ON THE MOMENTS OF SOME ONE-SIDED STOPPING RULES

by

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Mimeograph Series Number 42
June, 1965

*This work has been supported by the National Science Foundation under Grant GP-04590.
1. Introduction. The moments of stopping rules (or stopping times) have been discussed in [1, 3, 4], and the following results have been proved. Let \( x_n \) be independent random variables with \( \text{Ex}_1 = \alpha, \text{Ex}_n^2 = 1 \), and \( S_n = x_1 + \cdots + x_n \). For \( c > 0 \) and \( m = 1, 2, \ldots \), define \( t_m \) to be the first \( n \geq m \) such that \( |S_n| > c n^{1/2} \). If \( c \geq 1 \), then \( \text{Et}_1 = \infty \).

If \( P[|x_n| \leq K] = 1 \) for some \( K < \infty \) and \( n = 1, 2, \ldots \), then \( \text{Et}_m < \infty \) for every \( m \) if \( c < 1 \), \( \text{Et}_m^2 < \infty \) for every \( m \) if \( c < 3 - \sqrt{6} \), and \( \text{Et}_m^2 = \infty \) for some large \( m \) if \( c > 3 - \sqrt{6} \).

In this note, we are interested in the following one-sided stopping rules, instead of the above stated two-sided stopping rules. For \( c > 0 \) and \( 1 > p > 0 \), define

\[
s = \text{first } n \geq 1 \text{ such that } S_n \geq c n^p .
\]

One of the results states that, if \( x_n \) are independent, \( \text{Ex}_n = \mu > 0 \), and \( \text{Ex}_n^2 - \mu^2 = \sigma^2 < \infty \), then \( \text{Es}^2 < \infty \) and

\[
\lim_{c \to \infty} \frac{\mu^2 \text{Es}^2}{(c^2 \text{Es}^{2p})} = \lim_{c \to \infty} \frac{\mu \text{Es}^2}{(c \text{Es}^{1+p})} = 1 .
\]
When \( p = o \), \( E s^2 < \infty \) implies that \( P[S_1 < c, \ldots, S_n < c] = P[s > n] = o(n^{-2}) \) as \( n \to \infty \), which completes a result of Morimura [8]. Also (1) extends the elementary renewal theorem from first moments to second moments and generalizes some results due to Chow and Robbins [2] and Hatori [6].

2. The first moment.

Let \( (\Omega, \mathcal{F}, P) \) be a probability space and \( x_n \) be a sequence of integrable random variables. Let \( \mathcal{F}_n \) be the Borel field generated by \( x_1, \ldots, x_n \) and \( \mathcal{F}_0 = \{ \emptyset, \Omega \} \). Put \( S_n = x_1 + \ldots + x_n \), \( S_0 = 0 \), \( m_n = E(x_n | \mathcal{F}_{n-1}) \) and \( T_n = \sum_{j=1}^{n} m_j \). Assume that for some constant \( \alpha > \mu > 0 \) and for some null set \( N \),

\[
(2) \lim_{n \to \infty} \frac{T_n}{n} = \mu, \text{ uniformly on } \Omega-N.
\]

For \( c > 0 \) and \( 1 > p \geq c \), define

\[
s = \text{first } n \geq 1 \text{ such that } S_n \geq c n^p.
\]

Theorem 1. (i) If for some \( 0 < \delta < \mu/3 \), \( P[x_n \leq m_n + n\delta] = 1 \) for all large \( n \), then \( E s < \infty \). (ii) If \( E[(x_n - m_n)^+ | \mathcal{F}_{n-1}] \leq K < \infty \) for some \( \alpha > 1 \), then \( E s < \infty \) and

\[
(3) \lim_{c \to \infty} \frac{\mu E s}{(c E s^p)} = l = \lim_{c \to \infty} \frac{E s}{(c E s^p)}.
\]
Proof. (i) Set \( t = \min (s, k) \) for \( k = 1, 2, \ldots, T \). Then by the Wald identity for martingales [see 5, p. 302; or 3],

\[
\mathbb{E}T_t = \mathbb{E}S_t = \mathbb{E}(S_{t-1}^+ x_t) \leq c \mathbb{E}T^p + \mathbb{E}(m_t + 6t).
\]

Let \( \delta < \epsilon < \delta \). As \( k \to \infty \), by (2)

\[
\mathbb{E}T_t \geq (\mu - \epsilon) \mathbb{E}t + O(1), \quad \mathbb{E}m_t = O(1) + o(\mathbb{E}t).
\]

Hence

\[
(\mu - \epsilon) \mathbb{E}t \leq c \mathbb{E}T^p + \delta \mathbb{E}T + O(1) + o(\mathbb{E}t),
\]

\[
\int_{[s \leq k]} s \, dP + k \mathbb{P}[s > k] = \mathbb{E}t = O(1),
\]

as \( k \to \infty \). Therefore \( \mathbb{P}[s < \infty] = 1 \) and \( \mathbb{E}s < \infty \).  

