A Martingale Convergence Theorem of Ward's Type

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Introduction. The martingale convergence theorems were first utilized by Doob [2; p. 343] in giving a new proof of the Lebesgue differentiation theorem of functions of bounded variation on a real line. Later Chow [1] gave a proof of the Lebesgue differentiation theorem of interval functions of bounded variation by applying convergence theorems of partially ordered martingales. In 1959, Ward's differentiation theorem [3; p. 137, p. 141], among other things, have been generalized by Rutowitz [7] to cell functions by introducing the concept of the p-bordering property. In this paper, by following Doob's approach in [3], we are able to obtain a convergence theorem [Theorem I], which includes some martingale convergence theorems and extends a theorem of Rutowitz [7; Theorem II] to non-atomic basis. Theorem IV puts the above cited Ward's theorem into Martingale setting.

1. Definitions and notation.

Suppose that \((\Omega, \mathcal{F}, P)\) is a complete measure space with \(P(\Omega) = 1\). A stochastic basis \((\mathcal{F}_\delta, \Delta)\) is a net, where \(\Delta\) is a directed set, \(\mathcal{F}_\delta\) is a sub-\(\sigma\)-algebra of \(\mathcal{F}\) for each \(\delta \in \Delta\), and \(\mathcal{F}_\delta \subseteq \mathcal{F}_{\delta'}\), if \(\delta < \delta'\). A stochastic process \((x_\delta, \mathcal{F}_\delta, \Delta)\) is a triple, where \((\mathcal{F}_\delta, \Delta)\) is a stochastic basis and \(x_\delta\) is an \(\mathcal{F}_\delta\)-measurable function. \(P^*\) is the outer measure induced by \(P\) and the integral \(\int_A x\) will mean \(\int_A x dP\). For a set \(A\), the \(\mathcal{F}_\delta\)-cover of \(A\) is denoted by \(A_\delta^*\) and the \(\mathcal{F}\)-cover by \(A^*\). \(A-B\) will be the proper difference of sets \(A\) and \(B\), and \(I(A)\) the indicator (or characteristic) function of the set \(A\). The function \(x_\delta\) is sometimes written as \(x(\delta)\). ||x||_q is the \(L_q\)-norm of \(x\). For sets \(A\) and \(B\), \(A \in \mathcal{F}_\delta B\), if \(A \subseteq B\) and \(A \in \mathcal{F}_\delta\).

Definition 1. A stochastic basis is said to satisfy the Vitali condition
$V_q$ for $1 \leq q \leq \infty$, if for every $\epsilon > 0$, every set $A$ and every net $(K_\delta, \Delta)$ of $\mathcal{F}_\delta$-sets such that $\limsup_{\Delta} K_\delta \not\subset A$ a.e., there exist $\delta_i > \delta$ for any given $\delta$, and $\mathcal{F}_\delta$-sets $L_1 \subset K_{\delta_i}$ so that

$$P^*(A-B) < \epsilon$$

where $B = \bigcup_{\delta_i} L_1$, and that

$$\left\{ \sum_{\delta_i} \mathcal{I}(L_1) - \mathcal{I}(\phi) \right\}_q < \epsilon.$$

The conditions $V_1$ and $V_\infty$ are called respectively the weak and the strong Vitali conditions. If $\Delta$ is a countable linearly ordered set, then any stochastic basis $(\mathcal{F}_\delta, \Delta)$ satisfies $V_\infty$. The ordinary differentiation basis satisfies the strong Vitali condition $V_\infty$ (See [1] or [4; p.209], in [1] $V_\infty$ has been denoted by $V_0$), and the strong differentiation basis has the property $V_1$ (see [4; p.210]).

A stochastic basis is said to satisfy the Vitali condition $V_q^*$, if it satisfies the conditions of Definition 1, replacing $\limsup$ by $\text{ess lim sup}$ and $A$ by $A^*$. Both definitions of $V_q$ and $V_q^*$ are due to Krickeberg ([5], [6]). He denotes $V_q$ and $V_q^*$ by $V_q^*$ and $V_q^*$.

Definition 2. Let $b > 0$, $1 \leq q \leq \infty$ and $V = \{ \sup |x(\delta)| < b \}$.

