On the Distribution of Linear Functions and Ratios of Linear Functions of Ordered Correlated Normal Random Variables with Emphasis on Range

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0. Summary

This paper deals with the distribution of linear functions and ratios of linear functions of order statistics from an equally correlated set of normal random variables. Some special cases of this family of linear functions are considered. The case of range is studied in great detail both for the case of equal correlations as well as in the general case. For the general case, formulae for the distribution of range are obtained for sample sizes of two, three and four. In the equally correlated case, expressions are obtained for the probability integral, percentage points and moments of the linear functions in terms of the corresponding expressions for the uncorrelated cases. It is also known that ratios of certain linear functions of the order statistics have a distribution independent of this common correlation.

1. Introduction

Let $X_1, X_2, \ldots, X_n$ be jointly normally distributed random variables with $EX_i = 0, EX_i^2 = 1, EX_iX_j = \rho_{ij}, (i \neq j = 1, 2, \ldots, n)$. Let $X_{(k)}$ be the kth order statistic when $X_i$'s are arranged in an increasing order as follows:

\[(1.1) \quad X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(1)} \leq \cdots \leq X_{(n)}.
\]

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The distribution of these order statistics is of interest, for instance, in problems of multiple decisions and life testing. Owen and Steck (1962) have obtained the moments of $X\langle k \rangle$ when $\rho_{ij} = \rho$, all $i, j$. Gupta (1963a) discussed the evaluation of the probability integral of $X\langle n \rangle$, and has given tables of this integral for selected values of $\rho_{ij} = \rho$. In a separate bibliography Gupta (1963b) gives references to other papers in this area. The papers by Stuart (1958), Ruben (1961), Steck and Owen (1962), Steck (1962), and Thigpen (1962) are of more direct interest.

In this paper, we shall consider a class of linear functions of $X\langle i \rangle$ given by $Y = \sum_{i=1}^{n} a_{i}X\langle i \rangle$. This class covers a number of useful statistics that arise in various problems of statistical inference. For instance, when $a_{1} = -1$, $a_{n} = 1$ and $a_{2} = a_{3} = \ldots = a_{n-1} = 0$, the above linear function of order statistics reduces to the range of the $n$ random variables $X_{i}$. We discuss the latter case in great detail. Another special case is that of the Nair statistic (1943) which is discussed briefly. When $\rho_{ij} = \rho$, we use the reduction first given by Dunnett and Sobel (1955) [see also Stuart (1958)] to obtain the moments of $Y$ and to derive formulae for the probability integral and the percentage points of $Y$.

We also consider statistics of the form $Y_{1}/Y_{2}$ which are used in rejection of outliers and as a substitute for the $F$-statistic in testing equality of variances.

In Section 2, we discuss the distribution of $Y$ for the equicorrelated case. Section 3 deals with moments of $Y$. In Section 4, we discuss the Studentized case when/variables have a common unknown variance. In Section 5, we discuss the distribution of certain ratios of $Y$'s. In Section 6, we discuss the distribution of the range, a particular case
of $Y$. For $n = 2, 3, \text{ and } 4$, the distribution of the range is obtained in the general case when the correlations are not necessarily equal.

2. Distribution of $Y$ when $X_i$'s are equally correlated

If $X_i$'s are equally correlated, then it is well-known that this common $\rho \geq -\frac{1}{n-1}$. Let us consider the cases $\rho \geq 0$ and $-\frac{1}{n-1} \leq \rho < 0$, separately. Following Owen and Steck (1962b), we generate the random variables \{X_i\} from the random variables \{Z_i\} as follows.

\[
(2.1) \quad X_i = (\rho)^{1/2} Z_0 + (1 - \rho)^{1/2} Z_i, \quad \rho \geq 0, \quad i = 1, 2, \ldots, n
\]

where $Z_0, \ldots, Z_n$ are ind. $N(0,1)$ and
(2.2) \( X_i = (-\rho)^{1/2} Z_0 + (1-\rho)^{1/2} Z_i \), \( \rho < 0, \ i = 1, 2, \ldots, n \)
where \( Z_i (i = 1, 2, \ldots, n) \) are ind. \( N(0,1) \), \( Z_0 \) is \( N(0,1) \) and
\[ E(Z_0 Z_i) = (-\rho)^{1/2} / (1-\rho)^{1/2}. \]

