On Some Selection and Ranking Procedures with Applications to Multivariate Populations*

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On Some Selection and Ranking Procedures with Applications to Multivariate Populations

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Introduction and Summary

This paper is concerned with ranking and selection of $k$ multivariate normal populations. The selection and ranking problem is formulated in terms of suitably defined scalar functions. For $k$ multivariate normal populations with mean vectors $\mu_i (i=1,2,\ldots,k)$ each of which has $p$ components, a function that arises naturally, is the scalar quantity $\lambda_i = \mu_i^T \Sigma_i^{-1} \mu_i$ where $\Sigma_i$ is the covariance matrix of the $i$th population. With suitably defined statistics the ranking of multivariate normal populations in terms of $\lambda_i$ can be reduced to the ranking of non-centrality parameters of non-central chi-square or non-central $F$ distributions.

We are interested in selecting the populations with large (small) values of the parameters $\lambda_i$. The procedures to be defined select a non-empty subset which is small and yet large enough to guarantee a certain basic probability requirement. This requirement is that the population with the largest value of the parameter is included in the selected subset with probability at least equal to a given number $P^* (1/k < P^* < 1)$. This type of problem has been studied in a number of recent papers. For a

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rather complete bibliography, reference should be made to Gupta (1965).

In Section 2, a formal statement of the problems is given and procedures are defined for selecting populations with the largest and smallest parameters. Probability of a correct selection and its partial infimum are evaluated.

Section 3 deals with a general result concerning the infimum of the probability of a correct selection. In Section 4 applications to multivariate populations are given.

2. Formal Statement of the Problem

Suppose each of the $k$ population, $\pi_1, \pi_2, \ldots, \pi_k$ has an observable random variable $Y_i (i=1, 2, \ldots, k)$ whose density function is $f_{\lambda_i}(y), y \geq 0, \lambda_i \geq 0$. We assume that the density function $f_\lambda(y)$ has a monotone likelihood ratio. This implies that the expected value of $Y$ is a monotone increasing function of $\lambda$. In all specific cases to be considered the mean value will be a linear increasing function of $\lambda$.

Let the ranked $\lambda$'s be denoted by

\begin{equation}
\lambda[1] \preceq \lambda[2] \preceq \cdots \preceq \lambda[k].
\end{equation}

It is assumed that there is no a priori information available about the correct pairing of the ordered $\lambda[i]$ values and the $k$ given populations. Any population associated with $\lambda[k]$ ($\lambda[1]$) will be called a best population. A correct selection is defined as the selection of any subset of the $k$ populations which includes a best population. Our problem is to define a selection procedure which selects a small, non-empty subset of the $k$ populations and guarantees that a best population has been included with
probability at least $P^* (1/k < P^* < 1)$. If $CS$ stands for a correct selection then our goal is to define a decision rule $R$ such that

$$\inf_{\Omega} P[CS|R] \geq P^*$$

where $\Omega = \{ (\lambda_1, \lambda_2, \ldots, \lambda_k) : \lambda_i \geq 0, \text{ all } i \}$.

Selection Procedures

Let $y_i$ be an observation on $Y_i (i=1,2,\ldots,k)$. Then the procedures for selecting the population with the largest value $\lambda_{[k]}$ is

$R$: Select $\pi_i$ iff

$$cy_i \geq y_{\text{max}}, \quad c > 1$$

where $c = c(k, P^*)$ is the minimal value for which (2.2) is satisfied.

Similarly, the procedure $R'$ for selecting a subset containing the population with the smallest value $\lambda_{[1]}$ is defined to be

$R'$: Select $\pi_i$ iff

$$y_i \leq b y_{\text{min}}, \quad b > 1$$

where $b = b(k, P^*)$ is again the minimal value for which (2.2) is satisfied.

