On Optimal Stopping Rules for \( s_n/n \)

by

Y.S. Chow and Herbert Robbins

Purdue University and Columbia University

Department of Statistics
Division of Mathematical Sciences
Purdue University
Mimeograph Series No. 16
May, 1964
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1. Introduction. Let

(1) $x_1, x_2, \ldots$

be a sequence of independent, identically distributed random variables on a probability space $(\Omega, \mathcal{F}, P)$ with

(2) $P(x_1 = 1) = P(x_1 = -1) = 1/2,$

and let $s_n = x_1 + \ldots + x_n$. Let $i = 0, \pm 1, \ldots$ and $j = 0, 1, \ldots$ be two fixed integers. Assume that we observe the sequence (1) term by term and can decide to stop at any point; if we stop with $x_n$ we receive the reward $(i + s_n)/(j + n)$. What stopping rule will maximize our expected reward?

Formally, a stopping rule $t$ of (1) is any positive integer valued random variable such that the event $t = n$ is in $\mathcal{F}_n$ ($n \geq 1$) where $\mathcal{F}_n$ is the Borel field generated by $x_1, \ldots, x_n$. Let $T$ denote the class of all stopping rules; for any $t$ in $T$, $s_t$ is a well-defined random variable, and we set

(3) $v_j(i|t) = E\left(\frac{i + s_t}{j + t}\right)$, $v_j(i) = \sup_{t \in T} v_j(i|t)$.

It is by no means obvious that for given $i$ and $j$ there exists a stopping rule $\tau_j(i)$ in $T$ such that

(4) $v_j(i|\tau_j(i)) = v_j(i) = \max_{t \in T} v_j(i|t)$;
such a stopping rule of (1) will be called optimal for the reward sequence

\[ \frac{i+s_1}{j+1}, \frac{i+s_2}{j+2}, \ldots. \]

Theorem 1 below asserts that for every \( i = 0, 1, \ldots \) and \( j = 0, 1, \ldots \) there exists an optimal stopping rule \( \tau_j(i) \) for the reward sequence (5).

We remark that for any \( t \) in \( T \) and any \( i = 0, 1, \ldots \) and \( j = 0, 1, \ldots \) the random variable

\[ \tau' = \begin{cases} t & \text{if } i+s_t \geq 1, \\ \text{first } n > t \text{ such that } i+s_n = 1 & \text{if } i+s_t < 0 \end{cases} \]

is in \( T \) and

\[ \tau'_t \geq 1, \quad 0 < \mathbb{E}\left(\frac{i+s_{\tau'_t}}{j+t}\right) \geq \mathbb{E}\left(\frac{i+s_t}{j+t}\right). \]

It follows that

\[ \tau_j(i) = \sup_{\tau \in T} \mathbb{E}\left[\frac{(i+s_{\tau})^+}{j+t}\right], \]

where by definition \( a^+ = \max(0, a) \).

2. Reduction of the problem to the study of bounded stopping rules. For any fixed \( N = 1, 2, \ldots \) let \( T_N \) denote the class of all \( t \) in \( T \) such \( t \leq N \). By the usual backward induction (see e.g. [1]) it may be shown that in \( T_N \) there exists a minimal optimal stopping rule of (1) for the reward sequence

\[ \tau'_t \geq 1, \quad 0 < \mathbb{E}\left(\frac{(i+s_1)^+}{j+1}\right), \quad \mathbb{E}\left(\frac{(i+s_2)^+}{j+2}\right), \ldots; \]

that is, an element \( \tau_j^N(i) \) of \( T_N \) such that, setting
\begin{align}
\tag{2} w_j(i|t) &= E \left[ \frac{(i+s_t)^+}{j+t} \right],
\end{align}

we have

\begin{align}
\tag{3} w_j(i|\tau_N^j(i)) &= \max_{t \in T_N} w_j(i|t),
\end{align}

and such that if \( \tilde{t} \) is any element of \( T_N \) for which

\begin{align}
\tag{4} w_j(i|\tilde{t}) &= \max_{t \in T_N} w_j(i|t),
\end{align}

then \( \tau_N^j(i) \leq \tilde{t} \). The sequence \( \tau_N^j(i), \tau_N^{j,2}(i), \ldots \) is such that as \( N \to \infty \),

\begin{align}
1 \leq \tau_N^j(i) \leq \tau_N^{j,2}(i) \leq \ldots \quad \rightarrow \quad \tau^{*}_N(i) \leq \infty,
\end{align}