It follows from (2.1) and (2.2) that the linear function \( Y = \Sigma a_i X_i \) can
be expressed in terms of the corresponding linear function \( \Sigma a_i Z_i \) (\( Z_i \)'s uncorrelated) and a term involving \( Z_0 \). Hence, we have for \( \rho \geq 0 \),
\[
(2.3) \quad P\{Y \leq y\} = P\left\{ \sum_{i=1}^{n} a_i Z_i \leq -\left[\frac{-\rho}{1-\rho}\right]^{1/2} \left( \sum_{i=1}^{n} a_i \right) Z_0 + \frac{y}{(1-\rho)^{1/2}} \right\}
\]
and for \( \rho < 0 \)
\[
(2.4) \quad P\{Y \leq y\} = P\left\{ \sum_{i=1}^{n} a_i Z_i \leq -\left[\frac{-\rho}{1-\rho}\right]^{1/2} \left( \sum_{i=1}^{n} a_i \right) Z_0 + \frac{y}{(1-\rho)^{1/2}} \right\}.
\]

From (2.3) we obtain by putting \( \left[\frac{-\rho}{1-\rho}\right]^{1/2} = c \)
\[
(2.5) \quad P\{Y \leq y\} = H(y; \rho) = \int_{-\infty}^{\infty} H\left( -cx(Z) + \frac{y}{(1-\rho)^{1/2}} ; 0 \right) \phi(x) dx
\]
where \( \phi(x) \) denotes the density of a standard normal random variable.

It is interesting to note that if \( \sum_{i=1}^{n} a_i = 0 \), then for both negative and positive \( \rho \), we have one result, namely,
\[
(2.6) \quad H(y; \rho) = H\left( \frac{y}{(1-\rho)^{1/2}} ; 0 \right).
\]

Using (2.6) we can evaluate the cdf of \( Z^i X_i(1) \) when \( \sum a_i = 0 \), a case of common
practical interest. We mention the following four cases.

(a) \( a_1 = -1; \ a_n = 1; \ a_i = 0, \ i = 2, 3, \ldots, n-1. \)
In this case the cdf of the range of correlated random variables reduces to the
already well-studied case (see Pearson and Hartley (1942), Gumbel (1949),
Pillai (1952), Harter (1960)) of the range of uncorrelated variables. Letting
\( y_{\alpha} (n; \rho) \) denote the \( \alpha \) percentage points of the range in the correlated case, we see that
(2.7) \[ y_a(n; \rho) = y_a(n; 0)/(1 - \rho)^{1/2}. \]

(b) Quasi Range, a particular case when \( a_i \) and \( a_{n-i+1} \) \( (i = 2, 3, \ldots, \lfloor \frac{n}{2} \rfloor) \) are equal to \(-1\) and \(+1\), respectively, and the remaining \( a_i \)'s are equal to zero. The formulae (2.6) and (2.7), again, give the distribution of the correlated case.

(c) Nair's statistic (Nair, 1948) is obtained by putting \( a_n = 1 - \frac{1}{n} \) and \( a_1 = a_2 = \cdots = a_{n-1} = -\frac{1}{n} \). Thus formulae (2.6) and (2.7) are valid for reducing the distribution of Nair Statistic from the correlated to the uncorrelated case, the latter being tabulated by Nair (1948), Pillai (1959) and by others.

(d) Tests for outliers based on \( X(n) - X(n-1) \) (see Grubbs (1950)) or similar differences of successive order statistics. These are again special cases of

\[ Y = \sum_{i=1}^{n} a_i X(i) \]

and can be solved by the same formulae (2.6) and (2.7).

3. Moments of \( Y \)

Owen and Steck (1962) have given formulae and tables for the moments and cumulants of \( X(k) \) in terms of the corresponding moments and cumulants of \( X(k) \) and \( Z_0 \). More directly, from (2.1) and (2.2) we can write

\[
(3.1) \quad X(k) = (\pm \rho)^{1/2} Z_0 + (1 - \rho)^{1/2} Z(k)
\]

where in the first term on the right in (3.1) we take + or - sign according as \( \rho \geq 0 \) or \( \rho < 0 \).