Probability of a Correct Selection and Its Infimum

We will now derive an expression for the probability of a correct selection and its infimum. Let $y_{(i)} (i=1,2,\ldots,k)$ be the observation which has come from the population $\pi_i$ with parameter $\lambda_{[i]}$. It should be
noted that \( y_{(1)} \) is one of the numbers \( y_i \) \((i=1,2,\ldots,k)\) though it is not known to us. For selecting the population associated with \( \lambda_{[k]} \), we then have

\[
(2.5) \quad P[\text{Selecting } \pi_{(k)} | R] = P[cY_{(k)} \geq y_{\text{max}}]
\]

\[
= P[cY_{(k)} \geq y_{(j)}, j=1,2,\ldots,k-1]
\]

\[
= \int_0^\infty \left[ \prod_{j=1}^{k-1} F_{\lambda_{[j]}}(cy) \right] f_{\lambda_{[k]}}(y) \, dy.
\]

Since \( f_{\lambda}(y) \) is assumed to have a monotone likelihood ratio, it follows that \( F_{\lambda}(y) \leq F_{\lambda'}(y) \) for all \( \lambda > \lambda' \) and each \( y \). In this case

\[
(2.6) \quad P[\text{Selecting } \pi_{(k)}] \geq \int_0^\infty \left[ F_{\lambda_{[k]}}(cy) \right]^{k-1} f_{\lambda_{[k]}}(y) \, dy.
\]

Since \( P[CS|R] \geq P[\text{Selecting } \pi_{(k)} | R] \), we conclude that

\[
(2.7) \quad \inf_{\Omega} P[CS|R] \geq \inf_{\lambda} \int_0^\infty F_{\lambda_{[k]}}(cy) f_{\lambda}(y) \, dy.
\]

For the problem of selecting the population with the smallest \( \lambda_{[1]} \), a similar argument shows that

\[
(2.8) \quad \inf_{\Omega} P[CS|R'] \geq \inf_{\lambda} \int_0^\infty [1-F_{\lambda_{[1]}}(y)]^{k-1} f_{\lambda}(y) \, dy.
\]
In the next Section we discuss a general theorem dealing with the infima of the expressions on the right sides of (2.7) and (2.8).

3. A Result Concerning the Infima of Probability of a Correct Selection

Let $g_j(x), j=0,1,2,...$ be a sequence of density functions on the interval $[0,\infty)$ and define

$$f_\lambda(x) = \sum_{j=0}^{\infty} \frac{\lambda^j e^{-\lambda}}{j!} g_j(x), \quad x \geq 0. \tag{3.1}$$

For a fixed integer $k \geq 2$ and $c > 1$ let

$$I(\lambda) = \int_{0}^{\infty} [F_\lambda(cx)]^{k-1} f_\lambda(x) \, dx. \tag{3.2}$$

and

$$J(\lambda) = \int_{0}^{\infty} [1-F_\lambda(cx)]^{k-1} f_\lambda(x) \, dx. \tag{3.3}$$

The purpose of this section is to provide sufficient conditions on the sequence $g_j(x), j=0,1,...$ which guarantee that the functions $I(\lambda)$ and $J(\lambda)$ attain their minimum value on $[0,\infty)$ at the point $\lambda = 0$.

**Theorem 3.1.**

1. If for each integer $k \geq 0$

$$\sum_{i=0}^{k} \frac{1}{i!(k-i)!} \left[ (G_{i+1}(cx)-G_i(cx))g_{k-i}(x) - cG_i(cx)G_{k-i+1}(x)-G_{k-i}(x) \right] \geq 0 \tag{3.4}$$
then the functions $I(\lambda)$ and $J(\lambda)$ defined in (3.2) and (3.3) are non-decreasing in $\lambda$.

(ii) If strict inequality holds in (3.4) for some integer $k$ then $I(\lambda)$ and $J(\lambda)$ are strictly increasing.

**Corollary 3.1.a**

Let

$$g_j(x) = \frac{x^{\mu+j-1}e^{-x}}{\Gamma(\mu+j)}, \quad j = 0, 1, \ldots$$

where $\mu > 0$. Then the functions $I(\lambda)$ and $J(\lambda)$ defined in (3.2) and (3.3) are strictly increasing.

