On the Non-Central Multivariate Beta Distribution and the Moments of Traces of Some Matrices

By K. C. Sreedharan Pillai

Purdue University

1. Introduction and summary. Let $A_1$ and $A_2$ be two positive definite symmetric matrices of order $p$, $A_1$ having a Wishart distribution \([3, 15]\) with $f_1$ degrees of freedom and $A_2$ having an independent non-central Wishart distribution with $f_2$ degrees of freedom, corresponding to the linear case \([1, 2]\). Now let

$$A_1 = C L C'$$

where $C$ is a lower triangular matrix such that

$$A_1 + A_2 = C C'$$

It has been shown \([6]\) that the density function of $L$ is given by

$$f(L) = Ke^{-\lambda^2/2} \frac{1}{\Gamma(f_1 f_2/2)} \frac{1}{\lambda^2 (1-f_{11})} |L|^{(f_1 - p - 1)/2} |I - L|^{(f_2 - p - 1)/2}$$

\[1.1\]
\[ K = \pi^{-(p+1)/4} \prod_{i=1}^{p} \Gamma\left[\frac{1}{3}(f_1^i + f_2^i + 1 - i)\right] / \Gamma\left[\frac{1}{3}(f_1^i + 1 - i)\right] \Gamma\left[\frac{1}{3}(f_2^i + 1 - i)\right]. \]

\( \lambda^2 \) is the single non-centrality parameter in the linear case, \( \lambda_{11} \) is the element in the top left corner of the \( L \) matrix, and \( _2F_1 \) denotes the confluent hypergeometric function.

In this paper, the density function of \( L \) given by (1.1) has been observed to be a product of density functions of \( p(p+1)/2 \) independent beta variables, explicit expressions for these variables being given for \( p = 2, 3, 4 \) and 5. In view of the independence of the beta variables, it has been shown how the moments of the trace of \( L \) (say \( W^{(p)} \)) and of \( I-L \) (say \( V^{(p)} \), which is actually Pillai's \( V^{(s)} \) criterion with \( s = p \) [8]) can be computed from those of the beta variables. Again, if we denote the characteristic roots of \( I-L \) by \( \theta_1 (i = 1, 2, \ldots, p) \), a method has been given for computing the moments of \( U^{(2)} = \sum_{i=1}^{2} [\theta_1 / (1-\theta_1)] = \sum_{i=1}^{2} \lambda_1 \) (a constant times Hotelling's \( \tau^2 \), \( s = 2 \), [8]), also from those of the independent beta variables. The case of \( p = 2 \) has been considered in detail, deriving the first four moments of \( W^{(2)}, V^{(2)} \) and \( U^{(2)} \) and suggesting approximate distributions for them.

In addition, for tests of the hypothesis: \( H_0: \lambda = 0 \) against \( H_1: \lambda > 0 \) based on the three criteria \( V^{(2)}, U^{(2)} \) and Wilks' criterion \( \Lambda = \prod_{i=1}^{2} (1-\theta_i) \), [16] comparison of power functions has been carried out for different values of \( f_1 \) and \( f_2 \) using the moments of these criteria. Further, such comparison has been extended to include also Roy's largest root criterion in testing.
the hypothesis $H_0: \rho = 0$ against $H_1: \rho > 0$ where $\rho$ is the single non-null population canonical correlation coefficient.

2. Independent beta variables. Let

$$L = T T'$$

where $T$ is a lower triangular matrix $[t_{ij}]$. It has been shown [4] that then the diagonal elements $t_{ii}$ are independently distributed and that $t_{ii}^2 (i = 2, 3, \ldots, p)$ follows the distribution

$$f_1(t_{ii}^2) = (t_{ii}^2)^{(r_{11} + 1 - i) - 1} (1 - t_{ii}^2)^{(r_{22} - 1 - i) - 1} / \beta(\frac{1}{2}(r_{11} + 1 - i), \frac{1}{2}r_{22})$$

$$0 \leq t_{ii}^2 \leq 1,$$

while $t_{11}^2$ is distributed as

$$f_1(t_{11}^2) = e^{-\frac{t_{11}^2}{2}} (t_{11}^2)^{-\frac{r_{11}}{2} - 1} (1 - t_{11}^2)^{\frac{r_{22}}{2} - 1} f_1(\frac{1}{2}(r_{11} + r_{22}), \frac{1}{2}r_{22}, \frac{1}{2}r_{11}) / \beta(\frac{1}{2}r_{11}, \frac{1}{2}r_{22})$$

$$0 \leq t_{11}^2 \leq 1.$$

(i) $p = 2$. Now, if $p = 2$, it can be shown that

$$f(t_{11}, t_{22}, t_{21}) = f_1(u_{11})f_2(u_{22})f_{21}(u_{21})$$

where

$$u_{11} = t_{11}^2, u_{22} = t_{22}^2 \text{ and } u_{21} = t_{21}^2 / (1 - t_{11}^2)(1 - t_{22}^2)$$

$f_1(u_{11})$ is given by (2.2), $f_2(u_{22})$ by (2.1) with $i = 2,$
\[ f_{21}(u_{21}) = u_{21}^{1-u_{21}}(1-u_{21})^{u_{21}-1} / \beta\left(1, \frac{1}{2}(f_2-1), \right), \ 0 \leq u_{21} \leq 1. \]

Thus, from (2.3) it may be seen that \( u_{11}, u_{22} \) and \( u_{21} \) are independently distributed.