$(x_\delta, \mathcal{F}_\delta, \Delta)$ is said to satisfy the condition $(A,b)_q$, if for every $\delta_0 \in \Delta$

there exists $0 < c < \infty$ such that for any given $\delta_0, \delta_1, \ldots, \delta_m$ in $\Delta(\delta > \delta_0)$

and $L_i \in V_{\delta_i}, \mathcal{F}_{\delta_i}$, there are $\eta \in (\delta_i, i = 1, 2, \ldots, m)$ and $\mathcal{F}_{\eta}$-measurable functions $y' = y'(\eta), y'' = y''(\eta)$ with $||y'||_q \leq c, ||y''||_q \leq c$ so that there exist $n_i, 1 = n_i, \eta_1, \ldots, \eta_{k_i} = \eta$ and $\mathcal{F}_{\eta_i}$-measurable functions $x_i = x_i(\eta), x_i'' = x_i''(\eta)$ satisfying for $i = 1, 2, \ldots, n$ and $j = n + 1, \ldots, m$

$$x_i''' = x_i(\eta) = x_i''' \text{ in } V, \ x_i' \leq \text{inf } L_1, \ x_i'' \geq c \text{ in } L_j,$$

$$\int_{L_1} x_i(\delta_1) \leq \int_{L_1} y' + \int_{L_1 - A_1} x_i'',$$

$$\int_{L_1} x_i(\delta_1) \leq \int_{L_1 + A_1} y'' - \int_{L_1 - A_1} x_i''.$$
\[
\sum_{i} x^{(i)} = \sum_{j} y^{(j)} + \sum_{j} x^{(j)}
\]

where \( A_{i} = \{k \leq k_{i} \mid x_{i,k} < b\} \) and \( B_{j} = \{k \leq k_{j} \mid x_{j,k} \leq -b\} \).

**Definition 3.** A stochastic process \((\mathbf{x}_{0}, \mathcal{F}_{0}, \Delta)\) is a martingale, if \((\mathcal{F}_{0}, \Delta)\) is a stochastic basis, \(x_{0}\) is integrable, and if for \(n \leq \mathbf{3}\)

\[ x(n, \mathcal{F}_{0}) = x_{0}, \text{ a.e., where } x(n, \mathcal{F}_{0}) = \text{Radon-Nikodym derivative of the integral of } x_{0} \text{ relative to } \mathcal{F}_{0}. \]

If \((\mathbf{x}_{0}, \mathcal{F}_{0}, \Delta)\) is a martingale and \(\sup_{\Delta} ||x_{0}||_{q} < k < \infty\), then the condition \((A; b)_{q}\) is satisfied for every \(b > 0\), by taking \(\eta \geq \delta_{i}\)

\(i = 1, 2, \ldots, m\), \(y' = y' \cdot x(\eta), \eta_{1,2} = \eta, x_{i} = \min \{x(\eta), b\}, x_{i}' = \max \{x(\eta), -b\}, \) and \(c = \max \{b, k\}\).

2. Martingale convergence theorems

**Theorem 1.** If \(1 \leq q < \infty, p^{-1} + q^{-1} = 1\) and \((\mathbf{x}_{0}, \mathcal{F}_{0}, \Delta)\) is a stochastic process satisfying the Vitali condition \(V_{q}\), then \(x_{0}\) converges a.e. where \(\sup_{\Delta} |x_{0}| < b\), provided \((A; b)_{p}\) is satisfied for some \(b > 0\).

**Proof.** Suppose that it is false and \(\delta_{0} \in \Delta\). Then there exist two real numbers \(a < d\) and a set \(V\) with \(P^{*}(V) > 0\) such that

\[
\sup_{\Delta} |x_{0}| < b, \limsup_{\Delta} x_{0} > d > a > \liminf_{\Delta} x_{0}
\]
on \(V\). Put

\[
K_{0} = V^{*}(x_{0} > d).
\]

Then \(\limsup K_{0} \supset V\). By the Vitali condition \(V_{q}\), for \(1 > \varepsilon > 0\) there exist \(\delta_{i} > \delta_{0}\) and \(L_{i} \in K_{0} \mathcal{F}_{0}, i = 1, \ldots, n\), such that

\[
\mathbb{P}^{*}(I - A) < \varepsilon, \left|\left|\sum_{1}^{n} I(L_{i}) - I(A)\right|\right|_{q} < \varepsilon,
\]

where \(A = \bigcup_{1}^{n} L_{i}\). Put
\[ (2.4) \quad H_0 = AV^*_0 \quad (x_0 < \varepsilon). \]