**Proof:** For $g_j(x)$ defined by (3.5). Integrating by parts we see that

$$g_i(x) - g_{i+1}(x) = g_{i+1}(x), \quad i = 0, 1, \ldots$$

For $k \geq 1$ we insert the above expression in (3.4) and combine the terms $i$ and $k-i$. It may easily be shown that (3.4) reduces to

$$\sum_{i=0}^{\left[\frac{k-1}{2}\right]} \frac{e^{-x(c+1)}x^{\mu+k-1}(c^{-2i})c^{\mu+i}c^{\mu+k-2i-1}}{\Gamma(\mu+i+1)\Gamma(\mu+k-i+1)i!(k-i)!}.$$ 

For $k \geq 1$ the above expression is strictly positive. For $k = 0$ equation (3.4) reduces to zero. The corollary thus follows from Theorem 3.1.
Corollary 3.1.b. Let

\[(3.7) \quad g_j(x) = \frac{\Gamma(\mu + v + 1)}{\Gamma(v) \Gamma(\mu + j)} \frac{x^{\mu + j - 1}}{(1 + x)^{\mu + v + j}}, x \geq 0,\]

where \(\mu > 0, v > 0\) and \(j = 0, 1, \ldots\). Then the functions \(I(\lambda)\) and \(J(\lambda)\) defined in (3.2) and (3.3) are strictly increasing in \(\lambda\).

Proof. The proof in this case proceeds as in Corollary 3.1.a. The expression corresponding to (3.6) is

\[(3.8) \quad g_j(x) - g_{j+1}(x) = \frac{\Gamma(\mu + v + 1)}{\Gamma(v) \Gamma(\mu + j + 1)} \frac{x^{\mu + j}}{(1 + x)^{\mu + v + j}}.\]

Combining terms as in Corollary 3.1.a, equation (3.4) can be reduced to

\[
\sum_{i=0}^{\left[\frac{\ell-1}{2}\right]} \frac{\Gamma(\mu + v + 1)(i+1)}{i! (\ell-1)! \Gamma(\mu + i + 1)} \frac{x^{2\mu - 2i - 1} e^{\mu + i}}{[(1 + x)(1 + cx)]^{\mu + v + i}} \left[\frac{(cx)^{\ell - 2i}}{(1 + cx)^{\ell - 2i}} - \frac{x^{\ell - 2i}}{(1 + x)^{\ell - 2i}}\right].
\]

For \(\ell \geq 1\) the above expression is positive since the function \([x/(1+x)]\) is strictly increasing in \(x\). For \(\ell = 0\), (3.4) can be checked separately.

In order to prove Theorem 3.1 we first consider a number of elementary lemmas. For each integer \(\alpha \geq 0\) we define \(A(\alpha)\) as the set of \(k\)-tuples \((\alpha_1, \alpha_2, \ldots, \alpha_k)\) where \(\alpha_i (i=1, \ldots, k)\) are non-negative integers and

\[
\sum_{i=1}^{k} \alpha_i = \alpha.\]

The multinomial coefficient \(\frac{\alpha!}{\alpha_1! \alpha_2! \ldots, \alpha_k!}\) will be denoted by \(\binom{\alpha}{\alpha_1, \alpha_2, \ldots, \alpha_k}\).
\begin{equation}
(\alpha_1 \alpha_2 \ldots \alpha_k) \text{ as usual.}
\end{equation}

**Lemma 3.1:** The functions $I(\lambda)$ and $J(\lambda)$ defined in (3.2) and (3.3) can be expressed as

\begin{equation}
I(\lambda) = e^{-\lambda k} \sum_{\alpha=0}^{\infty} a_\alpha \lambda^\alpha
\end{equation}

and

\begin{equation}
J(\lambda) = e^{-\lambda k} \sum_{\alpha=0}^{\infty} b_\alpha \lambda^\alpha
\end{equation}

where

\begin{equation}
\alpha! a_\alpha = \sum_{A(\alpha)} \left( \alpha_1 \ldots \alpha_k \right) \int_0^\infty \left\{ \prod_{i=1}^{k-1} g_{\alpha_i}(cx) \right\} g_{\alpha_k}(x) dx
\end{equation}

and

\begin{equation}
\alpha! b_\alpha = \sum_{A(\alpha)} \left( \alpha_1 \ldots \alpha_k \right) \int_0^\infty \left\{ \prod_{i=1}^{k-1} \left[ 1 - g_{\alpha_i}(\xi) \right] \right\} g_{\alpha_k}(x) dx.
\end{equation}

**Proof.** Equation (3.9) follows easily by inserting the expression for $f_\lambda(x)$ from (3.1) in (3.2). Equation (3.10) follows in the same manner after observing that

\begin{equation}
1-F_\lambda(x) = \sum_{j=0}^{\infty} \frac{\lambda^j e^{-\lambda}}{j!} (1-g_j(x)).
\end{equation}
Lemma 3.2: The functions $I(\lambda)$ is nondecreasing provided

\[(3.13) \quad (\alpha+1) \quad a_{\alpha+1} - k \quad a_\alpha \geq 0, \quad \alpha = 0, 1, \ldots \]

If strict inequality holds for some $\alpha$ then $I(\lambda)$ is strictly increasing. Similar statements hold for $J(\lambda)$ if $a_\alpha$ is replaced by $b_\alpha$.