(ii) \( p = 3 \). When \( p = 3 \), it can be shown that

\[ f_3(\lambda_{11}, \lambda_{22}, \lambda_{33}, \lambda_{21}, \lambda_{32}, \lambda_{31}) = f_2(u_{11}) f_2(u_{22}) f_3(u_{33}) f_{21}(u_{21}) f_{21}(u_{32}) f_{31}(v_{31}) \]

where \( u \)'s are defined in a similar manner as in (2.4), \( v_{31} \) is defined by

\[ v_{31} = \left(\sqrt{u_{31}} + \sqrt{u_{21} u_{32} u_{22}}\right)^2 / \left[(1-u_{21})(1-u_{32})\right]. \]

\( f_1(u_{11}) \) follows (2.2), \( f_i(u_{ii})(i = 2, 3) \) is given in (2.1), \( f_{21}(u_{21}) \) and \( f_{21}(u_{32}) \) both follow the form as in (2.5) and \( f_{31}(v_{31}) \) is given by

\[ f_{31}(v_{31}) = v_{31}^{1-v_{31}}(1-v_{31})^{v_{31}-1} / \beta\left(1, \frac{1}{2}(f_2-2), \right), \ 0 \leq v_{31} \leq 1. \]

(iii) \( p = 4 \). Now, if \( p = 4 \),

\[ f(L) = \prod_{i=1}^{4} f_i(u_{ii}) f_{21}(u_{21}) f_{21}(u_{32}) f_{21}(u_{43}) f_{31}(v_{31}) f_{31}(v_{42}) f_{41}(v_{41}). \]
where \( u \)'s are similarly defined as before, \( v_{31} \) is given in (2.7), \( v_{42} \) is given by

\[
(2.9) \quad v_{42} = \left( \sqrt{u_{42}} + \sqrt{u_{43}u_{32}} \right)^2 / \left[ (1-u_{32})(1-u_{43}) \right],
\]

\[
(2.10) \quad v_{41} = \left( \sqrt{v_{41}} + \sqrt{v_{31}z_{42}} \right)^2 / \left[ (1-v_{31})(1-v_{42}) \right],
\]

where

\[
(2.11) \quad v_{41} = \left( \sqrt{u_{41}} + \sqrt{u_{42}u_{22}} \right)^2 / \left[ (1-u_{21})(1-u_{43}) \right],
\]

\[
(2.12) \quad z_{42} = \left( \sqrt{u_{42}u_{32}} + \sqrt{u_{43}u_{32}} \right)^2 / \left[ (1-u_{32})(1-u_{43}) \right],
\]

and where \( f_1(u_{11}) \) as before is given by (2.2), \( f_i(u_{11})(i = 2, 3, 4) \) by (2.1), \( f_{21}(u_{21}), f_{21}(u_{32}) \) and \( f_{21}(u_{43}) \) follow the form (2.5), \( f_{31}(v_{31}) \) and \( f_{31}(v_{42}) \) follow the form (2.3) and

\[
(2.13) \quad f_{41}(v_{41}) = \left. \frac{1}{2} - 1 \right| \left. v_{41} \right| ^{\frac{1}{2}} \frac{1}{2} (f_2 - 3) - 1 / \left. (\frac{1}{2}, \frac{1}{2} (f_2 - 3)) \right| (0 \leq v_{41} \leq 1).
\]

(iv) \( p = 5 \). When \( p = 5 \),

\[
(2.14) \quad f(L) = \left[ \prod_{i=1}^{5} f_i(u_{11}) \right] \left[ \prod_{i=1}^{4} f_{21}(u_{1,i+1}) \right] \left[ \prod_{i=1}^{3} f_{31}(v_{1,i+2}) \right] x \left[ \prod_{i=1}^{2} f_{41}(v_{1,i+3}) \right] f_{51}(x_{51})
\]
where \( u \)'s are defined as before, \( v_{31} \) and \( v_{42} \) are given by (2.7) and (2.9) respectively \( v_{53} \) is given by

\[
(2.15) \quad v_{53} = \frac{\sqrt{u_{53} + u_{43} u_{54}}}{\sqrt{(1-u_{43})(1-u_{54})}}^2
\]

\( w_{41} \) is defined in (2.10), \( w_{52} \) is given by

\[
(2.16) \quad w_{52} = \frac{\sqrt{v_{52} + v_{42} z_{53}}}{(1-v_{42})(1-v_{53})}^2
\]

where

\[
(2.17) \quad v_{52} = \frac{\sqrt{u_{52} + u_{32} u_{53}}}{\sqrt{(1-u_{32})(1-u_{53})}}^2
\]

\[
(2.18) \quad z_{53} = \frac{\sqrt{u_{53} u_{43} + u_{54} u_{44}}}{\sqrt{(1-u_{43})(1-u_{54})}}^2
\]

and where \( x_{51} \) is given by

\[
(2.19) \quad x_{51} = \frac{\left[ (\sqrt{v_{51} + v_{31} z_{52}})(1-v_{42}) + \sqrt{v_{41} + v_{31} z_{42}}\sqrt{v_{42} v_{52}} + z_{53} \right]^2}{(1-v_{31})(1-v_{42})(1-v_{53})(1-v_{41})(1-v_{52})}
\]

and where

\[
(2.20) \quad v_{51} = \frac{\sqrt{u_{51} + u_{21} u_{52} u_{22}}}{\sqrt{(1-u_{21})(1-u_{54})}}^2
\]

\[
(2.21) \quad z_{52} = \frac{\sqrt{u_{52} u_{32} + u_{53} u_{33}}}{\sqrt{(1-u_{32})(1-u_{54})}}^2
\]
Here again \( f_{1}(u_{11}) \) is given by (2.2), \( f_{1}(u_{11})(i = 2, 3, 4, 5) \) is given by (2.1), \( f_{21}(u_{21}), f_{21}(u_{22}), f_{21}(u_{43}) \) and \( f_{21}(u_{54}) \) follow the form (2.5), \( f_{31}(v_{31}), f_{31}(v_{42}) \) and \( f_{31}(v_{53}) \) follow the form (2.8), \( f_{41}(w_{41}) \) and \( f_{41}(w_{52}) \) follow the form (2.13) and \( f_{51}(x_{51}) \) is given by

\[
(2.22) \quad f_{51}(x_{51}) = x_{51}^{\frac{1}{2}-1} (1-x_{51})^{\frac{1}{2}(f_{2}^{2}-4)-1} /\beta(\frac{1}{2}, \frac{1}{2}(f_{2}^{2}-4)), \quad 0 \leq x_{51} \leq 1.
\]