\begin{align}
\tag{5} 0 \leq w_j(i|\tau_N^j(i)) \leq w_j(i|\tau_N^{j,2}(i)) \leq \ldots \quad \rightarrow \quad \sup_{t \in T} w_j(i|t) = v_j(i),
\end{align}

the last equality following from (1.8). It is shown in [1] that there exists an optimal element in \( T \) for the reward sequence (1.5) if and only if

\begin{align}
\tag{6} \tau^{*}_N(i) &= \lim_{N \to \infty} \tau_N^j(i)
\end{align}

is in \( T \) that is, if and only if

\begin{align}
\tag{7} P(\tau^{*}_N(i) < \infty) &= 1
\end{align}

and when (7) holds \( \tau^{*}_N(i) \) is the minimal element of \( T \) which satisfies (1.4).

The remainder of the present paper is devoted to proving that (7) holds.
3. The constants \( a_n^N(i) \) and \( a_n(i) \). In order to study the nature of the optimal bounded stopping rules \( \zeta_j^N(i) \) of Section 2 we proceed as follows. Define for \( n = 1, 2, \ldots \) and \( i = 0, \pm 1, \ldots \) the constants

\[
b_N^N(i) = \frac{i + \frac{}{N}}{N},
\]

(1)

\[
b_N^N(i) = \max \left( \frac{i + b_N^{N+1}(i+1) + b_N^{N-1}(i-1)}{2} \right) \quad (n = 1, 2, \ldots, N-1).
\]

Then

(2)

\[
b_n^N(i) = \max \left( \frac{i + \sup_{t \in T_{N-n}} E\left[ \frac{(i+s_t)^+}{n+t} \right]}{n} \right) \quad (n = 1, 2, \ldots, N-1),
\]

and

(3)

\[
\zeta_j^N(i) = \text{first } n \geq 1 \text{ such that } b_j^{N+1}(i+s_n) = \frac{(i+s_n)^+}{j+N},
\]

In view of (2) and (3) it is convenient to introduce the constants \( a_n^N(i) \) defined for \( N = 1, 2, \ldots; i = 0, \pm 1, \ldots; n = 1, 2, \ldots, N \) by

(4)

\[
a_n^N(i) = b_n^N(i) - \frac{i + \frac{}{N}}{N};
\]

then (3) becomes

(5)

\[
\zeta_j^N(i) = \text{first } n \geq 1 \text{ such that } a_j^{N+1}(i+s_n) = 0.
\]

From (5) and (1) it follows that the constants \( a_n^N(i) \) satisfy the recursion relations
\[ a_n^N(1) = 0 \quad (\text{all } i), \]

\[ a_n^N(i) = \left[ \frac{a_{n+1}^N(i+1) + a_{n+1}^N(i-1)}{2} + \frac{(i+1)^+ + (i-1)^+}{2(n+1)} - \frac{i^+}{n} \right] \quad (n = 1, 2, \ldots, N-1) \]

from which they may be successively computed for \( n = N, N-1, \ldots, 1 \).

Moreover, from (2) and (4) we have

\[ a_n^N(i) = \sup_{t \in T_{N-n}} E^+ \left[ \frac{(i+s_t)^+}{n+t} - \frac{i^+}{n} \right] \quad (n = 1, 2, \ldots, N-1) \]

and

\[ \sup_{t \in T_N} E \left[ \frac{(i+s_t)^+}{j+t} \right] = \frac{1}{2} \left[ a_{j+1}^N(i+1) + a_{j+1}^N(i-1) + \frac{(i+1)^+ + (i-1)^+}{j+1} \right]. \]

For any \( i = 0, 1, \ldots \) and \( n = 1, 2, \ldots \) we have

\[ 0 = a_n^N(i) \leq a_n^{n+1}(i) \leq \ldots, \]

and letting \( N \to \infty \) we obtain constants

\[ a_n(i) = \lim_{N \to \infty} a_n^N(i) \]

such that

\[ a_n^N(i) \uparrow a_n(i) = \sup_{t \in T} E^+ \left[ \frac{(i+s_t)^+}{n+t} - \frac{i^+}{n} \right], \]

while for \( j = 0, 1, \ldots \)

\[ \sup_{t \in T} E \left[ \frac{(i+s_t)^+}{j+t} \right] = \sup_{t \in T} E \left( \frac{i+s_t}{j+t} \right) = v_j(i) = \frac{1}{2} \left[ \frac{(i+1)^+ + (i-1)^+}{j+1} + a_{j+1}(i+1) + a_{j+1}(i-1) \right]; \]

moreover \( \zeta_j^N(i) \uparrow \zeta_j^*(i) \) where
Thus (2.7) holds if and only if

\[
(13) \quad P(a_{j+n(i+s_n)} = 0 \text{ for some } n \geq 1) = 1.
\]