Proof: The above statements follow readily by differentiating the expressions (3.9) and (3.10).

Lemma 3.3:

(1) For each set of integers $\alpha_1, \alpha_2, \ldots, \alpha_k$ we have

\[(3.14) \quad \int_0^\infty \prod_{i=1}^{k-1} g_{\alpha_i}(cx) \frac{g_{\alpha_k+1}(x)}{g_{\alpha_k}} dx = \int_0^\infty \prod_{i=1}^{k-1} g_{\alpha_i}(cx) \frac{g_{\alpha_k}}{g_{\alpha_k+1}} dx
\]

\[= \int_0^\infty \frac{d}{dx} \left[ \prod_{i=1}^{k-1} g_{\alpha_i}(cx) \right] (g_{\alpha_k+1}(x) - g_{\alpha_k}(x)) dx\]

(ii) Equation (3.14) remains true if the $k-1$ functions $G_{\alpha_i}$, $i=1, \ldots, k-1$ are replaced by $1 - G_{\alpha_i}$, $i = 1, \ldots, k-1$.

Proof: To prove part (i) we first integrate the left side of (3.14) by parts to obtain

\[(3.15) \quad \int_0^\infty \prod_{i=1}^{k-1} \frac{g_{\alpha_i}(cx)}{g_{\alpha_k+1}(x)} dx = 1 - \int_0^\infty \frac{d}{dx} \left[ \prod_{i=1}^{k-1} g_{\alpha_i}(cx) \right] G_{\alpha_k+1}(x) dx\]
The right side of (3.15) is then written as

\begin{equation}
(3.16) \quad 1 - \int_0^\infty \frac{d}{dx} \left[ \prod_{i=1}^{k-1} g_{\alpha_i}(cx) \right] a_{\alpha_k}(x) dx - \int_0^\infty \frac{d}{dx} \left[ \prod_{i=1}^{k-1} g_{\alpha_i}(cx) \right] \left( a_{\alpha_{k+1}}(x) - a_{\alpha_k}(x) \right) dx .
\end{equation}

Now applying (3.15), with \( \alpha_{k+1} \) replaced by \( \alpha_k \), to the first two terms of (3.16), the desired result (3.14) follows. Part (ii) is obtained by a similar argument.

We now proceed with the Proof of Theorem 3.1. We first show that if (3.4) holds then \( I(\lambda) \) is nondecreasing. From (3.11) we have

\begin{equation}
(3.17) \quad (\alpha+1)! \cdot a_{\alpha+1} = \sum_{\lambda(\alpha+1)} \left( \alpha_{\lambda} \right)_{\alpha+1} \int_0^\infty \left\{ \prod_{i=1}^{k-1} g_{\alpha_i}(cx) \right\} g_{\alpha_k}(x) dx .
\end{equation}

Since 

\[ \left( \alpha_{\lambda} \right)_{\alpha+1} = \left( \alpha_{\lambda-1}, \alpha_2, \ldots, \alpha_k \right) + \left( \alpha_{\lambda}, \alpha_{\lambda-1}, \alpha_3, \ldots, \alpha_k \right) + \cdots \]

\[ \left( \alpha_{\lambda}, \alpha_{\lambda-1}, \alpha_k-1 \right) \]