(v) **General case \((p)\).** In this subsection, for convenience, let us relabel \( u_{i+1,i}^{(2)} \) as \( u_{i+1,i}^{(2)} \), \( i = 1, 2, \ldots, p-1; \) \( v_{i+2,i}^{(2)} \) as \( u_{i+2,i}^{(3)} \), \( i = 1, \ldots, p-2; \) \( w_{i+3,i}^{(4)} \) as \( u_{i+3,i}^{(4)} \), \( i = 1, \ldots, p-3; \) \( x_{i+4,i}^{(5)} \) as \( u_{i+4,i}^{(5)} \), \( i = 1, \ldots, p-4; \) etc. Now from (2.4),

\[
(2.23) \quad 1 - u_{21}^{(2)} = |I - L|(p=2)/[(1-u_{11})(1-u_{22})],
\]

where \( L = TT' \). Further, \( u_{32}^{(2)} \) is obtained from \( u_{21}^{(2)} \) by adding simultaneously unity to both suffixes of each of the t's involved in \( u_{21}^{(2)} \), which is reflected in the notation 32 which replaces 21. Similarly \( u_{43}^{(2)} \) is obtained from \( u_{32}^{(2)}, u_{54}^{(2)} \) from \( u_{43}^{(2)} \) etc. Again,

\[
(2.24) \quad 1 - u_{31}^{(3)} = \frac{|I - L|(p = 3)}{(1-u_{11})(1-u_{22})(1-u_{33})(1-u_{21})(1-u_{32})}.
\]
Further, \( u_{31}^{(3)} \) is obtained from \( u_{31}^{(3)} \) by increasing as before both suffixes in each of the t's in \( u_{31}^{(3)} \) by unity. Similarly \( u_{53}^{(3)} \) is obtained from \( u_{42}^{(3)} \) etc. Following this pattern, it is easy to see that

\[
(2.25) \quad 1 - u^{(p)}_{pl} = \frac{|I - L|}{\prod_{i=1}^{p-1} \left(1 - u_{i+1, i}^{(2)}\right) \prod_{i=1}^{p-2} \left(1 - u_{i+2, i}^{(3)}\right) \cdots \prod_{i=1}^{p-1} \left(1 - u_{i+p-2, i}^{(p-1)}\right)} .
\]

Hence it may be seen that in the case of \( p \) variables *

\[
(2.26) \quad f(L) = \prod_{i=1}^{p} f_i(u_{i i}) \prod_{i=1}^{p-1} f_{2 i}(u_{i i+1, i}) \times \prod_{i=1}^{p-2} f_{3 i}(u_{i i+2, i}) \cdots \prod_{i=1}^{p-1} f_{p-1, i}(u_{i i+p-2, i}) f_{pl}(u_{pl}) ,
\]

where

\[
(2.27) \quad f_{j l}(u_{i+j-1, i}) = (u_{i+j-1, i})^{\frac{1}{2} - 1} (1 - u_{i+j-1, i})^{\frac{1}{2} - (f_2 - j + 1) - 1} / \beta\left(\frac{1}{2}, \frac{1}{2}(f_2 - j + 1)\right) ,
\]

\[0 \leq u_{i+j-1, i} \leq 1, \quad j = 2, 3, \ldots, p.
\]

Further, it may be noted that \( K \) in (1.1) equals

\[
(2.28) \quad \prod_{i=1}^{p} \beta\left(\frac{1}{2}(f_1 + 1 - i), \frac{1}{2}(f_2)\right) \beta\left(\frac{1}{2}, \frac{1}{2}(f_2 - i)\right)^{p - i} .
\]

*Since this paper was written, a theorem was proved to establish this. (See Khatri, C.G. and Pillai, K.C.S. (1965) "Some Results on the Non-Central Multivariate Beta Distribution and Moments of Traces of Two Matrices", Ann. Math. Statist., 36, October Issue.)
3. Traces of some matrices as functions of independent beta variables.

First, consider the trace of \( L \) when \( p = 2 \). Noting that

\[
(3.1) \quad \lambda_{11} + \lambda_{22} = t_{11} + t_{22} + t_{21}^2
\]

and using (2.4) we get

\[
(3.2) \quad W^{(2)} = \lambda_{11} + \lambda_{22} = u_{11} + u_{22} + u_{21}(1-u_{11})(1-u_{22}) .
\]

Similarly

\[
(3.3) \quad v^{(2)} = 2 - W^{(2)} = (1-u_{11}) + (1-u_{22}) - u_{21}(1-u_{11})(1-u_{22}) .
\]

When \( p = 3 \),

\[
(3.4) \quad W^{(3)} = u_{11} + u_{22} + u_{33} + u_{21}(1-u_{11})(1-u_{22}) + u_{32}(1-u_{22})(1-u_{33})
\]

\[
\quad + (1-u_{11})(1-u_{33})[v_{31}(1-u_{21})(1-u_{32}) + u_{21}u_{23}u_{32}
\]

\[
\quad - 2\sqrt{v_{31}(1-u_{21})(1-u_{32})u_{21}u_{23}u_{32}}
\]

and \( v^{(3)} = 3 - W^{(3)} \).

Similarly, \( W^{(4)} \), \( v^{(4)} \), and \( W^{(5)} \) and \( v^{(5)} \) can be expressed explicitly as functions of independent beta variables.
Now consider \( u^{(2)} = \sum_{i=1}^{2} \lambda_i \). It may be seen that

\[
(3.5) \quad u^{(2)} = \sum_{i=1}^{2} \left[ \lambda_i / (1-\theta_i) \right] = \left[ (1-\theta_1) + (1-\theta_2) \right] / [(1-\theta_1)(1-\theta_2)] \cdot -2.
\]

Noting that \( (1-\theta_1) + (1-\theta_2) = W^{(2)} \) and \( (1-\theta_1)(1-\theta_2) = u_{11}u_{22} \) we get

\[
(3.6) \quad u^{(2)} = \frac{1-u_{11}}{u_{11}} + \frac{1-u_{22}}{u_{22}} + \frac{u_{21}(1-u_{11})(1-u_{22})}{u_{11}u_{22}}.
\]