In the next section we shall prove (lemma 4) that there exists a positive integer \( n_0 \) such that \( n \geq n_0 \) and \( i > 13 \sqrt{n} \) together imply that \( a_n(i) = 0 \). Hence

\[
(14) \quad P(a_{j+n(i+s_n)} = 0 \text{ for some } n \geq 1) \geq P(s_n > 13 \sqrt{n-1} \text{ for some } n \geq n_0).
\]

The law of the iterated logarithm implies that the latter probability is 1 and this establishes (13); hence \( \mathcal{I}_j^*(i) \) defined by (12) is in \( T \) and is optimal for the reward sequence (1.5). We thus have the following

**Theorem 1.** For the sequence (1.1) with the distribution (1.2) and the reward sequence (1.5) there exists an optimal stopping rule \( \mathcal{I}_j^*(i) \) defined by (12); the expected reward in using \( \mathcal{I}_j^*(i) \) is

\[
(15) \quad v_j(i) = \max_{t \in T} E \left( \frac{i+t}{j+t} \right) = \frac{1}{2} \left[ \frac{(i+1)^+ + (i-1)^+}{j+1} + a_{j+1}(i+1) + a_{j-1}(i-1) \right]
\]

\((i = 0, \pm 1, \ldots ; j = 0, 1, \ldots)\). The constants \( a_n(i) = \lim_{n \to \infty} a_n^N(i) \) which occur in (12) and (15) are determined by (7).
4. **Lemmas.**

**Lemma 1.** \( a_n(0) \leq \frac{1}{\sqrt{n}} \) \( (n = 1, 2, \ldots) \).

**Proof.** From (3.7) we have

\[
\begin{align*}
\left( a_n^N(i+1) + a_n^N(i-1) \right) \frac{n+1}{2} + \frac{1}{2(n+1)} \quad (i \leq -1), \\
\left( a_n^N(i+1) + a_n^N(i-1) \right) \frac{n+1}{2} - \frac{i}{n(n+1)} \quad (i = 0), \\
\left( a_n^N(i+1) + a_n^N(i-1) \right) \frac{n+1}{2} + \frac{a_n^N(i+1) + a_n^N(i-1)}{2} \quad (i \geq 1)
\end{align*}
\]

Hence

\[
a_n^N(0) = \frac{a_n^N(1) + a_n^N(-1)}{2} + \frac{1}{2(n+1)} \leq \frac{1}{2} \left[ a_n^N(2) + 2a_n^N(0) + a_n^N(-2) \right] + \frac{1}{2(n+1)} \leq \frac{1}{2} \left[ a_n^N(3) + 3a_n^N(1) + 3a_n^N(-1) + a_n^N(-3) \right] + \frac{1}{2(n+1)} + \frac{(2)}{2^3(n+1)}
\]

\[
\leq \cdots \leq \sum_{k=0}^{\infty} \frac{(2k)}{2^{2k+1}(n+2k+1)}
\]

since \( a_n^N(i) = 0 \). By Stirling's formula

\[
\left( \begin{array}{c} 2k \\ k \end{array} \right) < \frac{2^{2k}}{\sqrt{2k}}
\]

and

\[
\left( \begin{array}{c} 2k \\ k \end{array} \right) < \frac{2^{2k}}{\sqrt{2k}}
\]
\[ \sum_{k=n}^{\infty} \frac{1}{2\sqrt{k\pi} (n+2k+1)} \leq \frac{1}{2\sqrt{\pi}} \int_{r-\frac{1}{2}}^{\infty} \frac{x}{\sqrt{x} (n+2x+1)} dx = \frac{1}{\sqrt{2\pi(n+1)}} \left( \frac{\pi}{2} - \tan^{-1} \sqrt{\frac{2r-1}{n+1}} \right). \]

Hence

\[ a_n(0) = \lim_{N \to \infty} a_n^N(0) \leq \sum_{k=0}^{\infty} \frac{2^k}{2k+1(n+2k+1)} + \frac{1}{\sqrt{2\pi(n+1)}} \left( \frac{\pi}{2} - \tan^{-1} \sqrt{\frac{2r-1}{n+1}} \right). \]