It follows then

\[
(3.2) \quad E(Y_j) = E \left[ \left( \sum_{i} a_i \right) \left( \pm \rho \right)^{j/2} Z_0 + (1 - \rho)^{j/2} \sum_{i=1}^{n} a_i Z(i) \right]^{j/2} \]

\[
= \sum_{p=0}^{j} \binom{j}{p} (\pm \rho)^{p/2} (1 - \rho)^{(j-p)/2} E \left[ Z_0^p \left( \sum_{i} a_i Z(i) \right)^{j-p} \right].
\]
If $p \geq 0$, then we get

\[(3.3) \quad E(Y^j) = \sum_{a=0}^{[\frac{j}{2}]} (\sum_{i=1}^{n} a_i)^{2a} (1-p)^{a} (2a+1)_{a} \frac{(2a)!}{a!2^a} E(\sum_{i=1}^{n} Z(i))^{j-2a} \frac{(2a)!}{a!2^a} E(\sum_{i=1}^{n} Z(i))^{j-2a} \frac{(2a)!}{a!2^a}
\]

where $[x]$ is the largest integer less than or equal to $x$.

In (3.3), one can substitute from the tables (Ruben (1954), Eicker (1956), Harter (1961)) the expected values of the powers of the $i$th order statistic in a random sample of size $n$ from a $N(0,1)$. Note that $\sum_{i=1}^{n} a_i = 0$.

\[(3.4) \quad E(Y^j) = (1-p)^{j/2} \sum_{i=1}^{n} E[Z(i)]^j; \quad \sum_{i=1}^{n} a_i = 0; \quad \text{all } p.
\]

In particular, for the moments of range $W_n$, we get

\[(3.5) \quad E(W_n^j) = 2(1-p)^{j/2} E[Z(n) - Z(1)]^j
\]

and, generally,

\[(3.6) \quad E(W_n^j) = (1-p)^{j/2} E[Z(n) - Z(1)]^j
\]

Note that

\[(3.7) \quad EX = (1-p)^{1/2} \sum_{i=1}^{n} a_i E[Z(i)], \quad \text{all } p.
\]

4. **Studentized Case**

If $X_i$ are assumed to be equally correlated normal random variables with $EX_i = 0$, $EX_i^2 = \sigma^2$ ($i = 1, 2, \ldots, n$), and if $s^2_v$ is an estimate of $\sigma^2$, which is distributed as $\sigma^2 \chi^2/v$ and which is independent of $\sum_{i=1}^{n} a_i X(i)$, then we can write

\[(4.1) \quad P \left( \frac{\sum_{i=1}^{n} a_i X(i)}{s_v} \leq y' \right) = \int_{0}^{y'} H(y'; x, \rho) g_v(x)dx
\]

where $g_v(x)$ is the density function for the random variable $\chi_v$ and

where $H(y'; \rho)$ is the integral defined by (2.5). If $Ea_i = 0$, then we have
(4.2) \( P \left\{ \sum_{i=1}^{n} a_i^2 X(i) / s_v \leq y \right\} = \int_{\mathbb{V \backslash \{Y \}}^{0}} \mathcal{G}_v(x) \, dx, \quad \Xi a_i = 0. \)

It follows from (4.2) that the percentage point \( y'_\alpha (n, \nu; \rho) \) of the studentized correlated case is related to the percentage point \( y'_\alpha (n, \nu; 0) \) of the studentized uncorrelated case by

\[
y'_\alpha (n, \nu; \rho) = y'_\alpha (n, \nu; 0)/(1-\rho)^{1/2},
\]

which is analogous result to (2.7).

We remark that the moments of \( \Xi a_i X(i) / s_v \) will exist only if the negative moments of \( s_v \) exist. We have

\[
(4.4) \quad E(\Xi a_i X(i) / s_v)^j = E(\Xi a_i X(i) / \sigma)^j \cdot E(s_v / \sigma)^{-j} \quad \text{if} \quad j < \nu
\]

\[
= E(\Xi a_i X(i) / \sigma)^j \frac{\Gamma(\frac{\nu-1}{2})(\frac{\nu}{2})^{3/2}}{\Gamma(\frac{\nu}{2})}, \quad j < \nu.
\]

It should be pointed out that in (4.4) the first factor in the second line of (4.4) can be evaluated by using formulae derived in Section 3. In particular we can obtain the moments of the studentized range, studentized Nair statistic and studentized quasi ranges by using the formula (4.4) and the formulae of Section 3. The probability integral and percentage points for these particular cases can be evaluated by using (4.1) - (4.3).