4. Moments of \( W^{(2)}, V^{(2)} \) and \( U^{(2)} \). The first four moments of \( W^{(2)} \) will be given by

\[
(4.1) \quad \mu_1(W^{(2)}) = \left\{ 2f_1 e^{-\lambda^2/2}/(v-1) \right\} \sum_{i=0}^{\infty} a_i \left( \frac{1}{2\lambda^2} \right)^i / i!
\]

where

\[
(4.2) \quad a_i = (v+i-1)/\sigma_1
\]

\[ v = (f_1+f_2) \quad \text{and} \quad \sigma_1 = v + 2i. \]

\[
(4.3) \quad \mu_2(W^{(2)}) = \left\{ 4f_1 e^{-\lambda^2/2}/(v^2-1) \right\} \sum_{i=0}^{\infty} b_i \left( \frac{1}{2\lambda^2} \right)^i / i!
\]

where
\begin{align*}
\tag{4.4} b_1 &= \left[ f_1 v^2 + (i+1) f_1^2 + (i^2+3i-1) f_1 + (2i+3) f_1 f_2 + f_2^2 + (2i-1) f_2 + 2(i^2-1) \right] / e_0,
\end{align*}

where \( e_0 = g_1 (g_1 + 2) \).

\begin{align*}
\tag{4.5} \mu_1^{(\eta(2))} &= \left[ 8 f_1 e^{-\lambda^2/2}/\{(v^2-1)(v+3)\} \right] \sum_{i=0}^{\infty} c_i (\lambda^2)^i / i!,
\end{align*}

where

\begin{align*}
\tag{4.6} c_i &= e_1 / e_2,
\end{align*}

and where

\begin{align*}
e_1 &= f_1 v^2 + (3i+9) f_1^2 + (6i+21) f_1 f_2^2 + (3i+15) f_2^2 + (3i^2+21i+25) f_1^2 \\
&+ (3i^2+30i+41) f_1 f_2^2 + (i^3+18i^2+44i+15) f_2^2 + (9i+18) f_1 f_2 \\
&+ 3 f_1 f_2^2 + 2 f_2^3 + (12i^2+39i+9) f_1 f_2 + 6if_2^2 \\
&+ (6i^3+30i^2+18i-26) f_1 + (12i^2+6i-26) f_2 + 8i^3+12i^2-26i-24,
\end{align*}

and

\begin{align*}
e_2 &= g_1 (g_1 + 2) (g_1 + 4).
\end{align*}
(4.7) \[ \mu_n^4(\nu^2) = [\sqrt{\nu^2 - 1}/2/[(\nu^2 - 1)(\nu^2)(\nu+3)]\sum_{i=0}^{\infty} a_i (\nu^2)^i/i! \]

where

(4.8) \[ d_i = e_3/e_4 \]

and where

\[ e_3 = (\nu+5)[(f_1+2)(f_1+4)(\nu+3)(\nu+5)f_1^2 + 4f_1g_1 + 23f_1 + 6\nu - 4g_1 - 30] \]

\[ + 4(f_1+2)(\nu+3)h_1(f_1^3 + 3f_1g_1 + 19f_1 + 4\nu - 3g_1 - 14) \]

\[ + 2(f_1^2 - 1)(g_1 + 4)(g_1 + 6)(2f_1g_1 + 3f_1^2 + 6\nu + 6g_1 + 30) \]

\[ + 12(f_1 - 1)(g_1 + 6)h_1(f_1^2 + 4f_1g_1 + 3h_1 + 10) \]

\[ + 6h_1(h_1 + 2)(3f_1^2 + 9f_1 + 6\nu + 10h_1 + 58)] \]

\[ + (g_1 + 5)[(f_1 - 1)(g_1 + 6)(f_1^2 + h_1 + 4)(f_1)(g_1 + 5) + 12h_1 + 6] + 45h_1(h_1 + 2)] \]

\[ + 105h_1(h_1 + 2)(h_1 + 4)] \]

and

\[ e_4 = g_1(g_1 + 2)(g_1 + 4)(g_1 + 6) \]

and where

\[ h_1 = \nu^2 + 2i \]
It may be observed that the moments of $V(2)$ can be obtained from those of $W(2)$ using the relation $V(2) = 2W(2)$, which is given in terms of the $u$'s in (3.3).

Now consider the moments of $U(2)$. From (3.6)

\[(4.9) \quad U(2) = z_1 + z_2 + z_1z_2u_{22}\]

where $z_1 = (1-u_{11})/u_{11}$ and $z_2 = (1-u_{22})/u_{22}$.

From (2.2) we get

\[(4.10) \quad f(z_1) = e^{-\lambda^2/2}z_1^{\frac{1}{2}f_2-1}I_1\left\{v,\frac{1}{2},\frac{1}{2}z_1/(1+z_1)\right\}/\left\{1+z_1\right\}^{\frac{v}{2}}\beta\left(\frac{1}{2}f_2,\frac{1}{2}f_1\right)\].

Similarly from (2.1),

\[(4.11) \quad f(z_2) = z_2^{\frac{1}{2}f_2-1}/\left\{(1+z_2)^{\frac{1}{2}(v-1)}\beta\left(\frac{1}{2}f_2,\frac{1}{2}(f_1-1)\right)\right\}.

Now using (4.10), (4.11) and (2.5) we obtain the first four moments of $U(2)$ as follows:

\[(4.12) \quad \mu_i(U(2)) = [2 e^{-\lambda^2/2}/(f_1-3)] \sum_{i=0}^{\infty} (f_2+1)(\frac{1}{2}\lambda^2)^i/i!,

= (2f_2+\lambda^2)/(f_1-3).

Similarly
\[(4.13) \quad \mu_2^*(u^{(2)}) = \left[ \lambda^4(f_1-2)+4(\lambda^2+f_2)(f_1+f_2(f_1-3)-1) \right] / \left[(f_1-2)(f_1-3)(f_1-5)\right], \]

\[(4.14) \quad \mu_3^*(u^{(2)}) = \left[ \lambda^6(f_1-2)+3\lambda^4\left[(f_1-2)(f_2+4)+f_2(f_1-6)+4\right] \right. \]
\[\quad + (3\lambda^2+2f_2)(f_2+2)(f_1-2)(f_2+4) \]
\[\left. + 3(f_2(f_1-6)+4) \right] / \left[(f_1-2)(f_1-3)(f_1-5)(f_1-7)\right] , \]

and

\[(4.15) \quad \mu_4^*(u^{(2)}) = \left[ \lambda^8b+\lambda^6\left[(12+s_1)b+6\right] + \lambda^4\left[(28+6s_1+s_2)b+12(f_2+4)B+6A\right] \right. \]
\[\quad + \lambda^2\left[(8+4s_1+2s_2+s_3)b+16(f_2+2)(f_2+4)B+12(f_2+2)A\right] \]
\[\left. + 2f_2(f_2+2)(f_2+6)b \right] \]
\[+ 4(2f_2+4)(3+3A)] / [(f_1-2)(f_1-3)(f_1-4)(f_1-5)(f_1-7)(f_1-9)] \]

where \(s_i\) is the \(i\)th \((i=1,2,3)\) elementary symmetric function in the arguments \(f_2, f_2+2, f_2+4\) and \(f_2+6,\)

\[A = f_2^2(f_1-6)(f_1-8)+2f_2(f_1-4)(f_1-6)+16f_1-72 , \]

\[B = (f_1-4)(f_2(f_1-8)+6) \quad \text{and} \quad b = (f_1-2)(f_1-4) . \]

