Moments of Randomly Stopped Sums

by

Y. S. Chow
Purdue University

and

Herbert Robbins
Columbia University

and

Henry Teicher
Purdue University

Department of Statistics
Division of Mathematical Sciences
Mimeograph Series No. 32
November, 1964
Moments of Randomly Stopped Sums

1. Introduction. Let \((\Omega, \mathcal{F}, P)\) be a probability space, let \(x_1, x_2, \ldots\) be a sequence of random variables on \(\Omega\), and let \(\mathcal{F}_n\) be the \(\sigma\)-algebra generated by \(x_1, \ldots, x_n\), with \(\mathcal{F}_0 = (\emptyset, \Omega)\). A stopping variable (of the sequence \(x_1, x_2, \ldots\)) is a random variable \(t\) on \(\Omega\) with positive integer values such that the event \([t=n] \in \mathcal{F}_n\) for every \(n \geq 1\). Let \(S_n = \sum_{i=1}^{n} x_i\); then

\[
S_t = S_t(\omega) = \sum_{i=1}^{t} x_i
\]

is a randomly stopped sum. We shall always assume that

\[
E|x_n| < \infty, \quad E(x_{n+1} | \mathcal{F}_n) = 0, \quad (n \geq 1).
\]

The moments of \(S_t\) have been investigated since the advent of Sequential Analysis, beginning with Wald [9], whose theorem states that for independent, identically distributed (i.i.d.) \(x_i\) with \(E x_i = 0\), \(E t < \infty\) implies that \(ES_t = 0\). For higher moments of \(S_t\), the known results [1,3,4,5,10] are not entirely satisfactory. We shall obtain theorems for \(ES_t^r\) \((r = 2, 3, 4)\); the case \(r = 2\) is of special interest in applications. In the case of i.i.d. \(x_i\) with \(E x_i = 0\) and \(E x_i^2 = \sigma^2 < \infty\), we shall show that \(E t < \infty\) implies \(ES_t^2 = \sigma^2 E t\).

2. The second moment. It follows from assumption (1) that

\[
(S_n, \mathcal{F}_n; n \geq 1)\text{ is a martingale; i.e., that }
\]

\[
E|S_n| < \infty, \quad E(S_{n+1} | \mathcal{F}_n) = S_n \quad (n \geq 1).
\]
The following well-known fact [3, p. 302] will be stated as

**Lemma 1.** Let \( (S_n, \mathcal{F}_n; n \geq 1) \) be a martingale and let \( t \) be any stopping variable such that

\[
E|S_t| < \infty, \quad \lim \inf_{t \geq n} |S_n| = 0; \tag{3}
\]

then

\[
E(S_t | \mathcal{F}_n) = S_n \quad \text{if} \quad t \geq n \quad (n \geq 1) \quad \tag{4}
\]

and hence \( E S_t = E S_1 \).

**Lemma 2.** If \( E \sum_1^t |x_i| < \infty \), then (3) holds.

**Proof.** \( |S_t| \leq \sum_1^t |x_i| \), so that \( E|S_t| < \infty \), and

\[
\lim \inf_{t \geq n} |S_n| \leq \lim \inf_{t \geq n} \sum_1^t |x_i| = 0. \tag{5}
\]

In this section we shall suppose, in addition to (1) that

\[
E x_n^2 < \infty \quad (n \geq 1) \quad \tag{6}
\]

and we define for \( n \geq 1 \)

\[
Z_n = S_n^2 - \sum_1^n x_i^2. \tag{7}
\]

The sequence \( (Z_n, \mathcal{F}_n; n \geq 1) \) is also a martingale, with \( EZ_1 = 0 \).

For any stopping variable \( t \), let \( t(n) = \min (n, t) \); then Lemma 1 applies to \( Z_n \) and \( t(n) \), so that \( EZ_{t(n)} = 0 \), and hence

\[
E S_{t(n)}^2 = E \sum_1^{t(n)} x_i^2. \tag{8}
\]

Letting \( n \to \infty \) we have a.e. \( S_{t(n)}^2 \to S_t^2 \) and \( \sum_1^t x_i^2 \uparrow \sum_1^t x_i^2 \).
Hence, by Fatou's lemma and (7),

$$\lim E_t^2 < \lim E_t^2(n) = \lim E_t \sum_1^t x_1^2 = E_t \sum_1^t x_1^2. \quad (8)$$

The question now arises under what circumstances equality holds in (8).

(By Lemma 1 this will be the case if (3) holds with $S$ replaced by $Z$, but, as we shall see, this requirement is unnecessarily stringent.)

According to (8), we need only consider the case in which $E_t^2 < \infty$, and it will suffice to prove that

$$E_t^2 \geq E_t^2(n) \quad (n \geq 1). \quad (9)$$

**Lemma 3.** If

$$\lim \inf_{[t > n]} |S_n| = 0, \quad (10)$$

then $E_t^2 = E_t \sum_1^t x_1^2$.