(ii) For any \( \delta < \delta < \mu/3 \), define \( x_n^\prime = \min (x_n, m_n^\prime + n\delta) \),

\[
m_n^\prime = \mathbb{E}(x_n^\prime | \mathcal{F}_{n-1}), \quad \text{and} \quad T_n^\prime = m_1^\prime + \ldots + m_n^\prime.
\]

Let \( \mathbb{I}(A) \) be the indicator function of the set \( A \). Then

\[
o \leq m_n^\prime - m_n = \mathbb{E}((x_n - m_n - n\delta) \mathbb{I}[x_n > m_n + n\delta] | \mathcal{F}_{n-1})
\]

\[
\leq \mathbb{E}((x_n - m_n) \mathbb{1}[x_n > m_n + n\delta] | \mathcal{F}_{n-1})
\]

\[
\leq \mathbb{E}^{1/\alpha'} ([(x_n - m_n)^+] \mathbb{1}[x_n > m_n + n\delta] | \mathcal{F}_{n-1}) p^{1/\alpha'} (x_n - m_n > n\delta | \mathcal{F}_{n-1}) (\alpha + \alpha' = \alpha')
\]

\[
\leq K(n\delta)^{-\alpha/\alpha'}
\]
Therefore $\lim_{n \to \infty} T'_n/n = \mu$ uniformly on $\Omega-N$. Define $t = \text{first } n \geq 1 \text{ such that } x'_1 + \ldots + x'_n \geq c \cdot n^p$.

Then $s \leq t$. By (1), $E_t < \infty$. Therefore $E_s < \infty$ and it follows by the Wald identity again [5, p. 302; or 3] that

$$E(c \cdot s^p + x'_s) \geq ES_s = ET_s \geq c \cdot ES^p.$$

Let $Z_n = n \sum_{j=1}^{n} [(x'_j - m_j)^+]^\alpha$. Then by Lemma 6 of [3],

$$E^{\alpha}((x'_s - m_s)^+) \leq EZ_s = E \sum_{j=1}^{n} E([[(x'_j - m_j)^+]^\alpha]_{j=1}^{\infty}) \leq K \cdot ES.$$

Since (2) implies that as $c \to \infty$, $E_m = O(1) + o(ES)$ and $ET_s = O(1) + (\mu + O(1)) ES$, we have

$$Ex_s = O(E^{1/\alpha} s) + o(ES) + O(1)$$

from (6), and

$$\lim_{c \to \infty} \mu \cdot E_s/(c \cdot ES^p) = \lim_{c \to \infty} ET_s/(c \cdot ES^p) = \lim_{c \to \infty} ES_s/(c \cdot ES^p) = 1$$

from (5) and (7), since $\lim_{c \to \infty} ES = \infty$. The proof is completed.

When $p = \infty$, part (ii) of Theorem 1 reduces to an elementary renewal theorem, which was proved in [2], in a slightly restricted form by requiring that $m_n = E(x'_n)$ for each $n$. 
3. The second moment.

Assume that $\mathbb{E}X_n^2 < \infty$ for each $n$, let $V_n = \sum_{j=1}^{n} \mathbb{E}((X_i-m_i)^2|\mathcal{F}_{j-1})$ for $n = 1, 2, \ldots$, and define $\mathcal{S}$ as before. For a random variable $y$, put $||y|| = (\mathbb{E}y^2)^{1/2}$.

**Theorem 2.** If (2) holds and $\mathbb{E}((X_n-m_n)^2|\mathcal{F}_{n-1}) \leq K < \infty$, then $\mathbb{E}\mathcal{S}^2 < \infty$, $\mathbb{E}\mathcal{S}_g^2 < \infty$, and as $c \to \infty$,

\begin{equation}
\mathbb{E}\mathcal{S}_g^2 + \mathbb{E}T_g^2 = \mathbb{E}V + 2\mathbb{E}\mathcal{S}_g T_g,
\end{equation}

\begin{equation}
\lim \mathbb{E}\mathcal{S}_g^2/\mathbb{E}T_g^2 = 1,
\end{equation}

\begin{equation}
\lim \mu^2 \mathbb{E}\mathcal{S}^2/(c^2 \mathbb{E}\mathcal{S}^2) = 1,
\end{equation}

\begin{equation}
\lim \mathbb{E}\mathcal{S}_g^2/(c^2 \mathbb{E}\mathcal{S}^2) = 1,
\end{equation}

\begin{equation}
\lim \mu \mathbb{E}\mathcal{S}^2/(c \mathbb{E}\mathcal{S}^{1+p}) = 1.
\end{equation}