5. Distribution of the range of correlated normal variables

The joint distribution of \( X(1), X(2), \ldots, X(n) \) defined in (1.1) can be written in the form

\[
(5.1) \quad f(X_1, X_2, \ldots, X_n) = \frac{n! e^{-X'A^{-1}X}}{(2\pi)^{n/2} |A|^{1/2}}
\]

where \( X \) is the column vector of the \( X(i) \)'s and \( A \), the matrix of simple correlation coefficients between pairs of unordered \( X \)'s and \( |A| \), the determinant of \( A \).
5. Distribution of $Y_1/Y_2$ when $X_i$'s are equally correlated

Let $W = X(n) - X(1)$, and let $W_1, W_2, \ldots, W_k$ be independently, identically distributed variables with the same distribution as $W$.

We shall show that the random variables

$$R = \left( X(h) - X(1) \right) / \left( X(j) - X(k) \right), \quad (h \neq i, j \neq k)$$

$$S = \max \frac{W_i}{W_1 + \ldots + W_k}$$

$$T = \frac{W_1}{W_2}$$

all have distributions independent of $\rho$. We shall also find the distribution of $T$ in the more general case where $W_i (i = 1, 2)$ is the range in an equicorrelated sample of size $n_i$ with $\rho = \rho_i$.

In the case $\rho = 0$, these random variables have been discussed by several authors. Statistics of the form of $R$ are discussed and their percentage points tabulated by Dixon (1951) in connection with testing the consistency of suspected observations with the whole sample. The statistic $S$ is suggested and the 5% points tabulated by Bliss, Cochran and Tukey (1956) for testing the same hypothesis in a different situation. For more discussion of these statistics see Dixon (1962). Link (1950) suggested the use of $T$ as an alternative to the usual F-test for equality of variances and partially tabulated its percentage points. More recently, Pillai and Buenaventura (1961) and Harter (1963) have provided more extensive tabulation of the percentage points.
The distribution of $R$ is independent of $\rho$ since
\[ P(R \leq r | \rho) = P(X_{(i)} - rX_{(j)} + rX_{(k)} + \leq 0 | \rho) = P(\{X_{(i)} - rX_{(j)} + rX_{(k)} \} \cup (1 - \rho) \leq 0 | 0) = P(R \leq r | 0) \]
by (2.6).

The distribution of $S$ is determined by the distribution of the maximum of $k$ identically distributed variables of the form $U_k = W_1/(W_2 + \ldots + W_k)$ (See Bliss, Cochran, and Tukey (1956)), and we now show that the distribution of $U_k$ is independent of $\rho$. The characteristic function of $W_i$ is
\[ \phi_{W_i}(t | \rho) = \phi_{W_i}(t/(1 - \rho) | 0) \] as seen above. Consequently, the joint characteristic function of the $\{W_i\}$ is
\[ \phi_{W_1, W_2, \ldots, W_n}(t_1, t_2, \ldots, t_n | \rho) = \prod \phi_{W_i}(t_i | \rho) = \prod \phi_{W_i/(1-\rho)}(t_i | 0). \] (5.1)

Since the distribution of $W_1/(W_2 + \ldots + W_k)$ and $aW_1/(aW_2 + \ldots + aW_k)$ are the same it follows from (5.1) that the distribution of $U_k$ is independent of $\rho$.

The distribution of $T$ in the more general case is argued as follows. The joint characteristic function of $W_1$ and $W_2$ is
\[ \phi_{W_1, W_2}(t, u | \rho_1, \rho_2, n_1, n_2) = \phi_{W_1/(1-\rho_1), W_2/(1-\rho_2)}(t, u | 0, 0, n_1, n_2). \]
This implies that
\[ P(W_1/W_2 \leq r | \rho_1, \rho_2, n_1, n_2) = P(W_1/W_2 \leq r/(1-\rho_2)/(1-\rho_1)) | 0, 0, n_1, n_2). \]
Thus the distribution of $W_1/W_2$ in equicorrelated samples can be found easily from the distribution in independent samples.
distribution of the range of correlated normal random variables

Let \( X_1, X_2, X_3, X_4 \) be normally distributed variables with zero means, unit variances and \( \text{EX}_iX_j = \rho_{ij} (i \neq j) \). Also, let

\[
W = \max X_i - \min X_i,
\]

\[
a_{ij} = \sqrt{2(1 - \rho_{ij})},
\]

\[
\Delta_{ijk} = 1 - \rho_{ij}^2 - \rho_{ik}^2 - \rho_{jk}^2 + 2 \rho_{ij}\rho_{ik}\rho_{jk}.
\]

The distribution of \( W \) is given by

\[(6.1) \quad P(W \leq w) = P(W \leq w \text{ and } X_1 < X_2 < X_3 < X_4) + \ldots + \text{ (24 terms).} \]