**Proof.** We may suppose that $E_t^2 < \infty$ whence, by (10) and Lemma 1, (4) holds. Hence

$$E_t^2 = \int_{[t \leq n]} S_t^2 + \int_{[t > n]} (S_n + (S_t - S_n))^2$$

$$\geq \int_{[t \leq n]} S_t^2 + \int_{[t > n]} S_n^2 + 2\int_{[t > n]} S_n E(S_t - S_n|\mathcal{F}_n) = E_t^2(n).$$

**Lemma 4.** If

$$\lim \inf_{[t > n]} |S_n| < \infty, \quad (11)$$

then (10) holds.

**Proof.** Suppose (10) does not hold; then

$$\lim \inf_{[t > n]} |S_n| = \varepsilon > 0.$$
Hence for any constant $0 < a < \infty$,

$$\liminf_{[t > n]} S_n^2 > a \liminf_{[t > n], |S_n| > a} |S_n| = a \in$$

which contradicts (11), since $a$ may be arbitrarily large.

**Lemma 5.** If $E \sum_1 x_i^2 < \infty$, then (11) holds.

**Proof.** Setting $S_0 = 0$ we have

$$\sum_{i=1}^n S_i^2 = \sum_{i=1}^n (S_i^2 - S_{i-1}^2) \leq \sum_{i=1}^\infty x_i^2 = E \sum_1 x_i^2 < \infty.$$  

From Lemmas 1-5 we have

**Theorem 1.** Let $(S_n, \mathcal{F}_n; n \geq 1)$ be a martingale with $ES_n^2 < \infty$ and let $t$ be any stopping variable. Set $x_1 = S_1$, $x_{n+1} = S_{n+1} - S_n$. Then

$$E S_t^2 \leq E \sum_1 x_i^2.$$  

If any one of the four conditions

$$\liminf_{[t > n]} |S_n| = 0, \liminf_{[t > n]} S_n^2 < \infty, E \sum_1 x_i^1 < \infty, E \sum_1 x_i^2 < \infty$$  

holds, then

$$E S_t^2 = E \sum_1 x_i^2.$$  

If $E \sum_1 x_i^2 < \infty$, then (3) and (4) hold.

Theorem 1 generalizes (a) and (b) of Theorem II of [1]. In order to apply it, we first verify

**Lemma 6.** For any stopping variable $t$ and any $r > 0$,

$$E \sum_1 x_i^r = E \sum_1 E(|x_i^r|_1 \mathcal{F}_{i-1}).$$
**Proof.** \[ E \sum_{i=1}^{\infty} |x_i|^r = \sum_{j=1}^{\infty} \sum_{i=1}^{j} |x_i|^r = \sum_{i=1}^{\infty} \sum_{t \geq i} |x_i|^r = \sum_{i=1}^{\infty} \sum_{t \geq i} E(|x_i|^r | \mathcal{F}_{i-1}) = E \sum_{l=1}^{t} E(|x_l|^r | \mathcal{F}_{l-1}) . \]

For independent \( x_n \), we have from Theorem 1 and Lemma 6

**Theorem 2.** Let \( x_1, x_2, \ldots \) be independent with \( E x_n = 0, E|x_n| = a_n \),

\[ Ex_n = \sigma_n^2 < \infty \quad (n \geq 1) \] and let \( S_n = \sum_{l=1}^{n} x_l \). Then either of the two relations

\begin{align}
(15) \quad & E \sum_{l=1}^{t} a_l < \infty, \quad E \sum_{l=1}^{t} \sigma_n^2 < \infty \\
(16) \quad & E S_n^2 = E \sum_{l=1}^{t} x_l^2 = E \sum_{l=1}^{t} \sigma_n^2 .
\end{align}

implies

\begin{align}
(15) \quad & E \sum_{l=1}^{t} a_l < \infty, \quad E \sum_{l=1}^{t} \sigma_n^2 < \infty \\
(16) \quad & E S_n^2 = E \sum_{l=1}^{t} x_l^2 = E \sum_{l=1}^{t} \sigma_n^2 .
\end{align}

If \( \sigma_n^2 = \sigma^2 < \infty \), then \( E t < \infty \) implies

\begin{align}
(17) \quad & E S_t^2 = E \sum_{l=1}^{t} x_l^2 = \sigma^2 E t .
\end{align}

Some stronger sufficient conditions for (16) have been given in [10,1,5,3 (p. 351), 4].

**Corollary 1.** Let \( x_1, x_2, \ldots \) be independent with \( E x_n = 0, Ex_n^2 = 1 \), and define \( t^* (\text{resp. } t_* ) = \text{st } n \geq 1 \) such that \( |S_n| > n^{1/2} (\text{resp. } <) = \infty \) otherwise.

Then \( E t^* = E t_* = \infty \).

**Proof.** If \( E t^* < \infty \), then \( t^* \) is a genuine stopping variable (i.e., \( P(t^* < \infty ) = 1 \) and by the definition of \( t^* \) and (17),

\[ E t^* = E S_{t^*}^2 > E t^* , \]

a contradiction; similarly for \( t_* \).

We note that \( t^* \) is a genuine stopping variable if the law of the iterated logarithm holds for \( x_1, x_2, \ldots \).