It may be observed that when \(\lambda = 0\), the moments given in this section reduce to those obtained by Pillai [8], [9], [10], [11] .
5. Approximations to the distributions of $W^{(2)}$, $V^{(2)}$ and $U^{(2)}$.

On the basis of the moments presented in the preceding section, the following approximation to the distribution of $W^{(2)}$ is suggested for small values of $\lambda$:

\[(5.1) \quad g_1(W^{(2)}) = (W^{(2)})^{p_1-1} (1-W^{(2)}/2)^{q_1-1} / [2^{p_1} \beta(p_1, q_1)] , \quad 0 < W^{(2)} < 2 , \]

where

\[p_1 = [(2K_1-K_2)K_1] / [2(K_2-K_1)] , \]

\[q_1 = [(2-K_1)(2K_1-K_2)] / [2(K_2-K_1)] , \]

where

\[K_1 = 2f_1[1-(\lambda^2/2)/(\nu+2)] / \nu \]

and

\[K_2 = 4f_1[\gamma_{\nu+\nu-2}[1-\lambda^2/(\nu+4)] / ([\nu-1] \nu(\nu+2)) . \]

A comparison of the lower order moments from (5.1) with the respective exact ones may be made from Table 1.

Since $V^{(2)} = 2-W^{(2)}$, an approximation to the distribution of $V^{(2)}$ can be obtained from (5.1) in the following form:

\[(5.2) \quad g_2(V^{(2)}) = (V^{(2)})^{q_1-1} (1-V^{(2)}/2)^{p_1-1} / [2^{q_1} \beta(q_1, p_1)] , \quad 0 < V^{(2)} < 2 . \]
Table 1

Moments (central) of $W^{(2)}$ from the exact and approximate distributions for different values of $f_1$ and $f_2$ and $\lambda = 2$.

<table>
<thead>
<tr>
<th>Moments</th>
<th>$f_1 = 10$</th>
<th>$f_2 = 5$</th>
<th>Ratio (A/E)</th>
<th>$f_1 = 100$</th>
<th>$f_2 = 5$</th>
<th>Ratio (A/E)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_1$</td>
<td>1.2134</td>
<td>1.1765</td>
<td>.9696</td>
<td>1.8708</td>
<td>1.8692</td>
<td>.9991</td>
</tr>
<tr>
<td>$\mu_2$</td>
<td>0.0506</td>
<td>0.0578</td>
<td>1.1404</td>
<td>0.02269</td>
<td>0.0284</td>
<td>1.0560</td>
</tr>
<tr>
<td>$\mu_3$</td>
<td>-0.02151</td>
<td>-0.0229</td>
<td>1.5230</td>
<td>-0.0419</td>
<td>-0.03113</td>
<td>1.2319</td>
</tr>
<tr>
<td>$\mu_4$</td>
<td>0.02731</td>
<td>0.02907</td>
<td>1.2405</td>
<td>0.04255</td>
<td>0.04303</td>
<td>1.1900</td>
</tr>
<tr>
<td>$\sqrt{\mu_2}$</td>
<td>0.2250</td>
<td>0.2403</td>
<td>1.0679</td>
<td>0.0518</td>
<td>0.0533</td>
<td>1.0276</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>0.0175</td>
<td>0.0273</td>
<td>1.5639</td>
<td>0.4350</td>
<td>0.5605</td>
<td>1.2886</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>2.8510</td>
<td>2.7192</td>
<td>0.9538</td>
<td>3.5265</td>
<td>3.7632</td>
<td>1.0671</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Moments</th>
<th>$f_1 = 20$</th>
<th>$f_2 = 20$</th>
<th>Ratio (A/E)</th>
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<th>$f_2 = 80$</th>
<th>Ratio (A/E)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_1$</td>
<td>0.9575</td>
<td>0.9524</td>
<td>.9947</td>
<td>1.0992</td>
<td>1.0989</td>
<td>.9997</td>
</tr>
<tr>
<td>$\mu_2$</td>
<td>0.0229</td>
<td>0.0232</td>
<td>1.0103</td>
<td>0.025419</td>
<td>0.025425</td>
<td>1.0012</td>
</tr>
<tr>
<td>$\mu_3$</td>
<td>0.03112</td>
<td>0.03100</td>
<td>0.8899</td>
<td>-0.04109</td>
<td>-0.04117</td>
<td>1.0722</td>
</tr>
<tr>
<td>$\mu_4$</td>
<td>0.02151</td>
<td>0.02154</td>
<td>1.0148</td>
<td>0.04872</td>
<td>0.04874</td>
<td>1.0020</td>
</tr>
<tr>
<td>$\sqrt{\mu_2}$</td>
<td>0.1514</td>
<td>0.1522</td>
<td>1.0052</td>
<td>0.0736</td>
<td>0.0737</td>
<td>1.0006</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>0.02105</td>
<td>0.0286</td>
<td>0.7678</td>
<td>0.03748</td>
<td>0.03856</td>
<td>1.1456</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>2.8849</td>
<td>2.8681</td>
<td>0.9942</td>
<td>2.9695</td>
<td>2.9688</td>
<td>0.9997</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Moments</th>
<th>$f_1 = 5$</th>
<th>$f_2 = 20$</th>
<th>Ratio (A/E)</th>
<th>$f_1 = 5$</th>
<th>$f_2 = 100$</th>
<th>Ratio (A/E)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_1$</td>
<td>0.3751</td>
<td>0.3704</td>
<td>0.9875</td>
<td>0.0935</td>
<td>0.0935</td>
<td>0.9991</td>
</tr>
<tr>
<td>$\mu_2$</td>
<td>0.0208</td>
<td>0.0203</td>
<td>0.9782</td>
<td>0.02162</td>
<td>0.02162</td>
<td>0.9976</td>
</tr>
<tr>
<td>$\mu_3$</td>
<td>0.02160</td>
<td>0.02167</td>
<td>1.0437</td>
<td>0.04522</td>
<td>0.04530</td>
<td>1.0136</td>
</tr>
<tr>
<td>$\mu_4$</td>
<td>0.02139</td>
<td>0.02136</td>
<td>0.9866</td>
<td>0.04102</td>
<td>0.04103</td>
<td>1.0075</td>
</tr>
<tr>
<td>$\sqrt{\mu_2}$</td>
<td>0.1442</td>
<td>0.1426</td>
<td>0.9890</td>
<td>0.0403</td>
<td>0.0403</td>
<td>0.9988</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>0.2848</td>
<td>0.3314</td>
<td>1.1637</td>
<td>0.6368</td>
<td>0.6591</td>
<td>1.0350</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>3.2123</td>
<td>3.2920</td>
<td>1.0248</td>
<td>3.8780</td>
<td>3.9262</td>
<td>1.0124</td>
</tr>
</tbody>
</table>
Again, consider $U^{(2)}$. An approximation to the distribution of $U^{(2)}$ for $f_1 > f_2$ and which is good even for very small values of $f_2$ is given below:

\[
(5.3) \quad g_3(U^{(2)}) = (U^{(2)})^{\frac{p_2-1}{p_2+q_2+1}} \left( \frac{1+U^{(2)}}{K_3} \right)^{p_2+q_2+1} K_3 P_3(p_2, q_2+1),
\]

\[0 < U^{(2)} < \infty,\]

where

\[p_2 = \frac{2q_2}{q_2(h-1)-2h}\]
\[q_2 = 2[c^2(f_1-5)h-(c+d)^2(f_1-3)]/[c^2(f_1-5)(h+1)-2(c+d)^2(f_1-3)]\]
\[K_3 = c[q_2(h-1)-2h]/[2(f_1-3)]\]
\[h = (c+1.99d)^3(f_1-3)/[(c+d)^2(f_1-7)c]\]
\[c = 2f_2+h^2 \quad \text{and} \quad d = (f_1-f_2-1)/(f_1-2).\]

A comparison of the moments from (5.3) with the respective exact ones may be made from Table 2.

6. Power functions of tests of hypothesis: $\lambda = 0$ against $\lambda > 0$ based on $V^{(2)}$, $U^{(2)}$ and $\Lambda$. Using the results on the moments of $W^{(2)}$ in section 4, and the relation $V^{(2)} = 2W^{(2)}$, the central moments $\mu_2$, $\mu_3$, and $\mu_4$, and the moment quotients $\beta_1$ and $\beta_2$ were computed for various values of $f_1$, $f_2$, and $\lambda$. Similar computations were made for $U^{(2)}$ and Wilks' criterion, using the expressions in section 4 for the
Table 2

Moments (central) of $U^{(2)}$ from the exact and approximate distributions for different values of $f_1 > f_2$ and $\lambda = 1, 3, \text{ and } 5$

<table>
<thead>
<tr>
<th>Moments</th>
<th>$f_1 = 10, f_2 = 2, \lambda = 1$</th>
<th>$f_1 = 15, f_2 = 5, \lambda = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Exact</td>
<td>Approximate</td>
</tr>
<tr>
<td>$\mu_1$</td>
<td>0.7143</td>
<td>0.7143</td>
</tr>
<tr>
<td>$\mu_2$</td>
<td>0.5041</td>
<td>0.4760</td>
</tr>
<tr>
<td>$\mu_3$</td>
<td>1.5792</td>
<td>1.5333</td>
</tr>
<tr>
<td>$\mu_4$</td>
<td>25.7893</td>
<td>27.0736</td>
</tr>
<tr>
<td>$\sqrt{\mu_2}$</td>
<td>0.7100</td>
<td>0.6899</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>19.4703</td>
<td>21.8049</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>101.4935</td>
<td>119.5121</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Moments</th>
<th>$f_1 = 50, f_2 = 10, \lambda = 1$</th>
<th>$f_1 = 100, f_2 = 10, \lambda = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Exact</td>
<td>Approximate</td>
</tr>
<tr>
<td>$\mu_1$</td>
<td>0.4468</td>
<td>0.4468</td>
</tr>
<tr>
<td>$\mu_2$</td>
<td>0.0258</td>
<td>0.0253</td>
</tr>
<tr>
<td>$\mu_3$</td>
<td>0.0^{2}373</td>
<td>0.0^{2}407</td>
</tr>
<tr>
<td>$\mu_4$</td>
<td>0.0^{2}296</td>
<td>0.0^{2}314</td>
</tr>
<tr>
<td>$\sqrt{\mu_2}$</td>
<td>0.1605</td>
<td>0.1591</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>0.8126</td>
<td>1.0207</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>4.4566</td>
<td>4.9032</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Moments</th>
<th>$f_1 = 100, f_2 = 20, \lambda = 3$</th>
<th>$f_1 = 100, f_2 = 20, \lambda = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Exact</td>
<td>Approximate</td>
</tr>
<tr>
<td>$\mu_1$</td>
<td>0.5052</td>
<td>0.5052</td>
</tr>
<tr>
<td>$\mu_2$</td>
<td>0.0155</td>
<td>0.0140</td>
</tr>
<tr>
<td>$\mu_3$</td>
<td>0.0^{2}116</td>
<td>0.0^{2}106</td>
</tr>
<tr>
<td>$\mu_4$</td>
<td>0.0^{2}874</td>
<td>0.0^{2}736</td>
</tr>
<tr>
<td>$\sqrt{\mu_2}$</td>
<td>0.1246</td>
<td>0.1184</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>0.3578</td>
<td>0.4111</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>3.6295</td>
<td>3.7458</td>
</tr>
</tbody>
</table>
moments of the former, and deriving the expressions for the moments of
the latter as the product of the respective moments of \( u_{11} \) and \( u_{22} \).