we rewrite (3.17), after a simple change of variables in the sums, as
\[
\begin{align*}
(\alpha+1)! \ a_{\alpha+1} &= \sum_{A(\alpha)} \sum_{\substack{1 \leq i_1 < \cdots < i_{k-1} \leq \alpha}} \left( \frac{\alpha}{\alpha_i} \right) \left[ \sum_{j=1}^{k-1} \int_0^\infty \left( g_{\alpha_{j+1}}(cx) - g_{\alpha_j}(cx) \right) \cdot \prod_{i=1}^{k-1} g_{\alpha_i}(cx) \right] \cdot \frac{\partial}{\partial x} \left( \prod_{i=1}^{k-1} g_{\alpha_i}(x) \right) dx \\
+ \int_0^\infty \left( \prod_{i=1}^{k-1} g_{\alpha_i}(cx) \right) \cdot \frac{\partial}{\partial x} \left( \prod_{i=1}^{k-1} g_{\alpha_i}(x) \right) dx \
+ \int_0^\infty \left( \prod_{i=1}^{k-1} g_{\alpha_i}(cx) \right) \cdot \frac{\partial}{\partial x} \left( \prod_{i=1}^{k-1} g_{\alpha_i}(x) \right) dx. 
\end{align*}
\]

Then

\[
(\alpha+1)! \ a_{\alpha+1} = (k-1)! \ a! \ a_\alpha \\
+ \sum_{A(\alpha)} \sum_{\substack{1 \leq i_1 < \cdots < i_{k-1} \leq \alpha}} \left( \frac{\alpha}{\alpha_i} \right) \left[ \sum_{j=1}^{k-1} \int_0^\infty \left( g_{\alpha_{j+1}}(cx) - g_{\alpha_j}(cx) \right) \cdot \prod_{i=1}^{k-1} g_{\alpha_i}(cx) \right] \cdot \frac{\partial}{\partial x} \left( \prod_{i=1}^{k-1} g_{\alpha_i}(x) \right) dx \\
+ \int_0^\infty \left( \prod_{i=1}^{k-1} g_{\alpha_i}(cx) \right) \cdot \frac{\partial}{\partial x} \left( \prod_{i=1}^{k-1} g_{\alpha_i}(x) \right) dx \
+ \int_0^\infty \left( \prod_{i=1}^{k-1} g_{\alpha_i}(cx) \right) \cdot \frac{\partial}{\partial x} \left( \prod_{i=1}^{k-1} g_{\alpha_i}(x) \right) dx. 
\]

For the last integral in the above expression we insert its value from Lemma 3.1 to obtain

\[
(\alpha+1)! \ a_{\alpha+1} = k! \ a! \ a_\alpha \\
+ \sum_{A(\alpha)} \sum_{\substack{1 \leq i_1 < \cdots < i_{k-1} \leq \alpha}} \left( \frac{\alpha}{\alpha_i} \right) \left[ \sum_{j=1}^{k-1} \int_0^\infty \left( g_{\alpha_{j+1}}(cx) - g_{\alpha_j}(cx) \right) \cdot \prod_{i=1}^{k-1} g_{\alpha_i}(cx) \right] \cdot \frac{\partial}{\partial x} \left( \prod_{i=1}^{k-1} g_{\alpha_i}(x) \right) dx \\
+ \int_0^\infty \left( \prod_{i=1}^{k-1} g_{\alpha_i}(cx) \right) \cdot \frac{\partial}{\partial x} \left( \prod_{i=1}^{k-1} g_{\alpha_i}(x) \right) dx \
+ \int_0^\infty \left( \prod_{i=1}^{k-1} g_{\alpha_i}(cx) \right) \cdot \frac{\partial}{\partial x} \left( \prod_{i=1}^{k-1} g_{\alpha_i}(x) \right) dx. 
\]
\[
\begin{align*}
= k \alpha! a_{\alpha} + \sum_{A(\alpha)} \left( \begin{array}{c}
\alpha\\
\alpha_1, \ldots, \alpha_k
\end{array} \right) \left[ \prod_{j=1}^{k-1} \int_{0}^{x} \left[ (g_{\alpha_j}(cx) - g_{\alpha_j}(cx)) g_{\alpha_k}(x) \right]
\right. \\
- c g_{\alpha_j}(cx) (g_{\alpha_{k+1}}(x) - g_{\alpha_k}(x)) \right] dx \right].
\end{align*}
\]

We now interchange the summations and fix \( \alpha_i (i=1, \ldots, k-1, i+1) \).