The example \( P[x_\infty = 1] = P[x_\infty = -1] = 1/2 \) shows that the \( > (\text{resp. } <) \) cannot
be replaced by $\geq (\leq)$, since $E_n = 0$, $E_n^2 = 1$, and $t^* = t_* = 1$. On the other hand, if $t^*$ is redefined as the first $n > 1$ for which $|S_n| \geq n^{1/2}$, $E_t^*$ is again infinite; similarly for $t^*$.

Corollary 1 is a generalization of Theorem 1 of [2]. The following corollary generalizes Theorem 2 of [2].

**Corollary 2.** Let $x_1, x_2, \ldots$ be independent with $E_n = 0$, $E_n^2 = 1$, $P[|x_n| \leq a < \infty] = 1$. For $0 < c < 1$ and $m = 1, 2, \ldots$, define

$$t = \text{first } n \geq m \text{ such that } |S_n| > cn^{1/2}.$$ 

Then $E_t < \infty$.

**Proof.** For $k = m, m + 1, \ldots$, put $t' = \min(t, k)$. Then $t'$ is a stopping variable and by Theorem 2,

$$E_t ' = E_{t'} = E_{t'}^{2} \leq \int_{t > k} S_{k}^{2} + \int_{t \leq k} (ct^{1/2} + a)^{2}$$ 

$$\leq c^{2}kP[t > k] + c^{2}\int_{t \leq k} t + 2ac\int_{t \leq k} t^{1/2} + a^{2}.$$ 

Hence

$$(1 - c^{2})(kP[t > k] + \int_{t \leq k} t) \leq 2ac\int_{t \leq k} t^{1/2} + a^{2}.$$ 

Therefore as $k \to \infty$, $\int_{t \leq k} t = O(1)$ and $P[t > k] = O(k^{-1}) = o(1)$, so that $t$ is a genuine stopping variable and $E_t < \infty$.

**Corollary 3.** If $x_1, x_2, \ldots$, are i.i.d. with $E_n = 0$, $E_n^2 = \sigma^2$, $P[|x_n| \leq a < \infty] = 1$, and if $E_{t}^{2} < \infty$ for a stopping variable $t$, then $E_t < \infty$ if and only if

$$(18) \quad \lim \inf nP[t > n] = 0.$$ 

**Proof.** The 'only if' part is obvious. Now suppose (18) holds. Then since

$$\int_{[t > n]} |S_n| \leq anP[t > n],$$
the first condition of (13) holds and hence $\sigma^2 E_t = ES^2_t < \infty$, so that $E_t < \infty$ if $\sigma^2 > 0$. (If $\sigma^2 = 0$, then $P[x_n = 0] = 1$ and hence $t$ is equal a.e. to a fixed positive integer, so $E_t < \infty$ in this case too.)

Applied to the case $P[x_1 = 1] = P[x_1 = -1] = 1/2$, with $t = \text{first } n \geq 1$ such that $S_t = 1$, we have by Wald's theorem $E_t = \infty$, but by Corollary 3 the stronger result $\liminf nP[t > n] > 0$.

**Corollary 4.** Let $(x_n, n \geq 1)$ satisfy $E(x_{n+1} | \mathcal{F}_n) = 0$ and let $E(x_{n+1} | \mathcal{F}_n) = \sigma_{n+1}^2 < \infty$ be constant for $n \geq 1$. Then for $\varepsilon > 0$,

$$P[\max_{n \leq m} |S_n| \geq \varepsilon] \leq \varepsilon^{-2} \sum_{l=1}^{m} \sigma_l^2.$$ 

If moreover $\sup_{n \geq 1} |x_n| = z$ with $Ez < \infty$, then

$$P[\max_{n \leq m} |S_n| \geq \varepsilon] \leq 1 - \frac{E(z+\varepsilon)^2}{\sum_{l=1}^{m} \sigma_l^2}.$$  

**Proof.** Define $t = \text{first } n \geq 1$ such that $|S_n| \geq \varepsilon$. Then $t' = \min(t, m)$ is a bounded stopping variable. Hence, by (14) and Lemma 6,

$$\varepsilon^2 P[\max_{n \leq m} |S_n| \geq \varepsilon] = \varepsilon^2 P[t \leq m] \leq ES^2_{t'} = \sum_{l=1}^{m} \sigma_l^2 \leq \sum_{l=1}^{m} \sigma_l^2.$$ 

If $Ez < \infty$, then

$$E(z+\varepsilon)^2 \geq ES^2_{t'}, = \sum_{l=1}^{m} \sigma_l^2 \geq \sum_{k=1}^{m} \sum_{j=1}^{k} \sigma_j^2 = \sum_{j=1}^{m} \sigma_j^2 P[t \geq j]$$ 

$$\geq (\sum_{j=1}^{m} \sigma_j^2) P[t \geq m]$$

and (19) holds.

The first part of Corollary 4 is a special case of submartingale inequalities [6, p. 391], and the second part generalizes slightly one of the Kolmogorov inequalities [6, p. 235] which requires that $z$ be constant.
3. The Fourth Moment.