By \( V_q \) again, for \( \delta_0' > \delta_1' \geq 1, \ldots, n \), there exist \( \delta_j > \delta_0' \) and \( L_j \in H_{\delta_j} \) for \( j = n+1, \ldots, m \), such that

\[ (2.5) \quad P^*(AV - B) < \varepsilon, \quad \bigg| \sum_{m+1}^n P(L_j) - I(E) \bigg|_q < \varepsilon, \]

where \( B = \bigcup_{n+1}^m L_j \). By the condition \( (A,b)_p \), there exist \( c, \eta, y', y'' \), \( x'_i \)

\( \eta_i, k \quad (i = 1, \ldots, n; \quad k = 1, \ldots, k_i) \) satisfying the conditions in \( (A,b)_p \). For each \( i = 1, \ldots, n \), let \( s_i \) be the first \( k \leq k_i \) such that \( x(s_i, k) \geq b \) if there is one, and \( s_i = \infty \) otherwise. Then for \( i = 1, \ldots, n \)

\[ (2.6) \quad \int_{L_i} x^{(s_i)} \leq \int_{L_i} (s_i < \infty) y' + \int_{L_i} (s_i = \infty) x'_i, \]

\[ \quad \sum_{i=1}^n P(L_i) \leq \sum_{i=1}^n \int_{L_i} (s_i < \infty) y' + \sum_{i=1}^n \int_{L_i} (s_i = \infty) x'_i. \]

Choose \( \delta_0 \) so large such that \( P(V^*_0 - V^*_\delta) < \varepsilon \) for \( \delta > \delta_0 \). Then

\[ \sum_{i=1}^n \int_{L_i} (s_i = \infty) x'_i \leq c \sum_{i=1}^n P(L_i (s_i = \infty) - V^*_\eta) \quad - \quad c P(U L_i (s_i = \infty) - V^*_\eta) \]

\[ \quad + \quad c P(U L_i (s_i = \infty) - V^*_\eta) \]

\[ \leq c \left( \sum_{i=1}^n P(L_i) - P(A) \right) + c P(U L_i (s_i = \infty) - V^*_\eta) \]

\[ \leq c \varepsilon + c P(A - V^*_\eta) \leq c \varepsilon + c P(V^*_0 - V^*_\eta) < 2c \varepsilon. \]

Hence

\[ (2.7) \quad \sum_{i=1}^n \int_{L_i} (s_i = \infty) x'_i < 2c \varepsilon. \]
Since $q < \infty$, we can assume that $\delta_0$ is so large that $P(V_{\delta_0}^* - V_{\delta}^*) < \varepsilon^q$ for every $\delta > \delta_0$. Then

$$
(2.8) \quad \int_{V_{\delta_0}^*}^{V_{\delta}^*} |y'| \leq ||y'||_p \leq c \varepsilon.
$$

Put $D = \bigcup_{1}^{n} L_i(s_i < \infty)$. Since $V_{\eta}^* \subset (s_i = \infty)$ for each $i$ and $D \subset A \subset V_{\delta_0}^*$,

$$
\int_{D} y' \leq \int_{A \setminus V_{\eta}^*} |y'| \leq \int_{V_{\delta_0}^*}^{V_{\eta}^*} |y'| \leq c \varepsilon.
$$

By (2.3),

$$
\sum_{1}^{n} \int_{L_i} (s_i < \infty) y' - \int_{D} y' \leq ||\sum_{1}^{n} I(L_i) - I(A)||_q ||y'||_p < c \varepsilon.
$$

Hence

$$
(2.9) \quad \sum_{1}^{n} \int_{L_i} (s_i < \infty) y' < 2c \varepsilon.
$$

From (2.9), (2.7) and (2.6),

$$
(2.10) \quad d \sum_{1}^{n} P(L_i) \leq 4c \varepsilon + \sum_{1}^{n} \int_{L_i V_{\eta}^*} x_i'.
$$

Similarly,

$$
(2.11) \quad a \sum_{n+1}^{m} P(L_j) \geq -4c \varepsilon + \sum_{n+1}^{m} \int_{L_j V_{\eta}^*} x_j'.
$$

Put $L_i^i = L_i$ and $L_i^i = L_i \setminus \bigcup_{1}^{i-1} L_i^i$ for $i = 2, \ldots, n$ and $L_{n+1} = L_{n+1}$ and