For \( r = 1 \) this gives

\[ a_n(0) \leq \frac{1}{2(n+1)} + \frac{1}{\sqrt{2n}} \leq \frac{1}{\sqrt{n}}. \]

**Lemma 2.** For \( n = 1, 2, \ldots \)

\[ 0 < a_n(-2) \leq a_n(-1) \leq a_n(0) \geq a_n(1) \geq a_n(2) \geq \ldots \geq 0, \]

\[ a_{n+1}(i) \geq \frac{n+1}{n+2} a_n(i) \quad \text{(all \( i \))}. \]

**Proof.** For \( i \leq 0 \) we have from (3.10) and (1.7)

\[ a_n(i) = \sup_{t \in T} E\left[ \frac{(i+s_i)^+}{n+t} \right] > 0; \]

Hence

\[ a_n(i) \geq \sup_{t \in T} E\left[ \frac{(i-1+s_i)^+}{n+t} \right] = a_n(i-1). \]

For \( i \geq 0 \) we have
\begin{equation}
\begin{align*}
(11) \quad a_n(i) &= \sup_{t \in T} E \left[ \frac{i+s_t}{n+t} - \frac{i}{n} \right] = \sup_{t \in T} E \left[ \frac{ns_t - it}{n(n+t)} \right] \\
&\geq \sup_{t \in T} E \left[ \frac{ns_t - (i+l)t}{n(n+t)} \right] = a_n(i+l) \geq 0.
\end{align*}
\end{equation}

(12) \quad \frac{n+2}{n+1} a_{n+1}^N(i) \geq a_n^N(i) \quad (\text{all } i);

(3) will follow from (12) on letting \( N \to \infty \). (12) is true trivially for \( n = N \)

since \( a_N^N(1) = 0 \). Assume now that (12) holds; for \( i \neq 0 \) we have by (1),

\begin{equation}
\begin{align*}
(13) \quad \frac{n+1}{n} a_n^{N+1}(i) &= \frac{n+1}{n} \left[ \frac{a_n^{N+1}(i+1) + a_n^{N+1}(i-1)}{2} - \frac{i^+}{n(n+1)} \right] \\
&\geq \frac{n+1}{n} \left[ \frac{a_n^N(i+1) + a_n^N(i-1)}{2} - \frac{i^+}{n(n+1)} \right] \\
&\geq \left[ \frac{a_n^N(i+1) + a_n^N(i-1)}{2} - \frac{i^+}{(n-1)n} \right] = a_n^{N-1}(1).
\end{align*}
\end{equation}

The case \( i = 0 \) is treated similarly. Thus (12) holds with \( n \) replaced by \( n-1 \),

and hence (12) holds for all \( n=N, N-1, \ldots, 2, 1 \).

\textbf{Lemma 3}. Let \( i \) and \( j \) be non-negative integers such that \( a_n(i+j) > 0 \).

Let \( \zeta_0 \) denote the first integer \( m \geq 1 \) such that \( s_m = j+1 \). Then for any
given \( t \) in \( T \) there exists a \( \zeta \) in \( T \) such that

\begin{equation}
\begin{align*}
(14) \quad \zeta \geq t, \quad \zeta \geq \zeta_0, \quad E\left( \frac{i+s_{\zeta}}{n+\zeta} \right) \geq E\left( \frac{i+s_t}{n+t} \right).
\end{align*}
\end{equation}
Proof. We have from (3.10) and (3.11) for $i \geq 0$,

$$a_n(i) = \left[ \sup_{t \in T} \mathbb{E} \left( \frac{i + s_t}{n+t} - \frac{i}{n} \right)^+ \right].$$

By (7) and (8) the inequality $a_n(i+j) > 0$ implies that for every positive integer $m$ and every integer $k \leq j$,

$$a_{n+m}(i+k) > 0,$$

and hence that there exists a stopping rule $t_{m,k}$ of the sequence $x_{m+1}, x_{m+2}, \ldots$ such that

$$\mathbb{E}\left( \frac{i + k + x_{m+1} + x_{m+2} + \ldots + x_{m+t_{m,k}}}{n+m+t_{m,k}} \right) > \frac{i+k}{n+m}.$$ 

Let $A$ be the event $\{ t < \mathcal{Z}_0 \}$, and define

$$t_1(\omega) = \begin{cases} t(\omega) & \text{if } \omega \notin A, \\ t(\omega) + t_{m,k}(\omega) & \text{if } \omega \in A, \quad t(\omega) = m, \quad s_t(\omega) = k \\ (m = 1, 2, \ldots; k \leq j). \end{cases}$$