The analysis in the case of the fourth moment of \( S_t \) is somewhat easier than that of the third moment and consequently is presented first. In this section \( \text{Ex}_n^4 \) will be supposed finite. Define for \( r = 1,2,3,4, \) and \( n = 1,2, \ldots \)

\[
\begin{align*}
  u_{r,n} &= E(x_n^r \mid \mathcal{F}_{n-1}) , & U_{r,n} &= \Sigma_{1}^{n} u_{r,j} , \\
  v_{r,n} &= E(|x_n|^r \mid \mathcal{F}_{n-1}) , & V_{r,n} &= \Sigma_{1}^{n} v_{r,j} , \\
  T_{r,n} &= \Sigma_{1}^{n} |x_j|^r , & T_{1,n} &= T_n .
\end{align*}
\]

(20)

In these terms, Lemma 6 asserts that \( E T_{r,t} = E V_{r,t} \).

Lemma 7. If \( ES_t^2 < \infty \) and \( \lim \inf \{ \frac{|S_n|}{n} : t > n \} = 0 \), then

\[
E(S_t^2 \mid \mathcal{F}_n) > S_n^2 \quad \text{and} \quad E(|S_t| \mid \mathcal{F}_n) > |S_n| \quad \text{for} \quad t > n .
\]

Proof. For any \( A \in \mathcal{F}_n \), by Lemma 1

\[
\int_{A[t > n]} S_t^2 = \int_{A[t > n]} [S_n^2 + 2S_n(S_t - S_n) + (S_t - S_n)^2] \geq \int_{A[t > n]} S_n^2 .
\]

Hence the first inequality of the lemma holds, and the second inequality follows immediately from Lemma 1 and the fact that

\[
E(|S_t| \mid \mathcal{F}_n) \geq E(S_t \mid \mathcal{F}_n) .
\]

Theorem 3. If \( t \) is a stopping variable such that

\[
\mathbb{E}(\sum_{1}^{t} E(x_j^4 \mid \mathcal{F}_{j-1})) < \infty , \quad \text{then} \quad \text{ES}_t^4 < \infty \quad \text{and}
\]

(21) \[
\text{ES}_t^4 = \text{EU}_4,t + 4\text{ES}_t^2U_3,t + 6\text{ES}_t^2U_2,t - 6\mathbb{E}_{1}^{t} u_{2,j}U_2,j .
\]

Proof. Set \( Y_n = S_n^4 - 6S_n^2U_2,n - 4S_nU_3,n - U_4,n + 6\sum_{j=1}^{n} u_{2,j}U_2,j .
\)
and \( t' = \min(t, k) \). Since \( \{Y_n, \mathcal{F}_n; n \geq 1\} \) is a martingale with \( EY_1 = 0 \), by Lemma 1,

\[
E^{4 \mathcal{F}_t} = 6ES^2_{t', U_2, t'} + 4ES^3_{t', U_3, t'} + EU_{4, t'} - 6E \left( \sum_{j=1}^{t'} u_{2, j}U_{2, j} \right) 
\]

\[
< 6(E^{1/2}S_{t'}^{1/2}S_{U_2, t'}) + 4(E^{1/4}S_{t'}^{1/4}S_{U_3, t'}) + EU_{4, t'} ,
\]

whence, if \( E^{4 \mathcal{F}_t} > 0 \),

\[
E^{1/2}S_{t'}^{1/2}S_{U_2, t'} + 4(E^{3/4}S_{U_3, t'})^{1/4} + (EU_{4, t'})(E^{4 \mathcal{F}_t})^{-1/2}.
\]

Now if \( p > 1, r > 0 \),

\[
V_{r, n} = \sum_{j=1}^{n} E\left( |Y_j|^p \right) \leq n^{\frac{p-1}{p}} \left( \sum_{j=1}^{n} E\left( |Y_j|^p \right) \right)^{\frac{1}{p}}.
\]

\[
\leq n^{\frac{p-1}{p}} \left( \sum_{j=1}^{n} E\left( |Y_j|^p \right) \right)^{\frac{1}{p}} = n^{\frac{p-1}{p}} V_{r, n}^{1/p}.
\]

and thus setting \( p = 2, r = 2 \) and then \( p = 4/3, r = 3, \)

\[
EU^2_{2, t} = EV^2_{2, t} \leq EtV_{4, t} < \infty, \quad EV_{4, t}^{4/3} \leq Et^{1/3} V_{4, t} < \infty.
\]

Moreover, \( EU_{4, t} \leq E(tU_{4, t}) < \infty \) and \( E(\sum_{j=1}^{t} u_{2, j}U_{2, j}) \leq EU^2_{2, t} < \infty \). Thus, the R.H.S. of (22) is a bounded function of \( k \), implying via Fatou's lemma that \( E^{4 \mathcal{F}_t} < \infty \).