$L_j^j = L_j \setminus \bigcup_{n+1}^{j-1} L_j^j$ for $j = n+2, \ldots, m$. Define $z' = x_i'$ on each $L_i^i$ and $z^j = x_j^j$ on each $L_j^j$. Then

$$
(2.12) \quad \sum_{1}^{n} \int_{L_i V_{\eta}^*} x_i' \leq \int_{A V_{\eta}^*} z' + c \left( \sum_{1}^{n} P(L_i) - P(A) \right) \leq \int_{A V_{\eta}^*} z' + c \varepsilon.
$$
Similarly,

\[(2.13)\]

\[\sum_{n=1}^{m} \int_{L_j} \eta^* \frac{\eta_j'}{\eta} \geq \int_{B_\eta^*} z'' \geq c \epsilon.\]

Hence

\[(2.14)\]

\[\int_{A\eta^*} z' - \int_{B\eta^*} z'' \leq c P[(A-B)\eta^*] \leq c P(A-B) \leq P^*\{A-B\} + \epsilon A - B < 2\epsilon.\]

From \((2.10)-(2.1k)\), we have

\[(2.15)\]

\[d \sum_{l=1}^{n} \ni \eta_l^* - a \sum_{n=1}^{m} \ni L_j < 12 \epsilon.\]

Thus we completed the proof.

Theorem 2. Let \((F_0, A)\) satisfy the Vitali condition \(V_q\) and \((x_0, F_0, A)\) be a martingale with \(\sup_{\Delta} ||x_0||_p < \infty\), where \(p > 1\) and \(p^{-1} + q^{-1} = 1\). Then \(x_0\) converges a.e.

Proof. For \(p = 1\), it follows immediately from Theorem 4.2 of [1] that \(\lim_{\Delta} x_0\) exists a.e., and for \(p > 1\) Theorem 1 states that \(\lim_{\Delta} x_0\) exists a.e. where both \(\limsup_{\Delta} x_0\) and \(\liminf_{\Delta} x_0\) are finite. Hence we need only to prove that under the conditions of Theorem 2, both \(\limsup_{\Delta} x_0\) and \(\liminf_{\Delta} x_0\) are finite a.e.

Assume that \(V = (\limsup_{\Delta} x_0 = \infty)\) and \(P^*(V) > a > 0\). Then by \(V_q\), for any \(0 < K < \infty, \epsilon > 0, \delta_0 \epsilon A,\) there exist \(\delta_1, \delta_2, \ldots, \delta_m\) and \(F_0\)-sets \(L_1 \subset [x_0 > K]\) such that \(\delta_1 > \delta_0\) and

\[(2.16)\]

\[P(A > a, ||\sum_{l=1}^{m} I(L_1) - I(A)||_q < \epsilon,\]

where \(A = \bigcup_{l=1}^{m} L_1\). Take \(\eta > \delta_1 (i = 1, 2, \ldots, m)\). Then
\[ K_a \leq \sum_{l}^{m} \int_{L_1} x(\delta_1) = \sum_{l}^{m} \int_{L_1} x(\eta) \leq \|\sum_{l}^{m} I(l) - I(A)\|_q \|x(\eta)\|_p + \|x(\eta)\|_p \]

\[ \leq (1 + \varepsilon) \|x(\eta)\|_p. \]

Hence we arrive at a contradiction and \( P(V) = 0 \). Similarly, \( P(\lim \inf_{\Delta} x_\delta = -\infty) = 0 \).

From the previous proofs, immediately we have:

Corollary 1. Both Theorems 1 and 2 hold, if we replace \( V_\mathcal{Q} \) by \( V_\mathcal{Q}^* \), sup by ess sup and convergence by essential convergence.

Corollary 1 completes a theorem due to Krickeberg [5, Theorem 3.5] on essential convergence of martingales of decreasing stochastic basis.

3. A convergence theorem of martingales generated by cell function.

Let \( \mathcal{G} \) be a family of \( \mathcal{F} \)-sets with positive measures. Each element in is called a cell. A partition of a set \( X \subset \Omega \) is a sequence of non-overlapping cells \( I_n \) with \( \bigcup_1^\infty I_n = X \) and any cell meets at most a finite number of \( I_n \).