Summing over \( \alpha_j \) and \( \alpha_k \) with \( \alpha_j + \alpha_k = \beta = \alpha = \sum_{i=1}^{k-1} \alpha_i \) we find that

\begin{equation}
(3.16) \quad \alpha! \left[ (\alpha+1) a_{\alpha+1} - k a_{\alpha} \right] \geq 0
\end{equation}

provided (3.4) holds. Therefore the function \( I(\lambda) \) is nondecreasing whenever (3.4) is satisfied for all integers \( \ell (\ell \geq 0) \). Moreover if strict inequality holds for some \( \ell \) in (3.4) then strict inequality holds for some \( \alpha \) in (3.18) and hence \( I(\lambda) \) is strictly increasing.

The proof of the statements concerning the function \( J(\lambda) \) are analogous and will be omitted.

4. Selection and Ranking of Multivariate Normal Populations in terms of

\[ \lambda_1 = \mu_1 \Sigma_1^{-1} \mu_1. \]

Let \( \pi_i : \mathbb{N}(\mu_i, \Sigma_i), i=1,2,\ldots,k \) be p-variate normal populations with mean vectors \( \mu_i \) and covariance matrix \( \Sigma_i \), respectively. Let

\[ \lambda_1 = \mu_1 \Sigma_1^{-1} \mu_1. \]
Case 1. $\Sigma_i$ known, $(i=1,2,\ldots,k)$.

We take a sample of $n$ independent observations from each of the $k$ populations. Let $x_{i,j}$ denote the $j$th observation of the $p$-dimensional random vector on the $i$th population; then for each $j = 1, 2, \ldots, n$, we compute

$$y_{i,j} = x_{i,j}^T \Sigma_i^{-1} x_{i,j}, \quad i=1, 2, \ldots, k; \quad j=1, 2, \ldots, n.$$  \hspace{1cm} (4.1)

Since $y_{i,j} = (1, 2, \ldots, n)$ correspond to the $n$ independent observations on a non-central $\chi^2$ for each $i$, then $Y_i = \sum_{j=1}^{n} y_{i,j}$ is distributed as a non-central $\chi^2$ with non-centrality parameter $\lambda'_i = n \mu'_i = n \mu'_i \Sigma_i^{-1} \mu_i$ and degrees of freedom $p' = np$. The proposed selection rule for the population with the largest value of $\lambda'_i$ is:

R: Select $\pi_i$ iff

$$c \sum_{j=1}^{n} y_{i,j} \geq \max_{i=1}^{n} \left\{ \sum_{j=1}^{n} y_{i,j} \right\}$$

where the constant $c = c(k, np, P^*)$ ($c > 1$), is determined to satisfy

$$\inf_{\lambda'_i} \int_0^\infty F_{\lambda'_i}^{k-1}(cy) f_{\lambda'_i}(y) \, dy = P^*$$  \hspace{1cm} (4.2)

where, now, $F_{\lambda'_i}(\cdot)$ and $f_{\lambda'_i}(\cdot)$ are the cdf and the density function of a non-central $\chi^2$ with $np$ d.f. Since the infimum of the above integral
takes place when \( \lambda' = 0 \), by Corollary 3.1.a, we have, the equation determining \( c \)

\[
(4.3) \quad \int_0^\infty H_{p'}^{-1} \left( cy \right) h_{p'}(y) dy = p^*, \quad p' = np
\]

where

\[
H_{p'}(x) = \int_0^x e^{-y} \frac{y^{p'-1}}{\Gamma(p')} dy \quad \text{and} \quad \frac{d}{dx} H_{p'}(x) = h_{p'}(x).
\]

The values of \( c' = 1/c \) satisfying (4.3) are given by Gupta (1963) for selected values of \( p' \) and \( P^* \) (see Table 1, \( p' = \sqrt{2} \)). Approximate \( c' \) values (obtained by using Wilson-Hilferty cube root transformation) are given by Gupta (1965) where the result concerning the infimum of \( P(CS|R) \) is proved for the case \( k=2 \). Armitage and Krishnaiah (1964) have extensive tables for \( c' \).