The other 23 terms are obtained from the first by permuting the indices of the \( \{X_i\} \). The first term, call it \( P_1 \), is

\[(6.2) \quad P_1 = P(X_1-X_2 \leq 0, X_2-X_3 \leq 0, X_3-X_4 \leq 0, 0 \leq X_4-X_1 \leq w) \]

\[= P(U_1 \leq 0, U_2 \leq 0, U_3 \leq 0, 0 \leq U_4 \leq w/a_{14}), \]

where

\[
U_1 = (X_1-X_2)/a_{12}, \quad U_2 = (X_2-X_3)/a_{23}, \quad U_3 = (X_3-X_4)/a_{34}, \quad U_4 = (X_4-X_1)/a_{14}.
\]

The \( \{U_i\} \) have zero means and unit variances, and if \( \xi_{ij} = EU_iU_j \),

then \( \xi_{12} = -(1 + \rho_{13} - \rho_{12} - \rho_{23})/a_{12}a_{23} \)

\( \xi_{13} = -(\rho_{23} + \rho_{14} - \rho_{13} - \rho_{24})/a_{12}a_{34} \)

\( \xi_{14} = -(1 + \rho_{24} - \rho_{12} - \rho_{14})/a_{12}a_{14} \)

\( \xi_{23} = -(1 + \rho_{24} - \rho_{23} - \rho_{34})/a_{23}a_{34} \)

\( \xi_{24} = -(\rho_{12} + \rho_{34} - \rho_{13} - \rho_{24})/a_{23}a_{14} \)

\( \xi_{34} = -(1 + \rho_{13} - \rho_{14} - \rho_{34})/a_{14}a_{34}. \)
Note that \( a_{12}U_1 + a_{23}U_2 + a_{34}U_3 + a_{14}U_4 = 0 \). This implies, since the \( \{a_{ij}\} \) are nonnegative, that \( P(U_1 \leq 0, U_2 \leq 0, U_3 \leq 0, U_4 \leq 0) = 0 \). Noting this, and substituting for \( U_4 \) in (6.2) gives
\[ P_1 = P(U_1 \leq 0, U_2 \leq 0, U_3 \leq 0, a_{12}U_1 + a_{23}U_2 + a_{34}U_3 \leq -w). \]

The variables \( \{U_i\} \) can be generated from independent \( N(0,1) \) variables \( \{X_i\} \) by
\[ U_1 = X_1 \]
\[ U_2 = \xi_{12}X_1 + \sqrt{1 - \xi_{12}^2}X_2 \]
\[ U_3 = \xi_{13}X_1 + \left( \frac{\xi_{23} - \xi_{12}\xi_{13}}{\sqrt{1 - \xi_{12}^2}} \right) X_2 + \frac{\sqrt{\Delta_{123}}}{(1 - \xi_{12}^2)} X_3, \]
and in terms of the \( \{X_i\} \) the plane \( a_{12}U_1 + a_{23}U_2 + a_{34}U_3 + w = 0 \) can be shown to be
\[ (6.3)- \xi_{14}X_1 - \left( \frac{\xi_{24} - \xi_{12}\xi_{14}}{\sqrt{1 - \xi_{12}^2}} \right) X_2 + \frac{\sqrt{\Delta_{124}}}{(1 - \xi_{12}^2)} X_3 + w/a_{14} = 0. \]

The distance from the origin to this plane is \( w/a_{14} \).

In terms of the \( \{X_i\} \) the volume of interest is that of the tetrahedron with vertices at
\[ A_0 = (0,0,0) \]
\[ A_1 = (0,0, -\frac{w}{a_{34}} \sqrt{\frac{1 - \xi_{12}^2}{\Delta_{123}}}) \]
\[ A_2 = \left(0, -\frac{w}{a_{23} \sqrt{1 - \xi_{12}^2}}, \frac{w}{a_{23}} - \frac{\xi_{23} - \xi_{12} \xi_{13}}{\sqrt{\Delta_{123} (1 - \xi_{12}^2)}}\right) \]

\[ A_3 = \left(-\frac{w}{a_{12}}, \frac{w \xi_{12}}{a_{12} \sqrt{1 - \xi_{12}^2}}, \frac{w}{a_{12}} - \frac{\xi_{13} - \xi_{12} \xi_{23}}{\sqrt{\Delta_{123} (1 - \xi_{12}^2)}}\right), \]

where \( A_1, A_2, A_3 \) lie on the plane whose equation is given by (6.3).