**Proof.** (1) First, assume that for some $0 < \delta < \mu/3$ and $0 < M < \infty$, $P(x_n \leq m_n + n\delta + M) = 1$. Set $t = \min(t, k)$ for $k = 1, 2, \ldots$. Then by Theorem 1 and Lemma 6 of [3],

\[ \mathbb{E}(S_t - T_t)^2 = \mathbb{E}V_t \leq K \mathbb{E}t. \]

Hence by Schwarz inequality

\begin{equation}
\mathbb{E}S_t^2 + \mathbb{E}T_t^2 \leq K \mathbb{E}t + 2||T_t|| \cdot ||S_t||.
\end{equation}
Assume, on the contrary, that $E_s^2 = \infty$. Then $\lim_{k \to \infty} E_t^2 = \infty$ and (2) implies that

$$E_{m_t}^2 = o(1) + o(E_t^2) = o(E_t^2),$$

as $k \to \infty$. Hence

$$||S_t|| \leq ||ot^P + m_t + 5t + M|| \leq c||t^P|| + 5||t|| + o(||t||) = (5 + o(1)||t||);$$

and from (2),

$$E_{T_t}^2 = o(1) + (\mu^2 + o(1)) E_t^2 = (\mu^2 + o(1)) E_t^2.$$

By (13), (14) and (15), we have

$$1 + ES_t^2/E_{T_t}^2 \leq 0(||t||^{-1}) + 2||S_t||/||T_t|| \leq 0(||t||^{-1}) + (25 + o(1))/\mu = 25/\mu + o(1).$$

Since $5 < \mu/8$, we have a contradiction when $k$ is large.

Therefore $E_s^2 < \infty$. From (14), (13) and Fatou's lemma $E_{S_s}^2 < \infty$ and $E_{T_s}^2 < \infty$.

(ii) For the general case, let $x_n' = \min (x_n, m_n + n5 + M)$ for an arbitrary constant $\epsilon > 0$ and $0 < 5 < \mu/8$. Define $m_n', T_n'$ and $t$ as in the proof of part (11) of Theorem 1. Then by (4) (for $\alpha = 2$), $0 \leq m_n - m_n' \leq K(n5)^{-1}$ Hence
\[
\lim T_n^i/n = \mu \quad \text{uniformly on } \Omega - N.
\]

It is not too difficult to see that

\[
E((x_n - m_n)^2|\gamma_{n-1}) - E((x'_n - m'_n)^2|\gamma_{n-1}) \geq E((x_n - m_n - n\delta - M)^+)^2|\gamma_{n-1}) - \nabla^2((x_n - m_n - n\delta - M)^+|\gamma_{n-1}) \geq 0.
\]

Therefore \( E((x_n - m_n)^2|\gamma_{n-1}) \leq K \). Since \( t \geq s \) and from part (i)
\( Et^2 < \infty \), we have that \( Es^2 < \infty \). By Theorem 1 and Lemma 6 of
[7] again,

\[
(16) \quad E(S_s - T_s)^2 = Ev_s \leq K Es.
\]

For \( \varepsilon > 0 \), (2) implies that there exists a constant \( \varepsilon > L > 0 \)
such that

\[
Et^2_s \leq L + (\mu^2 + \varepsilon) Es^2_s.
\]

Hence \( Et^2_s < \infty \) and from (16), \( Es_s^2 < \infty \). Thus (8) follows.

Now by (16),

\[
|Es_s^2 - Et_s^2| \leq E|S_s^2 - T_s^2| \leq \frac{|S_s - T_s|}{|S_s + T_s|} \leq (K Es)^{1/2} \frac{|S_s + T_s|}{|S_s|}.
\]

Since \( Es_s^2 \geq c_2 Es^2p \), from (3)

\[
|1 - Et_s^2/Es_s^2| \leq (K Es/Es_s^2)^{1/2}(1 + ||T_s||/||S_s||) = o(1) + o(||T_s||/||S_s||)
\]
as \( c \to 0 \). Hence (9) follows.
Since (2) implies that $ET \mathcal{S}^2 = O(1) + (\mu^2 + o(1)) Es^2$
as $c \to \infty$, from (9)

(17) \quad \lim_{c \to \infty} \mu^2 \frac{Es^2}{ET \mathcal{S}^2} = 1 = \lim_{c \to \infty} \mu^2 \frac{Es^2}{ES \mathcal{S}^2}.

