s_{m,n}| \leq M < \infty$.

Then $\sum a_k^2 < \infty$.

Proof. For $K > \max(M, 2\delta^{-1})$ and $m = 1, 2, \ldots$, put $b_n = b_n(m) = a_{m,n}$,

$s_n = s_{m,n} = \sum_{k=1}^{n} b_k x_k$, and

(4) $t = t(m) = \inf \{ n \geq 1 : s_n^2 > K^2 \}$.

For $j = 1, 2, \ldots$, put $\tau = \min(t, j)$. It is easy to see that (for example, see [1])

(5) $E S^2_\tau = E \sum_{k=1}^{\tau} b_k^2 E(d_k^2|s_{k-1}) \geq P[t=\infty] \sum_{k=1}^{\tau} b_k^2$.

On the other hand,
\[ E S_t^2 = \int [t > j] S_j^2 + \int [t \leq j, b_t^2 d_t^2 \leq K^4] S_t^2 + \]
\[ + \int [t \leq j, b_t^2 d_t^2 > K^4] (S_{t-1}^2 + 2 S_{t-1} b_t d_t + b_t^2 d_t^2) \]
\[ \leq (K + K^2)^2 + \int [t \leq j, b_t^2 d_t^2 > K^4] (2 S_{t-1} b_t d_t + b_t^2 d_t^2) \]
\[ \leq (K + K^2)^2 + (1 + 2K^{-1}) \int [t \leq j, b_t^2 d_t^2 > K^4] b_t^2 d_t^2. \]

Hence

\[(6) \quad (K + K^2)^2 \geq \sum_{k=1}^{\infty} b_k^2 \{P[t = \infty] - (1 + 2K^{-1}) \int [t = k, b_k^2 d_k^2 > K^4] d_k^2\}.\]

Since \( E d_k^2 = 1 \) and \( K > \max (M, 2 \delta^{-1}) \), we have

\[ \int [b_k^2 d_k^2 > K^4] |d_k| \leq |b_k| K^{-2} \int [b_k^2 d_k^2 > K^4] d_k^2 \leq \delta/2, \]
\[ \int [b_k^2 d_k^2 > K^4] d_k^2 = 1 - \int [b_k^2 d_k^2 \leq K^4] d_k^2 \leq 1 - (E|d_k| - \int [b_k^2 d_k^2 > K^4] |d_k|)^2 \]
\[ \leq 1 - \delta^2/4. \]
Choose \( K \) so large that \((1+2K^{-1})(1-\delta^2/4) \leq 1-\delta^2/8\). Then

\[
(K+K^2)^2 \geq \sum_{k=1}^{j} b_k^2 \left[P[t = \infty] - (1+2K^{-1})(1-\delta^2/4)\right]
\]

\[
\geq \sum_{k=1}^{j} b_k^2 (P[t = \infty] - 1 + \delta^2/8).
\]

The condition (2) implies that \( P[t = \infty] > 1-\delta^2/16 \) for all \( m = 1,2,\ldots \), if \( K \geq K_0 \) for some \( K_0 \). Let \( K = K_0 \). Then

\[
(K+K^2)^2 \geq (\delta^2/16) \sum_{k=1}^{j} a_k^2 = (\delta^2/16) \sum_{k=1}^{j} \sum_{m,k} a_{m,k}^2.
\]

Therefore \( (K+K^2)^2 \geq (\delta^2/16) \sum_{k=1}^{\infty} a_k^2 \), which completes the proof.

3. Some corollaries.

Corollary 1. If there exist positive constants \( \lambda \) and \( \gamma \) such that

\[
P(|d_k| > \lambda \sum_{k=1}^{j} a_k^2) \geq \gamma \quad \text{a.e.,}
\]

then \( \sum_k a_k^2 < \infty \), provided that (2) is satisfied.

Proof. Obviously (9) implies (1). To prove (3), assume that there exists a subsequence \( k_m \) such that \( |a_{n_k,m}| > m \) for \( m = 1,2,\ldots \). By Lévy's martingale version (for example, see [2], p. 324) of the Borel-Cantelli lemma, (9) implies that
(10) \( \Pr[|d_{k_m}| > \lambda \text{ i.o.}] = 1 \).

Hence

\[
\Pr[|a_{n_{k_m}, k_m}| > m \lambda \text{ i.o.}] = 1,
\]

which contradicts (2). Therefore (2) and (9) imply (3).