Then $t_1$ is a stopping rule, $t_1 \geq t$, and $t_1(\omega) \geq t(\omega) + 1$ if $\omega \in A$. Moreover

$$\mathbb{E}\left( \frac{i + s_{t_1}}{n+t_1} \right) = \int_{\Omega-A} \frac{i+s_t}{n+t} \, dP + \sum_{m,k} \int_{t=m, s_t=k, t < \mathcal{Z}_0} \frac{i+s_{t+t_{m,k}}}{n+t+t_{m,k}} \, dP$$

$$\geq \int_{\Omega-A} \frac{i+s_t}{n+t} \, dP + \sum_{m,k} \int_{t=m, s_t=k, t < \mathcal{Z}_0} \frac{i+k}{n+m} \, dP = \mathbb{E}\left( \frac{i+s_t}{n+t} \right).$$
Set $t_0 = t$ and $A_0 = A$. By a repetition of the preceding argument we may define a sequence of stopping rules $t_l$,\

(20) $t = t_0 \leq t_1 \leq t_2 \leq \cdots$ \\

and events $A_l = \{ t_l < \omega_0 \}$ with\

(21) $A = A_0 \supset A_1 \supset A_2 \supset \cdots$ \\

such that\

(22) $t_{l+1}(\omega) = \begin{cases} t_l(\omega) & \text{if } \omega \not\in A_l, \\ t_l(\omega) + 1 & \text{if } \omega \in A_l \end{cases}$ \\

Set\

(23) $\omega = \lim_{l \to \infty} \omega_l$ \\

then $\{ \omega = \infty \} = \{ \omega_0 = \infty \}$, so that $\omega$ is in $T$, and $\omega \geq \omega_0$, $\omega \geq t$.

By the Lebesgue dominated convergence theorem,

(24) $E \left( \frac{i + s_\omega}{n + \omega} \right) = \lim_{l \to \infty} E \left( \frac{i + s_{\omega_l}}{n + \omega_l} \right) \geq E \left( \frac{i + s_t}{n + t} \right)$,

and the proof is complete.

**Lemma 4.** There exists a positive integer $n_o$ such that $n \geq n_o$ and $i > 13 \sqrt{n}$ imply that $a_n(1) = 0$.

**Proof.** Let $i$ be a positive integer such that $s_n(2i) > 0$, and let $\omega$ denote the first integer $m \geq 1$ such that $s_m = i$. Then $[2; \text{p. 87}]$ as $i \to \infty$,

(25) $P(\omega \geq i^2) \to \sqrt{ \frac{2}{\pi} } \int_0^1 e^{- \frac{u^2}{2}} du > \sqrt{ \frac{2}{\pi e} } > \frac{1}{3}$.
Hence there exists \( i_o > 0 \) such that

\[
E \left( \frac{\xi}{1^2 + \zeta} \right) > \frac{1}{6} \quad (i \geq i_o),
\]

and therefore

\[
E \left( \frac{\xi}{n + \zeta} \right) > \frac{1}{6} \quad (i \geq i_o, 1 \leq n \leq i^2).
\]

By (7), \( a_n(i) > 0 \), and hence by Lemma 3 (putting \( j = 1 \)) there exists a \( t \in T \) such that \( t \geq \zeta \) and

\[
E \left( \frac{i + t}{n + t} \right) > \frac{i}{n}.
\]

Hence by Lemma 1 and (11),

\[
\frac{1}{\sqrt{n}} \geq a_n(0) \geq E \left( \frac{s_t}{n + t} \right) = E \left( \frac{i + t}{n + t} \right) > \frac{i}{n} - E \left( \frac{i}{n + t} \right) = \frac{i}{n} E \left( \frac{t}{n + t} \right)
\]

\[
> \frac{i}{n} E \left( \frac{\xi}{n + \zeta} \right) > \frac{i}{6n} \quad (i \geq i_o, 1 \leq n \leq i^2).
\]

Assume now that \( a_n(j) > 0 \) for some \( j > 13 \sqrt{n} \) and \( n > n_o = i_o^2 \).

Then by (7),

\[
a_n \left( 2 \left[ \frac{i}{2} \right] \right) > 0, \quad \left[ \frac{i}{2} \right]^2 \geq n \geq 1, \quad \left[ \frac{i}{2} \right] \geq i_o.
\]

Hence, setting \( i = \left[ \frac{i}{2} \right] \) in (29),

\[
\left[ \frac{i}{2} \right] < 6 \sqrt{n},
\]

and therefore

\[
j < 12 \sqrt{n} + 1 \leq 13 \sqrt{n},
\]
a contradiction. The proof of Lemma 4, and hence of Theorem 1, is complete.