Since

\[
|Y_n| \leq S_n^4 + 6S_n^2U_{2, n} + 4|S_n|V_{3, n} + U_{4, n} + 6 \sum_{j=1}^{n} u_{2, j}U_{2, j} = Y_n^4 \text{ (say)},
\]

it follows from the preceding that

\[
E|Y_t| \leq EY_t \leq ES^{4 \mathcal{F}_t} + 6(E^{1/2}S^{4 \mathcal{F}_t})(E^{1/2}S_{U_2, t'}) + 4(E^{1/4}S^{4 \mathcal{F}_t})(E^{3/4}V_{3, t})
\]

\[
+ EU_{4, t} + 6EU^2_{2, t} < \infty.
\]
From (24), \( E T_{2}, t = EU_{2}, t < \infty \). Thus, (8) of section 2 and Lemmas 4 and 5 are valid, whence by Lemma 7, \( E[S_{t}^{2}| \mathcal{F}_{k}] > S_{k}^{2} \) for \( t > k, k = 1, 2, \ldots \).

Consequently,

\[
\left\{ \begin{array}{l}
S_{t}^{h} \geq \left[ S_{n}^{h} + 2S_{n}^{2}(S_{t}^{2} - S_{n}^{2}) + (S_{t}^{2} - S_{n}^{2})^{2} \right] \geq \int S_{n}^{h} \\
+ 2 \int S_{n}^{2}E(S_{t}^{2} - S_{n}^{2} | \mathcal{F}_{n}) \geq \int S_{n}^{h}
\end{array} \right.
\]

implying \( \int S_{n}^{h} = o(1) \) and concomitantly

\[
\int S_{n}^{2u_{2}, n} \leq \left( \int S_{n}^{h} \right)^{1/2} \left( \int U_{2}^{2} \right)^{1/2} = o(1)
\]

\[
\int |S_{n}|V_{3,n} \leq \left( \int S_{n}^{h} \right)^{1/4} \left( \int V_{3}^{4/3} \right)^{3/4} = o(1)
\]

(25)

\[
\int U_{4,n} \leq \int U_{4,t} = o(1)
\]

\[
\int u_{2, j} \leq \int U_{2,t} = o(1).
\]

Thus, \( \int V_{n} = o(1) \) and by Lemma 1 \( EY_{t} = EY_{1} = 0 \).

Alternative expressions for \( ES_{t}^{h} \) are possible as indicated in

**Theorem 4.** If \( E(t \Sigma_{j=1}^{t} E[x_{j}^{h} | \mathcal{F}_{j-1}]) < \infty \), then setting \( S_{0} = 0, \)

\[
ES_{t}^{h} = 6E \Sigma_{j=1}^{t} S_{j-1}^{2} u_{2, j} + 4E \Sigma_{j=1}^{t} S_{j-1} u_{3, j} + EU_{4}, t.
\]

The proof of Theorem 4 is similar to that of Theorem 3 and will be omitted.

**Corollary.** If \( E(t U_{4,t}) < \infty \), then

\[
E(6 \Sigma_{j=2}^{t} S_{j-1}^{2} u_{2, j} + 4 \Sigma_{j=2}^{t} S_{j-1} u_{3, j}) = 6ES_{t}^{2} U_{2,t} + 4ES_{t} U_{3,t} - 6E( \Sigma_{j=1}^{t} u_{2, j} U_{2,t}^{2})
\]
It is intuitively clear that terms with like coefficients are equal, and indeed we have

Lemma 8. If $E(t \ U_4, t) < \infty$, then $ES_{t} U_3, t = E(\sum_{j=2}^{t} S_{j-1} u_3, j)$ and $E(S_{t}^2 U_2, t) = E(\sum_{j=2}^{t} S_{j-1}^2 u_2, j) + E(\sum_{j=1}^{t} u_2, j U_2, j)$.

Proof. It suffices to verify the first of the two relationships since the second will then follow from the corollary to Theorem 4.

Suppose first that

\begin{equation}
E(\sum_{n=1}^{t} x_n | V_r, j) < \infty
\end{equation}

Then

\[ \sum_{n=1}^{\infty} \int_{[t=k]}^{k} \sum_{j=1}^{\infty} x_j U_r, j = \sum_{j=1}^{\infty} \int_{[t \geq j]} x_j U_r, j = \sum_{j=1}^{\infty} \int_{[t \geq j]} E(x_j | F_{j-1} U_r, j = 0, \]

whence

\begin{equation}
E(\sum_{n=1}^{t} S_{j-1} u_r, j) = \sum_{k=1}^{\infty} \int_{[t=k]}^{k} [\sum_{j=2}^{k} S_{j-1} u_r, j + \sum_{j=1}^{k} x_j U_r, j] = \sum_{k=1}^{\infty} \int_{[t=k]}^{k} S_{k} U_r, k
\end{equation}