Corollary 2. Let \( a_{m,n} = a_n \) for all \( m \geq 1 \) and \( n \geq 1 \). If

(11) \( \Pr[\sum a_k d_k \text{ converges}] = 1 \),

then \( \sum a_k^2 < \infty \), provided that (1) is satisfied.

Proof. Obviously (11) implies (2). We will prove that (1) and (11) imply that \( \lim_n a_n = 0 \). Assume that there exist \( \epsilon > 0 \) and a subsequence \( k_m \) such that \( |a_{k_m}| \geq \epsilon \) for \( m = 1, 2, \ldots \). Then (11) implies that \( \Pr[\lim_{m} d_{k_m} = 0] = 1 \). Since \( \mathbb{E} d_{k_m}^2 = 1 \) implies that \( (d_k, k \geq 1) \) is uniformly integrable, we obtain \( \lim_{m} \mathbb{E}|d_{k_m}| = 0 \), which contradicts (1). Thus the proof is completed.

Corollary 2 reduces Gundy's condition (9) to condition (1), when the stochastic sequence \( (v_n, v_{n-1}, \ldots, n \geq 1) \) is a sequence of constants.

Corollary 3. Let \( d_1, d_2, \ldots \) be orthonormal, independent random variables with zero median. If

(12) \( \Pr[\lim_n s_{m,n} = T_m] = 1 \),
\[
(13) \quad P\left[ \sup_m |T_m| < \infty \right] = 1,
\]

and if (1) holds, then \( \sum a_k^2 < \infty \).

Proof. By Lévy's inequality (see, for example, [2], p. 106), (12) and (13) imply (2). Since \( d_n \) are independent and uniformly integrable, (1) implies (9) immediately. Therefore Corollary 3 follows from Corollary 1.

When \( P[d_n = \pm 1] = 1/2 \), Corollary 3 was proved by Zygmund [6].

4. Extension of a theorem of Kac and Steinhaus. Let \( (d_k, \mathcal{F}_k, k \geq l) \) be an orthonormal sequence of martingale differences such that \( (d_k^2, k \geq l) \) is uniformly integrable and let \( a_{m,n} \) and \( S_{m,n} \) be defined as in section 2.

Theorem 2. Under the conditions (2) and (3),

\[
(14) \quad \sum a_k^2 E(d_k^2 | \mathcal{F}_k-1) < \infty \quad \text{a.e.}
\]

Proof. For \( K > 0 \) and \( m,j = 1,2, \ldots \), define \( b_n, \tau \) and \( \tau \) as in section 2. Then, as before,

\[
(15) \quad ES^2_{\tau} = E \Sigma_{k=1}^{\tau} b_k^2 d_k^2 \geq \Sigma_{k=1}^{\tau} b_k^2 \int_{[t=\infty]} d_k^2 ,
\]

\[
(16) \quad ES^2_{\tau} \leq (K+K^2)^2 + (1+2K^{-1}) \Sigma_{k=1}^{\tau} b_k^2 \int_{[t=k]} d_k^2 .
\]

By (2) and the uniformly integrability of \( (d_k^2, k \geq l) \), for all \( K \geq K_0 \) and
\[ k \geq k_0, \text{ we have } \int_{[t=\infty]} d_k^2 > 1/2 \text{ and } \int_{[t=k]} d_k^2 < 1/4. \text{ Hence, as } j \to \infty,\]

\[ \sum_{k=k_0}^{j} b_k^2 = O(1). \text{ Therefore} \]

\[ O(1) = ES_T^2 = E \sum_{k=1}^{T} b_k^2 E(d_k^2|\mathcal{F}_{k-1}) \geq \int_{[t=\infty]} \sum_{k=1}^{j} b_k^2 E(d_k^2|\mathcal{F}_{k-1}). \]

Hence for all \( K \geq K_0, \)

\[ \int_{[t=\infty]} \sum_{k=1}^{\infty} a_k^2 E(d_k^2|\mathcal{F}_{k-1}) < \infty. \]

Since (2) implies that \( \lim_{K \to \infty} P[t=\infty] = 1, (14) \) follows immediately.

When \( d_1, d_2, \ldots, \) are independent random variables and \( a_{m,n} = a_n \) for \( m, n = 1, 2, \ldots, \) Theorem 2 was proved by Kac and Steinhaus [4].
References