5. Remarks.

1. If we define for \( n = 1, 2, \ldots \)

\[(1) \quad k_n = \text{smallest integer } k \text{ such that } a_n(k) = 0,\]

then from Lemma 2 it follows that

\[(2) \quad 0 < k_1 \leq k_2 \leq \ldots \]

and that

\[(3) \quad a_n(1) = 0 \text{ if and only if } i \geq k_n.\]

It is easily seen that

\[\mathcal{Z}_j^*(i) = \text{first } n \geq 1 \text{ such that } a_{j+n}(i+s_n) = 0\]

\[(4) \quad = \text{first } n \geq 1 \text{ such that } i+s_n = k_{j+n}.\]

Hence the stopping rules \( \mathcal{Z}_j^*(i) \) are completely defined by the sequence of positive integers \( k_n \). It is difficult to obtain an explicit formula for \( k_n \); by Lemma 4 we know that \( k_n = 0 (\sqrt{n}) \) as \( n \to \infty \). We note also that

\[(5) \quad \lim_{n \to \infty} k_n = \infty.\]

Otherwise we would have \( k_n < M \) for some finite positive integer \( M \) and every \( n = 1, 2, \ldots \). If so, let \( t = \text{first } m \geq 1 \text{ such that } s_m = M. \) Then since \( a_n(M) = 0, \)

\[(6) \quad E \left( \frac{M+t}{n+t} \right) \leq \frac{M}{n},\]
and hence

\[ E \left( \frac{2M}{n+c} \right) \leq \frac{M}{n}, \quad E \left( \frac{n}{n+c} \right) \leq \frac{1}{2}. \]

But as \( n \to \infty \),

\[ E \left( \frac{n}{n+c} \right) \to 1, \]

which contradicts (7).

2. We have from (3.15),

\[ v_0(0) = \max_{t \in T} E \left( \frac{s_t^2}{t} \right) = \frac{1}{2} \left[ 1 + a_1(1) + a_1(-1) \right]. \]

Now by (4.15), since \( s_t \leq t \),

\[ a_1(1) = \left[ \sup_{t \in T} E \left( \frac{1+t}{1+t} \right) - 1 \right] = 0, \]

and by (4.6) and (4.7),

\[ a_1(-1) \leq a_1(0) \leq \frac{1}{4} + \frac{1}{\sqrt{2}} < .96. \]

Hence

\[ v_0(0) < .98. \]

This inequality is very crude and can be greatly improved by a more detailed analysis of the term \( a_1(-1) \), but it is interesting to note that even (12) is not easy to prove directly from the definition of \( v_0(0) \).
3. In this connection let us define

\[(13) \quad v_N = \max_{t \in T_N} E \left[ \frac{s_t^+}{t} \right] ; \]

then as \(N \to \infty\)

\[(14) \quad v_N \uparrow v_0(0) = \max_{t \in T} \frac{s_t^+}{t} = \max_{t \in T} \frac{s_t}{t}. \]

Now for any fixed \(N = 1, 2, \ldots\) the value \(v_N\) can be computed by recursion; by (3.4) and (3.2),

\[(15) \quad v_N = \frac{1}{2} \left[ b_N^N(1) + b_N^N(-1) \right] = \frac{1}{2} \left[ 1 + b_N^N(-1) \right] , \]

where by (3.1)

\[(16) \quad b_N^N(i) = \frac{i^+}{N} , \]

\[b_n^N(i) = \max \left( \frac{i^+}{n}, \frac{b_n^{n+1}(i+1) + b_n^{n+1}(i-1)}{2} \right) \quad (n = 1, 2, \ldots, N-1). \]

The computation of the \(b_n^N(i)\) is easily programmed for a high speed computer; the following results were kindly supplied to us by R. Bellman and S. Dreyfus:

\[v_{100} = .5815 \]
\[v_{200} = .5835 \]
\[v_{500} = .5845 \]
\[v_{1000} = .5850 \]

4. It would be interesting to see whether the existence of an optimal stopping rule for \(s_n/n\) can be proved for sequences \(x_1, x_2, \ldots\) with a more general distribution than (1.2). We have some preliminary extensions of Theorem 1 to more general cases but no definite results as yet.
References


[2]. W. Feller: An introduction to probability theory and its applications,