Thus, if $t' = \min(t, N)$, (27) holds with $t$ replaced by $t'$ irrespective of (26). However,

\begin{equation}
ES_{t} U_3, t = \sum_{k=1}^{N} \int_{[t=k]}^{k} S_{k} U_3, k + \int_{[t > N]} S_{t} U_3, t
\end{equation}

\begin{equation}
= ES_{t'} U_3, t' - \int_{[t > N]} S_{N} U_3, N + \int_{[t > N]} S_{t} U_3, t,
\end{equation}

and analogously

\begin{equation}
E(\sum_{j=2}^{t} S_{j-1} u_3, j) = E(\sum_{j=2}^{t'} S_{j-1} u_3, j) - \int_{[t > N]}^{N} \sum_{j=2}^{t} S_{j-1} u_3, j
\end{equation}

\begin{equation}
+ \int_{[t > N]}^{t} \sum_{j=2}^{t} S_{j-1} u_3, j.
\end{equation}
Now \( E|S_t U_3, t| < EY_t < \infty \), and employing Lemma 7,

\[
E \sum_{l=1}^{t} |S_{j-l} u_3, j| = \sum_{k=1}^{\infty} \int_{[t=k]}^{\infty} \sum_{j=1}^{\infty} |S_{j-l} u_3, j| = \sum_{j=1}^{\infty} \int_{[t \geq j]} |S_{j-l} u_3, j| \leq \sum_{j=1}^{\infty} |S_t u_3, j| \leq E|S_t| V_3, t \leq EY_t < \infty.
\]

These facts plus (25) imply that all unwanted terms of (28) and (29) are \( o(1) \) and the result follows.

Identities and inequalities analogous to (27) abound and several of these will be catalogued as

**Lemma 2.** \( E(\sum_{n=1}^{t} S_n^2) \leq EtS_t^2 \) under the conditions of Lemma 7.

\[
E(\sum_{n=1}^{t} S_n^2) = EtS_t^2 \quad \text{if } EtT_t < \infty.
\]

\[
E(\sum_{n=1}^{t} T_n) \leq EtT_t \quad \text{if } EtT_t < \infty.
\]

**Proof.**

\[
E\sum_{n=1}^{t} S_n^2 = \sum_{k=1}^{\infty} \int_{[t=k]}^{\infty} \sum_{n=1}^{\infty} S_n^2 < \sum_{n=1}^{\infty} \int_{[t \geq n]} E(S_t^2 | \mathcal{F}_n)
\]

\[
= \sum_{n=1}^{\infty} \int_{[t \geq n]} S_t^2 = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \int_{[t=k]}^{\infty} S_t^2 = \sum_{k=1}^{\infty} \int_{[t=k]}^{\infty} S_t^2 = EtS_t^2
\]

employing Lemma 7. Similarly,

\[
E(\sum_{n=1}^{t} T_n) = \sum_{n=1}^{\infty} \int_{[t \geq n]} T_n < \sum_{n=1}^{\infty} \int_{[t \geq n]} T_t = EtT_t.
\]

Finally,

\[
E(\sum_{n=1}^{t} S_n) = \sum_{n=1}^{\infty} \int_{[t \geq n]} S_n = \sum_{n=1}^{\infty} \int_{[t \geq n]} E(S_t | \mathcal{F}_n) = \sum_{n=1}^{\infty} \int_{[t \geq n]} S_t = EtS_t
\]
in view of Lemmas 1 and 2 and the validity of interchanging the order
of summation and integration.

4. The Third Moment

In this section \( E(|x_n|^3) \) will be supposed finite. Define

\[
Y_n = S_n^3 - 3S_nU_2,n - U_3,n,
\]

\[
W_n = S_n^3 - \sum_{j=1}^{n} S_ju_2,j - U_3,n,
\]

\[
Z_n = S_n^3 - 3\sum_{j=1}^{n} S_ju_2,j - U_3,n.
\]

(30)

It is readily checked that \((Y_n, \mathcal{F}_n; n \geq 1), (W_n, \mathcal{F}_n; n \geq 1), (Z_n, \mathcal{F}_n; n \geq 1)\) are all martingales and that \(EY_1 = EW_1 = EZ_1 = 0\).

**Theorem 5.** If \(EV_{3,t} < \infty\) and \(EV_{1,t}^3 < \infty\), or equivalently if \(ET_{3,t}^3 < \infty\), then \(E|S_t|^3 < \infty\) and \(ES_{t}^3 = 3E(\sum_{j=1}^{t} S_{j-1}u_2,j) + EU_{3,t}\).

**Proof.** Suppose that \(EV_{3,t} < \infty\), \(EV_{1,t}^3 < \infty\) (Their equivalence with \(ET_{3,t}^3 < \infty\) will be deferred to Lemma 10). Then

\[
E|S_t|^3 = \sum_{k=1}^{\infty} \sum_{n=1}^{[t=k]} (|S_n|^3 - |S_{n-1}|^3) \lesssim \sum_{k=1}^{\infty} \sum_{n=1}^{[t=k]} (|x_n|^3).
\]

(31)

\[
+ 3|S_{n-1}|x_n^2 + 3S_{n-1}^2|x_n| \lesssim 6 \sum_{k=1}^{\infty} \sum_{n=1}^{[t=k]} (|x_n|^3 + S_{n-1}^2|x_n|)
\]

\[
= 6[E(\sum_{n=1}^{t} |x_n|^3) + E(\sum_{n=1}^{t} S_{n-1}^2|x_n|)].
\]

By Lemma 6,

\[
E(\sum_{n=1}^{t} |x_n|^3) = EV_{3,t} < \infty.
\]

(32)