For a given size \( \alpha \), using the \( \beta_1 \) and \( \beta_2 \) values computed for
fixed \( f_1 \) and \( f_2 \) for \( \lambda = 0 \), the critical region was determined for
each criterion referring to tables of "Percentage points of Pearson
curves for \( \beta_1 \) and \( \beta_2 \) expressed in standardized measure" [7]
Further, for the same values of \( f_1 \) and \( f_2 \) and a value of \( \lambda > 0 \),
the computed values of \( \beta_1 \) and \( \beta_2 \) were used to determine from the
same table by interpolation the power of the test based on the critical
region determined previously. The following table presents the results
of these computations.

Table 3

Powers of tests of hypothesis: \( \lambda = 0 \) against \( \lambda > 0 \)

<table>
<thead>
<tr>
<th>( f_1 )</th>
<th>( f_2 )</th>
<th>( \alpha )</th>
<th>( \lambda )</th>
<th>( V(2) )</th>
<th>( Y(2) )</th>
<th>Power</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>10</td>
<td>.005</td>
<td>1</td>
<td>.0076</td>
<td>.0076</td>
<td>.0113</td>
</tr>
<tr>
<td>50</td>
<td>10</td>
<td>.005</td>
<td>2</td>
<td>.0215</td>
<td>.0217</td>
<td>.0470</td>
</tr>
<tr>
<td>100</td>
<td>10</td>
<td>.01</td>
<td>1</td>
<td>.0156</td>
<td>.0156</td>
<td>.0217</td>
</tr>
<tr>
<td>100</td>
<td>20</td>
<td>.025</td>
<td>1</td>
<td>.0306</td>
<td>.0306</td>
<td>.0345</td>
</tr>
<tr>
<td>100</td>
<td>30</td>
<td>.025</td>
<td>1</td>
<td>.0300</td>
<td>.0300</td>
<td>.0303</td>
</tr>
<tr>
<td>100</td>
<td>50</td>
<td>.025</td>
<td>1</td>
<td>.0273</td>
<td>.0277</td>
<td>.0280</td>
</tr>
<tr>
<td>100</td>
<td>100</td>
<td>.025</td>
<td>1</td>
<td>.0270</td>
<td>.0270</td>
<td>.0270</td>
</tr>
<tr>
<td>50</td>
<td>50</td>
<td>.005</td>
<td>1</td>
<td>.0057</td>
<td>.0056</td>
<td>.0056</td>
</tr>
<tr>
<td>50</td>
<td>50</td>
<td>.005</td>
<td>2</td>
<td>.0085</td>
<td>.0083</td>
<td>.0085</td>
</tr>
<tr>
<td>50</td>
<td>50</td>
<td>.005</td>
<td>3</td>
<td>.0151</td>
<td>.0151</td>
<td>.0153</td>
</tr>
</tbody>
</table>
Table 3 shows that a) there is practically very little difference between the powers of tests based on $Y(2)$ and $y(2)$ and b) for small values of $f_2$ Wilks' criterion seems to have marked power compared to both $Y(2)$ and $y(2)$. This point needs further investigation.

7. Power functions for tests of hypothesis: $\rho = 0$ against $\rho > 0$ based on $Y(2)$, $y(2)$, $\Lambda$ and the largest root. In the case of relation between a $p$-set of variates, $X' = (x_1', \ldots, x_p')$, and a $q$-set, $Y' = (y_1, \ldots, y_q')$, from a $(p+q)$-variate normal population, where there is only one non-null population canonical correlation coefficient, $\rho$, and $p \leq q$, $(p+q) < n'$ where $n'$ is the sample size,

$$
\lambda^2 = \rho^2 \sum_{t=1}^{\nu} y_{1t}^2 / (1-\rho^2)
$$

where $y_{1t}(t=1,\ldots,\nu)$ are related to the sample observations of $Y_1$ and $Y$, here, is considered fixed [6]. Further, $f_2 = q$ and $f_1 = n'-q-1$ such that $\nu = f_1 + f_2$. If, however, $Y$ is not fixed, then

$$
\sum_{t=1}^{\nu} y_{1t}^2 \text{ in } \lambda^2 \text{ of (7.1) is a chi-square with } \nu \text{ degrees of freedom and, therefore, for obtaining the moments of } Y^{(2)} \text{ in this case the following changes may be made in the moments of } Y^{(2)} \text{ given in section 4:}
$$

$$
(7.2) \quad e^{-\lambda^2} \longrightarrow (1-\rho^2)^{\nu/2}, \quad (\lambda^2)^i \longrightarrow (\rho^2)^i \quad \text{and}
$$

$$(a_i, b_i, c_i, d_i) \longrightarrow [\nu(\nu+2) \ldots (\nu+2(i-1))] (a_i, b_i, c_i, d_i) .$$
Similar changes apply for Wilks' criterion. But for \( U_j^{(2)} (\lambda^2) \) is replaced by \((2\rho^2/(1-\rho^2))^{1/2} \nu \nu/(\nu^2+1)/\Gamma(\nu)/\Gamma(\nu/2)\). Now for the test of the hypothesis: \( \rho = 0 \) against \( \rho > 0 \) using \( \nu \nu \), \( U \nu \) and \( \Lambda \), powers were evaluated for \( \rho = .05 \) and \( \rho = .1 \) for certain values of \( \nu \nu \) and \( \nu \nu \nu \) using the method discussed in the foregone section. For the largest root, the power was computed using Constantine's form of the distribution of the canonical correlation coefficients [4], [5] in the following manner:

First the joint distribution for \( \rho = 2 \) and a single nonzero \( \rho \) was obtained as a series of determinants using a lemma by Pillai [12]. Further taking into account the first seven terms of the series and integrating out the smallest root by employing Pillai's method [8, 10], the following expression was obtained for the cdf of the largest canonical correlation coefficient, \( r_2 \).