Let $Z_n = \sum \limits_{1}^{n} (x_j - m_j)^2$. Applying Lemma 6 of [3], we have

$$E(x_s - m_s)^2 \leq EZ_s = E \sum \limits_{i}^{s} E((x_j - m_j)^2 | \mathcal{F}_{j-1}) \leq KEs.$$ 

From (2), $E_{s}^2 = O(1) + o(ES^2) = o(ES^2)$ as $c \to \infty$. Hence

(18) \quad Es^2 = E(x_s - m_s + m_s)^2 = o(ES^2), \quad ||x_s|| = o(||s||).$

Now from (18), as $c \to \infty$

(19) \quad c||s^p|| \leq ||S_s|| \leq c||s^p + x_s|| \leq c||s^p|| + ||x_s|| = c||s^p|| + o(||s||).

Therefore (10) follows from (17) and (19), and (11) follows from (17) and (10).

Now $ET \mathcal{S} = 0(ES_s) + (\mu + o(1)) Es S_s$ as $c \to \infty$. By the definition of $s$ and (18), as $c \to \infty$. 
(20) \[ C \cdot E_s^{1+p} \leq E_s S_s \leq C \cdot E_s^{1+p} + E_s x_s \leq c \cdot E_s^{1+p} + ||s|| \cdot ||x_s|| \leq C \cdot E_s^{1+p} + o(E_s^2) \].

Since \( E V_s \leq K E_s \), from (8), (9), (10), and (11), \( \lim E_s T_s/(\mu^2 E_s^2) = 1 \). Hence

\[ \lim E_s S_s/\mu E_s^2 = 1 \]

and then (20) implies (12).

4. Corollaries and comments.

In this section we assume that \( x_n \) is a sequence of random variables and \( p=0 \). Define \( S_n, m_n, T_n, \pi_n \) and \( s \) as in Section 2.

Corollary 1. If (2) holds and if

\[ E((x_n - m_n)^2 | \pi_{n-1}) \leq K < \infty, \]

then \( E_s^2 < \infty \) and

\[ \lim_{c \to \infty} E_s^{\alpha}/c^\alpha = \mu^{-\alpha} \text{ for } 0 \leq \alpha \leq 2. \]

Proof. Since (21) implies \( E(x_n - m_n)^2 \leq K \), from (2) and (21) it follows [7] that \( \lim S_n/n = \mu \text{ a.e.} \). Hence
\[ 1 \leq \lim \inf_{c \to \infty} \frac{S_s}{c} \leq \lim \sup_{c \to \infty} \frac{\mu_s}{c} = \lim \sup_{c \to \infty} \frac{\mu(s-1)}{c} = \lim \sup_{c \to \infty} \frac{S_{s-1}}{c} \leq 1. \]

Therefore, \( \lim \frac{s}{c} = \mu^{-1} \) a.e. Theorem 2 implies that
\[ \mathbb{E}(s/c)^2 \leq M < \infty \text{ for all } c > 0. \]
Hence [see 5, p. 629] for every \( 0 \leq \alpha < 2 \), \((s/c)^\alpha\) is uniformly integrable and

\[ \lim_{c \to \infty} \mathbb{E}|\mu^{-1} - s/c|^\alpha = 0, \quad \lim \mathbb{E}s^\alpha/c^\alpha = \mu^{-\alpha}. \]

Thus (22) follows from (23) and (10).

**Corollary 2.** Let \( x_n \) be a sequence of independent, identically distributed random variables such that \( \mathbb{E}x_1 > 0 \) and \( \mathbb{E}(x_1 - \mathbb{E}x_1)^2 < \infty \). Then for every \( c > 0 \), as \( n \to \infty \),

\[ P[S_1 < c, \ldots, S_n < c] = o(n^{-2}). \]

**Proof.** Since \([s > n] = [S_1 < c, \ldots, S_n < c], \mathbb{E}s^2 < \infty\) implies (24) and thus Corollary 2 follows from Corollary 1.

(22) has been proved by Hatori [6] for every \( \alpha > 0 \), by requiring, in addition to the assumptions of Corollary 1, that \( x_n \) be independent, \( P[x_n \geq 0] = 1 \) and \( m_n \geq L > 0 \) for each \( n \).
Under the conditions of Corollary 2, Morimura [8] proves that $P[S_1 < c, \ldots, S_n < c] = O(n^{-5})$ for $0 \leq \delta < (1 + \sqrt{5})/2$ and that there exists an example such that for some $D > 0$ and for each $\epsilon > 0$, $P[S_1 < c, \ldots, S_n < c] \geq D n^{-2-\epsilon}$ when $n$ is large enough. Thus (24) is the best possible. Clearly, Corollary 2 completes Morimura's work.

The counter example in [8] satisfies the condition $E s^{2+\epsilon} = \infty$ for every $\epsilon > 0$, since $P[s > n] \neq o(n^{-2-\epsilon})$. Therefore (22) can not be extended to the cases where $\alpha > 2$, without some conditions as $P[x_n \geq 0] = 1$ imposed in [6].
References