On the other hand, \(ES_{t}^2 \lesssim ET_{t}^2 \lesssim 1 + ET_{t}^3 < \infty\) and
\[
\int_{[t > k]} |S_k| \leq \int_{[t > k]} T_k \leq \int_{[t > k]} T_t \leq \int_{[t > k]} (1 + T_t^3) = o(1)
\]
in view of the asserted equivalence. Thus, Lemma 7 holds, whence

\[
E(\sum_{n=1}^{t} S_n \mid x_n) = \sum_{k=1}^{\infty} \sum_{n=1}^{k} \int_{[t = k]} S_n \mid x_n \mid = \sum_{n=1}^{\infty} \int_{[t > n]} S_{n-1} v_{1,n}
\]
\[
\leq \sum_{n=1}^{\infty} \int_{[t > n]} E(S_n^2) \mid S_{n-1} v_{1,n} = \sum_{n=1}^{\infty} \int_{[t > n]} S_{n-1}^2 v_{1,n} = E V_{1,t} = (E^{2/3} |S_t|^3)(E^{1/3} V_{1,t}^3).
\]

(33)

Replace \( t \) by \( t' = \min(t, k) \) in (31). Then from (32) and (33),

\[
E|S_t|^3 \leq 6EV_{3,t'} + 6(E^{2/3}|S_t|^3)(E^{1/3}V_{1,t'}^3) = o(1) + o(1)E^{2/3}|S_t|^3
\]
whence, by Fatou's lemma,

(34)

\[
E|S_t|^3 < \infty.
\]

Next, (34) implies that the expectation in the L.H.S. of (33) is finite
whence,

\[
E\left(\sum_{n=1}^{t} S_n \mid u_{2,n}\right) = \sum_{n=1}^{\infty} \int_{[t > n]} S_n \mid x_n \mid = E\left(\sum_{n=1}^{t} S_n \mid x_n^2\right)
\]
\[
\leq E\left[\sum_{n=1}^{t} (|x_n|^3 + |S_n| \mid 2 |x_n|)\right] < \infty.
\]

(35)

Combining (33), (34) and (35), \( E|W_t| < \infty \). Since, paralleling (31),

\[
\int_{[t > k]} |S_k|^3 \leq 6 \int_{[t > k]} \sum_{n=1}^{k} (|x_n|^3 + S_n^2 \mid x_n) = o(1),
\]
\[
\int_{[t > k]} |W_k| = o(1) \text{ and the theorem follows from Lemma 1.}
\]

**Corollary.** Under the same hypothesis, \( E(\sum_{n=1}^{t} x_{j, n} u_{2,j}) = 0 \).

**Proof.** Analogously, \( EZ_t = 0 \), whence \( E(W_t - Z_t) = 0 \).
Lemma 10. \( EV_{3,t} < \infty \) and \( EV_{1,t}^3 < \infty \) if and only if \( ET_{t}^3 < \infty \).

Proof. Suppose \( EV_{3,t} < \infty \) and \( EV_{1,t}^3 < \infty \). The argument of (31) with \( T_t \) replacing \( S_t \) yields
\[
ET_{t}^3 \leq 6 \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \int_{[t=k]} \left| x_n \right|^3 + T_{n-1}^2 \left| x_n \right|.
\]
The inequality of (33) also obtains with \( T \) replacing \( S \) in view of the fact that \( T_t > T_{n-1} \) on the set \([t \geq n]\). Thus, analogously,
\[
EV_{3,t}^3 \leq 0(1) + 0(1) E^{2/3} T_{t}^3, \text{ implying } ET_{t}^3 < \infty.
\]
Conversely, if \( ET_{t}^3 < \infty \), clearly \( EV_{3,t} = ET_{3,t} < ET_{t}^3 < \infty \). Moreover,
\[
EV_{1,t}^3 = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \int_{[t=k]} (V_{1,n}^3 - V_{1,n-1}^3) < \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \int_{[t=k]} (V_{1,n}^3 + 3V_{1,n-1}^2 V_{1,n} + 3V_{1,n-1}^2)
\]
\[
\leq 0(1) + 6 \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \int_{[t=k]} V_{1,n-1}^2 V_{1,n} = 0(1) + 6 \sum_{n=1}^{\infty} \int_{[t \geq n]} \left| x_n \right|^2 V_{1,n-1}^2 V_{1,n}
\]
\[
\leq 0(1) + 6 \sum_{n=1}^{\infty} \sum_{k=n}^{\infty} \int_{[t=k]} \left| x_n \right|^2 V_{1,t}^2 \leq 0(1) + 6ET_{t} V_{1,t}^2
\]
\[
\leq 0(1) + 0(1) E^{2/3} V_{1,t}^3,
\]
which implies, as earlier, that \( EV_{1,t}^3 < \infty \) and completes the proof.

Theorem 6. If \( ET_{t}^3 < \infty \) and \( E_t^{1/2} V_{3,t} < \infty \), \( ES_{t}^3 = 3ES_t U_{2,t} + EU_{3,t} < \infty \).