For a family \( \mathcal{G}_f \) of cells, each cell in \( \mathcal{G}_f \) is called a \( \mathcal{F} \)-cell. \( A(\mathcal{G}_f) \) will be the union of all \( \mathcal{G} \)-cells, \( \mathcal{G}^u \) the family of cells which are finite unions of \( \mathcal{G}_f \)-cells, and for a set \( X, \mathcal{G}^u X \) is the family of all \( \mathcal{G}_f \)-cells which are subsets of \( X \). A complex \( \mathcal{K} \) is a finite family of non-overlapping cells. For a complex \( \mathcal{K} \), define \( P(\mathcal{K}) = P(A(\mathcal{K})) \). For two families \( \mathcal{G} \) and \( \mathcal{H} \) of cells, if \( \mathcal{G} \subset \mathcal{H}^u \), we say that \( \mathcal{H} \) refines \( \mathcal{G} \), or \( \mathcal{H} \) is \( \mathcal{G} \)-fine, denoted by \( \mathcal{G} < \mathcal{H} \).

For two complexes \( \mathcal{K} \) and \( \mathcal{K}^b \), \( \mathcal{K}^b \) is said to be a bordering complex of \( \mathcal{K} \), if every \( \mathcal{K} \)-cell is contained in some \( \mathcal{K}^b \)-cell and no \( \mathcal{K}^b \)-cell is contained in \( \mathcal{K}^u \) (or equivalently \( A(\mathcal{K}) \)). For a cell \( I \), a partition \( \mathcal{J} \) of \( I \) is said to be \( p \)-bordering \( (p > 1) \), if for each cell \( J \in \mathcal{J}^u \) and each complex \( \mathcal{K} \subset \mathcal{J}^u \) with \( A(\mathcal{K}) \neq J \), there exists a bordering complex \( \mathcal{K}^b \) of \( \mathcal{K} \) with \( \mathcal{K}^b \subset \mathcal{J}^u \) and \( P(\mathcal{K}^b) \leq p \ P(\mathcal{K}) \). \( \mathcal{J} \) will be said to have the \( p \)-bordering property, if to every cell \( I \) and every complex \( \mathcal{K} \) of subcells of \( I \), there corresponds a \( \mathcal{K} \)-fine \( p \)-bordering partition of \( I \).
Assume that the family $\Lambda$ of all partitions $\lambda$ of $\Omega$ forms a directed set with respect to the order $\rightarrow$ (refinement). For each $\lambda \in \Lambda$, let $\mathcal{F}_\lambda$ be the $\sigma$-algebra generated by the $\lambda$-cells.

Theorem 3. Let $(x_\lambda, \mathcal{F}_\lambda, \Lambda)$ be a martingale and $\mathcal{I}$ have the p-ordering property with $1 < p < \infty$. Let $B$ be an $\mathcal{F}_{\lambda_0}$-cell $V = \left[ \sup_{\lambda > \lambda_0} |x_\lambda| < b \right]$ for $0 < b < \infty$, and $c = 2pb$. For any given $\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots, \lambda_m$ in $\Lambda(\lambda > \lambda_1)$ and $\mathcal{F}_{\lambda_1}$-sets $I_1 \subset BV_{\lambda_1}$, there exists $\eta > \lambda_1$, $i = 1, 2, \ldots, m$ in $\Lambda$ such that

$$\sum_{I_1} x(\lambda_1) \leq cP[I_1(x(\eta) > b)] + \sum_{I_1} x(\lambda_1) \leq \eta b] x(i)(\eta), i=1,\ldots,n,$$

$$\sum_{I_1} x(\lambda_1) \geq -cP[I_1(x(\eta) < -b)] + \sum_{I_1} x(\lambda_1) \geq -\eta b] x(i)(\eta), j=n+1,\ldots,m,$$

where

$$x(i)(\eta) = x(\eta) = x(j)(\eta), \text{ if } \omega \in I_1 \eta, IV \neq \emptyset,$$

$$i = 1, \ldots, n; j = n + 1, \ldots, m,$$

$$x(i)(\eta) = c = -x(j)(\eta), \text{ if } \omega \in I_1 \eta, IV = \emptyset, i=1,\ldots,n;$$