The rule for selecting the population with the minimum value of \( \lambda' \) is defined by

**R**: Select \( \pi_1 \) iff

\[
\sum_{j=1}^{n} y_{1j} \leq b \min_{i} \left\{ \sum_{j=1}^{n} y_{ij}; i = 1, 2, \ldots, k \right\}.
\]

It follows from the Corollary 3.1.a. that the constant \( b = b(k, np, P^*) \) is given by

\[
(4.4) \quad \int_0^\infty \left[ 1 - H_{p'} \left( \frac{y}{p'} \right) \right]^{k-1} h_{p'}(y) dy = P^*, \quad p' = np.
\]
The values $b^* = 1/b$ satisfying (4.4) are tabulated in Gupta and Sobel (1962) for selected values of $p^*$ and $F^*$ and more extensively by Krishnaiah and Armitage (1964).

Case 2. $\Sigma_i$ unknown ($i=1, 2, \ldots, k$).

If $\Sigma_i$'s are not known, we modify the rules $R$ and $R'$ as follows.

Let $z_i = \frac{z_i - \bar{X}_i}{s_i}$ where $\bar{X}_i$ is the sample mean vector of the $i$th population and where $S_i$ is the usual sample covariance matrix with $(n-1)$ as the divisor.

$R$: Select $\pi_i$ iff

$$c_i z_i \geq z_{\text{max}}$$

where $z_{\text{max}} = \max(z_1, z_2, \ldots, z_k)$ and where $c_i = c_i(k, p, n, F^*)$ is a constant (greater than unity) which satisfies

$$(4.5) \quad \int_0^p F_{p, n-p}^{k-1}(c_i x) f_{p, n-p}(x) \, dx = F^*$$

where $f_{p, n-p}$ is given by (3.7) with $j=0$, $\mu = p/2$ and $\nu = (n-p)/2$.

i.e. it is the density of a random variable which is $\frac{F^*}{n-p}$ times the central $F$ random variable. $F_{p, n-p}(\cdot)$ is the corresponding cdf. The modified procedure $R'$ is

$R'$: Select $\pi_i$ iff

$$z_i \leq b^* z_{\text{min}}$$
where \( b_1 = b_1(k,p,n,P) \) is a constant (greater than unity) determined by

\[
\int_0^\infty \left[ 1 - F_{p,n-p}(x/b_1) \right]^{k-1} f_{p,n-p}(x) dx = P^*.
\]

It should be pointed out that (4.5) and (4.6) are consequences of Corollary 3.1.b and the fact that each \( N T_i^2 (i=1,\ldots,k) \) (\( K = \text{Constant} \)) has the density (non-central F) given by (3.1) in conjunction with (3.7).

Case 3. \( \Sigma_1 = \Sigma_2 = \ldots = \Sigma_k = \Sigma \) (unknown).

In this case the usual pooled estimator \( S = (S_1 + S_2 + \ldots + S_k)/k \) is used in the procedures \( R \) and \( R' \) of Case 2. In this case the constants \( c_2 \) and \( b_2 \) are again determined by equations of the type (4.5) and (4.6), respectively, with degrees of freedom \( p, k(n-1) = p+1 \), respectively.

Remark 1: It should be pointed out that the procedures \( R \) and \( R' \) discussed under case 1 are not "strictly analogous" to those given for cases 2 and 3. If we use procedures based on \( \overline{X}_1, \Sigma_1, \overline{X}_1 \) in case 1, the corresponding constants \( c \) and \( b \) turn out to be independent of the number of observations which is undesirable.

Remark 2: The efficiency of these procedures in terms of expected size or related criteria has not been investigated here. Also the "indifference zone" approach, a different type of formulation, due to Bechhofer (1954) has not been discussed here.
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On Some Selection and Ranking Procedures with Applications to Multivariate Populations

Technical Report, October 1965

Gupta, Shanti S. and Studden, William J.

October 1965

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1. ABSTRACT

A problem of subset selection for parameters which are not necessarily scale or location parameters is considered. A general theorem dealing with the infimum of the probability of a correct selection for parameters occurring in densities which are Poisson mixtures of arbitrary densities on [0,∞) is proved. This theorem is applied to obtain the minimum value of the probability of a correct selection in several cases where multivariate normal populations are ranked according to $\lambda_i = \mu_i \Sigma_1^{-1} \Sigma_2^{-1} (1-i,2,\ldots,k)$ and $\mu_i$ is the unknown mean vector and $\Sigma_1$ (known or unknown) the covariance matrix of the $i$th $p$-variate normal population.
Multiple Decisions
Selection and Ranking
Multivariate Normal
Correct Selection
Distance Function

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