\[
P_r[r_2^2 \leq x] = K_2 \left\{ \sum_{j=0}^{6} (B_j x^{6-j}/(m+n+8-j)) \right\}
- x^6 I(x; m+1, n) \sum_{j=0}^{4} (C_j x^{4-j}/(m+n+7-j)) \right\}
- x^2 I(x; m+2, n) \sum_{j=0}^{2} (D_j x^{2-j}/(m+n+6-j)) \right\}
- x^3 I(x; m+3, n) E_0/(m+n+5) \right]
\[ +2I(x;2m+7,2n+1) \left\{ B_0)/(m+n+8)\right\} - \left\{ C_0)/(m+n+7)\right\} - \left\{ D_0)/(m+n+6)\right\} - \left\{ E_0)/(m+n+5)\right\} \]

\[ +2I(x;2m+6,2n+1) \left\{ B_1)/(m+n+7)\right\} - \left\{ C_1)/(m+n+6)\right\} - \left\{ D_1)/(m+n+5)\right\} \]

\[ +2I(x;2m+5,2n+1) \left\{ B_2)/(m+n+6)\right\} - \left\{ C_2)/(m+n+5)\right\} - \left\{ D_2)/(m+n+4)\right\} \]

\[ +2I(x;2m+4,2n+1) \left\{ B_3)/(m+n+5)\right\} - \left\{ C_3)/(m+n+4)\right\} \]

\[ +2I(x;2m+3,2n+1) \left\{ B_4)/(m+n+4)\right\} - \left\{ C_4)/(m+n+3)\right\} \]

\[ +2I(x;2m+2,2n+1) \left\{ B_5)/(m+n+3)\right\} + 2I(x;2m+1,2n+1) \left\{ B_6)/(m+n+2)\right\} \]

where

\[ f_1 = 2n+3, \quad f_2 = 2m+3, \]

\[ K_m = (1 - \rho^2)^{\nu/2} \cdot C(2, m, n), \]

\[ C(2, m, n) = \Gamma(2m+2n+5)/[4\Gamma(2m+2)\Gamma(2n+2)], \]

\[ I_0(m+1,n+1) = \int_0^\infty x^{m-1}(1-x)^{n-1} d\theta = \int_0^\infty e^{(1-x)d'} d'd, \]

\[ B_0 = 231A_3, B_1 = 63A_5 + (m+7)B_0/(m+n+8), B_2 = 35A_4 + (m+6)B_1/(m+n+7), \]

\[ B_3 = 5A_3 + (m+5)B_2/(m+n+6), B_4 = 3A_2 + (m+4)B_3/(m+n+5), \]

\[ B_5 = A_1 + (m+3)B_4/(m+n+4), B_6 = 1 + (m+2)B_5/(m+n+3), \]

\[ C_0 = 105A_6, C_1 = 28A_5 + (m+6)C_0/(m+n+7), C_2 = 15A_4 + (m+5)C_1/(m+n+6), \]

\[ C_3 = 2A_3 + (m+4)C_2/(m+n+5), C_4 = A_2 + (m+3)C_3/(m+n+4), \]

\[ D_0 = 21A_6, D_1 = 5A_5 + (m+5)D_0/(m+n+6), D_2 = 2A_4 + (m+4)D_1/(m+n+5), \]
\[ E_0 = 5A_6, \quad A_1 = \sqrt{2} \rho^2/2f_2, \]
\[ A_2 = \left[\sqrt{\nu(\nu+2)}\right]^2 \rho^4 / \left[ f_2(f_2+2)^2 \right], \]
\[ A_3 = A_2(\nu+4)^2 \rho^2 / [(f_2+4)2.31], \]
\[ A_4 = A_3(\nu+6)^2 \rho^2 / [f_2^6(f_2+6)], \]
\[ A_5 = A_4(\nu+8)^2 \rho^2 / [2^2.5(f_2+8)], \]
\[ A_6 = A_5(\nu+10)^2 \rho^2 / [2^3.6(f_2+10)]. \]

For \( \rho = 0 \), upper 1% points of the largest root were taken from Pillai’s tables [11] for values of \( m = 2 \) and 5 and \( n = 10, 15, 20, 25, 30, 40 \) and 60. Using these \( x_{.99} \) values to determine the critical region, the powers of the largest root test were computed for \( \rho = .05 \) and \( \rho = .1 \) for values of \( m \) and \( n \) given above. These are shown in Table 4.

Table 4

Powers of the largest root test for testing \( \rho = 0 \)
against \( \rho = .05 \) and \( \rho = .1 \) and \( \alpha = .01 \)

<table>
<thead>
<tr>
<th>n</th>
<th>( \rho = .05 )</th>
<th>( \rho = .05 )</th>
<th>( \rho = .1 )</th>
<th>( \rho = .1 )</th>
</tr>
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<td>( m = 5 )</td>
<td>( m = 2 )</td>
<td>( m = 5 )</td>
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<td>.011248</td>
<td>.019072</td>
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Now, a comparison of the powers of the test of hypothesis: 
\( \rho = 0 \) against \( \rho > 0 \) based on \( V^{(2)} \), \( U^{(2)} \), \( \Lambda \) and the largest root 
may be made from Table 5.

Table 5

<table>
<thead>
<tr>
<th>Power</th>
<th>( \rho = .05 )</th>
<th>( \rho = .1 )</th>
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</thead>
<tbody>
<tr>
<td>( f_1 )</td>
<td>( f_2 = 7 )</td>
<td>( f_2 = 13 )</td>
</tr>
<tr>
<td>( V^{(2)} )</td>
<td>( U^{(2)} )</td>
<td>( \Lambda )</td>
</tr>
<tr>
<td>53</td>
<td>.0103</td>
<td>.0107</td>
</tr>
<tr>
<td>83</td>
<td>.0115</td>
<td>.0115</td>
</tr>
<tr>
<td>123</td>
<td>.0123</td>
<td>.0120</td>
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<tr>
<td>53</td>
<td>.0135</td>
<td>.0135</td>
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<tr>
<td>83</td>
<td>.0165</td>
<td>.0165</td>
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<tr>
<td>123</td>
<td>.0202</td>
<td>.0200</td>
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</tbody>
</table>

Table 5 shows that a) the largest root has comparatively less power than the other test criteria b) \( V^{(2)} \) and \( U^{(2)} \) practically have equal power and c) Wilks' criterion as in the previous case seems to have greater power for the (small) values of \( f_2 \) considered here. Further investigation is being made to clear this point.

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REFERENCES