Proof. As in Theorem 3, after setting \( p = 3/2 \), \( r = 2 \) in (23) of section 3 to obtain
\[
ES_t U_{2,t} \leq (E^{1/3} S_{t}^3)(E^{2/3} U_{2,t}^{3/2}) \leq (E^{1/3} S_{t}^3)(E^{2/3} t^{1/2} V_{3,t}).
\]
Corollary. Under the conditions of Theorem 6, $\text{ES}_t U_2, t$
$= E(\Sigma_j S_{j-1} u_2, j)$.

The single requirement $ET_t^3 < \infty$, although equivalent to the two
conditions of Theorem 5, is difficult to check. The following single
condition is easily seen to imply all those of Theorems 5 and 6:

\begin{equation}
E(t^2 V_3, t) < \infty,
\end{equation}

and in addition yields

\begin{align*}
ET_t^3 &= 3ET_t^2 V_1, t + 3ET_t (V_2, t - 2 \Sigma_j V_1, j \Sigma_j V_2, j) + EV_3, t - 3E(\Sigma_j V_1, j \Sigma_j V_2, j) \\
&\quad - 3E(\Sigma_j V_2, j \Sigma_j V_1, j) + 6E(\Sigma_j V_1, j \Sigma_j V_1, j V_1, j).
\end{align*}

5. Sums of Independent Random Variables

In this section, the random variables $x_1, x_2, \ldots$ will be supposed
independent. If $Ex_n = 0$, all prior theorems are, of course, applicable
but may be reformulated in especially simple terms with conditions that
are susceptible of immediate verification. For example, from Theorems
3 and 6, we obtain:

**Theorem 7.** If $x_1, x_2, \ldots$ are independent with $Ex_n = 0$, $Ex_n^2 = \sigma^2$,
$Ex_n^3 = \gamma$, $Ex_n^4 = \beta < \infty$ and $t$ is a stopping rule with $E t^2 < \infty$, then $ES_t^4 < \infty$
and

$$ES_t^4 = 6 \sigma^2 E t S_t^2 + 4 \gamma E t S_t + \beta E t - 3\sigma^4 E t(t+1).$$

**Theorem 8.** If $x_1, x_2, \ldots$ are independent with $Ex_n = 0$, $Ex_n^2 = \sigma^2$,
$Ex_n^3 = \gamma$, $E|x_n|^3 \leq C < \infty$, and if $t$ is a stopping variable with $Et^3 < \infty$, then
$ES_t^3 = \gamma E t + 3\sigma^2 E t S_t < \infty.$
Proof. According to Theorem 6 and Lemma 10, it suffices to verify that

\[ EV_{3,t} \leq E(t^{1/2}v_{3,t}) \leq C E t^{3/2} < \infty, \]

\[ EV_{1,t}^3 \leq E(t(1+C))^3 < \infty. \]

In the final theorem, the requirement of Theorem 8 that \( Et^3 < \infty \) will be relaxed at the expense of increasing the moment assumptions on \( x_n \).

**Theorem 9.** If \( x_1, x_2, \ldots \) are independent with \( Ex_n = 0, Ex_n^2 = \sigma^2 \), \( Ex_n^3 = \gamma, Ex_n^4 \leq C < \infty \), and if \( t \) is a stopping variable with \( Et^2 < \infty \), then \( ES_t^3 = \gamma Et + 3\sigma^2 EtS_t \).

**Proof.** Here, the martingale \( Y_n \) of (30) simplifies to \( Y_n = S_n^3 - 3\sigma^2 S_n - ny \). The theorem will follow from Lemmas 1 and 2 once it is established that

\[ E \sum_{l=1}^{t} |Y_{n+l} - Y_n| = E \sum_{l=1}^{t} E(|Y_{n+l} - Y_n| \mid \mathcal{F}_n) < \infty. \]

Now

\[ E(|S_{n+1}^3 - S_n^3| \mid \mathcal{F}_n) \leq 6 E(|x_{n+1}|^3 + S_n^2|x_{n+1}| \mid \mathcal{F}_n) = O(1)S_n^2 + O(1), \]

\[ E(|n+1)S_{n+1} - nS_n| \mid \mathcal{F}_n) = E(|S_n + (n+1)x_{n+1}| \mid \mathcal{F}_n) \leq S_n^2 + n 0(1), \]

whence

\[ E(|Y_{n+l} - Y_n| \mid \mathcal{F}_n) = O(1)S_n^2 + n 0(1). \]

Next, Lemma 9 is applicable below since (17) insures \( ES_t^2 < \infty \) while Lemmas 6 and 2 guarantee (10). Consequently,
\[ \mathbb{E} \sum_{l}^{t} \mathbb{E}(|Y_{n+l} - Y_{n}| \mid \mathcal{F}_{n}) \leq o(1) \mathbb{E}(\sum_{l}^{t} S_{l}^{2}) + o(1) \mathbb{E}t \]

\[ \leq o(1) \mathbb{E}t S_{t}^{2} + o(1) \mathbb{E}t \]

\[ \leq o(1) (E^{1/2} t^{2})(E^{1/2} S_{t}^{4}) + o(1) \mathbb{E}t < \infty. \]
References