$$j = n + 1,\ldots,m.$$

Proof. We can and will assume that each $I_1$ is an $\mathcal{F}_{\lambda_1}$-cell. Let $\eta'$ be a partition of $\Omega$ such that $\eta' > \lambda_1$, $i = 1, \ldots, n, \ldots, m$. Let $\mathcal{J} = \eta'B$. Then $\mathcal{J}$ is a complex and $I_1 \subset \mathcal{J}^u$ for each $i = 1, \ldots, m$. By the p-ordering property of $\mathcal{I}^l$, there exists a $\mathcal{J}$-fine, p-ordering partition $\delta$ of $B$. Put $\eta = \eta'(\Omega - B) \cup \delta$. Then, $\eta \in \Lambda$ and $\eta > \eta' > \lambda_1$, $i = 1, \ldots, m$. For each $i = 1, \ldots, n$, let $K_1 = \left[ I \mid I \subset \eta \lambda_1, IV = \emptyset \right]$. If $K_1 = \emptyset$, then
\[ \int_{L_1} x(\lambda_1) = \int_{L_1} x(\eta) = \int_{L_1} \{ |x(\eta)| < b \} x(\eta) = \int_{L_1} \{ |x(\eta)| < b \} x^{(1)}(\eta) \]

Hence \( \lambda_1 = L_1 \), then since \( L_1 \subseteq V \)

\[ \int_{L_1} x(\lambda_1) \leq \int_{L_1} x(\eta) \leq \int_{L_1} \{ |x(\eta)| \geq b \} + \int_{L_1} \{ |x(\eta)| < b \} x^{(1)}(\eta) \]

\[ \leq \left( \int_{L_1} x(\eta) \geq b \right) + \int_{L_1} \{ |x(\eta)| < b \} x^{(1)}(\eta) = \mathbb{P}[L_1(x(\eta) \geq b)] + \]

\[ + \int_{L_1} \{ |x(\eta)| < b \} x^{(1)}(\eta) \]

Now assume that \( \lambda_1 \neq 0 \) and \( \lambda_2 \neq 0 \). Since \( A(\lambda_1) \neq L_1 \) is \( 0 \) and \( |x(\eta)| \leq b \), by the \( 0 \)-covering property of \( \delta \), there exists a complex \( \lambda_2 \subseteq \delta \) such that every \( \lambda_1 \)-cell is contained in some \( \lambda_2 \)-cell, \( IV \neq 0 \) for every \( \lambda_1 \)-cell \( I \), and that \( P(\lambda_1) \leq P(\lambda_1) \). Hence

\[ \int A(\lambda_1) x(\eta) = \int A(\lambda_2) x(\eta) - \int A(\lambda_2) - A(\lambda_2) x(\eta) \]

\[ \leq bP(\lambda_1) + b[\mathbb{P}(\lambda_1) - \mathbb{P}(\lambda_1)] \leq 2bP(\lambda_1) \leq cP(\lambda_1). \]

Therefore

\[ \int_{L_1} x(\lambda_1) = \int_{L_1} x(\eta) = \int A(\lambda_1) x(\eta) + \int_{L_1} - A(\lambda_1) x(\eta) \]

\[ \leq cP(\lambda_1) = \int_{L_1} - A(\lambda_1) x(\eta) \]

\[ \leq cP \{ A(\lambda_1)(|x(\eta)| \geq b) \} + cP[A(\lambda_1)(|x(\eta)| < b)] + \]

\[ + \int_{L_1} A(\lambda_1) \{ |x(\eta)| < b \} x(\eta) \leq cP[L_1(|x(\eta)| < b)] \]
+ \int L_{\lambda} |x(\eta)| < b \} x^{(1)}(\eta).

Since by (3.4)

\int L_{\lambda} \{ x(\eta) \leq -b \} x^{(1)}(\eta) = c^2 \{ L_{\lambda} (x(\eta) \leq -b) \},

\int L_{\lambda} x(\lambda) \leq c^2 \{ L_{\lambda} (x(\eta) > b) \} + \int L_{\lambda} \{ x(\eta) < b \} x^{(1)}(\eta).

Similarly we can prove (3.2).

Theorem 4. Let \( (x_{\lambda}, \bigvee_{\lambda}, \Lambda) \) be a martingale satisfying the weak Vitali condition \( V_{\Lambda} \) and \( \Lambda \) have the p-bordering property with \( 1 < p < \infty \). Then \( x_{\Lambda} \) converges a.e. where \( \sup_{\omega} |x_{\lambda}| < \infty \).

Proof. Theorem 3 states that \( (x_{\lambda}, \bigvee_{\lambda}, \Lambda) \) satisfies the condition \( (A, b)_\infty \) for every \( b > 0 \). Therefore, Theorem 4 follows from Theorem 1 immediately.

Theorem 3 includes Theorem II of Rutovitz' [7], which in turn (See [7, p.29]) includes a theorem of Ward [8, p.141].
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