The perpendicular from the origin to this plane intersects the plane at

\[ A_4 = \left(\frac{w \xi_{14}}{a_{14}}, -\frac{w \xi_{24} - \xi_{12} \xi_{14}}{a_{14} \sqrt{1 - \xi_{12}^2}}, -\frac{w \sqrt{\Delta_{124}}}{a_{14} (1 - \xi_{12}^2)}\right) \]

and the coordinates of \( A_4 \) are given by the following linear combination of the coordinates of the \( A_i \)

\[ A_4 = -\left(\frac{a_{34} \xi_{34}}{a_{14}} A_1 + \frac{a_{23} \xi_{24}}{a_{14}} A_2 + \frac{a_{12} \xi_{14}}{a_{14}} A_3\right), \]

where the sum of the weights is unity.

Imagine first that \( A_4 \) lies within the triangle \( A_1 A_2 A_3 \) (such will be the case if the weights are positive). If a perpendicular is drawn from \( A_4 \) to each of the sides of \( A_1 A_2 A_3 \) then the tetrahedron \( A_0 A_1 A_2 A_3 \) is divided into six regions and the probability that \( (X_1, X_2, X_3) \) lies in one of these regions is expressible in terms of the \( S \)-function
described by Steck (1958). Consequently, the probability content of the tetrahedron, \( P_1 \), is obtained by adding the contents of the six regions. If \( A_4 \) lies outside \( A_1 A_2 A_3 \) then \( P_1 \) is obtained as a linear combination of the contents of the regions, the weights being \( \pm 1 \).

The process of finding the requisite lengths is a tedious but straightforward process, though it helps to note equalities of the following type

\[
a_{14} + a_{12}e_{14} + a_{23}e_{24} + a_{34}e_{34} = 0
\]

\[
a_{14} \Delta_{124} = a_{34} \Delta_{123}.
\]

Let

\[
h = \frac{w}{a_{14}},
\]

\[
a_1 = \frac{\xi_{14}}{\sqrt{1 - \xi_1^2}}, \quad a_2 = \frac{\xi_{24}}{\sqrt{1 - \xi_2^2}}, \quad a_3 = \frac{\xi_{34}}{\sqrt{1 - \xi_3^2}},
\]

\[
b_{11} = \frac{\xi_{14} - \xi_{13} \xi_{14}}{\xi_{14} \sqrt{\Delta_{134}}}, \quad b_{21} = \frac{\xi_{14} - \xi_{12} \xi_{24}}{\xi_{24} \sqrt{\Delta_{124}}}, \quad b_{31} = \frac{\xi_{14} - \xi_{13} \xi_{34}}{\xi_{34} \sqrt{\Delta_{134}}},
\]

\[
b_{12} = \frac{\xi_{24} - \xi_{12} \xi_{14}}{\xi_{14} \sqrt{\Delta_{124}}}, \quad b_{22} = \frac{\xi_{34} - \xi_{24} \xi_{23}}{\xi_{24} \sqrt{\Delta_{234}}}, \quad b_{32} = \frac{\xi_{24} - \xi_{23} \xi_{34}}{\xi_{34} \sqrt{\Delta_{234}}},
\]
and let

\[
W(h,a,b) = \frac{\arctan b}{2\pi} (\xi(h) - \frac{1}{2}) + \frac{1}{4\pi} \arctan \frac{b}{\sqrt{1 + a^{2} + a^{2}b^{2}}} - S(h,a,b).
\]

Then \(W(h,a,b)\) is the probability content of a region of the tetrahedron and

\[
P_{1} = \sum_{k=1}^{3} \sum_{j=1}^{2} W(h_{k},a_{j},b_{k,j}).
\]

Corresponding results for the trivariate normal distribution can be obtained by letting \(X_{3} = X_{4}\). Then

\[
\rho_{14} = \rho_{13}, \quad a_{14} = a_{13}, \quad \xi_{12} = -(1 + \rho_{13} - \rho_{12} - \rho_{23})/a_{12}a_{23} = \theta_{12} \quad \text{(say)}
\]

\[
\rho_{24} = \rho_{23}, \quad a_{24} = a_{23}, \quad \xi_{14} = -(1 + \rho_{23} - \rho_{12} - \rho_{13})/a_{12}a_{13} = \theta_{13} \quad \text{(say)}
\]

\[
\rho_{34} = 1, \quad a_{34} = 0, \quad \xi_{24} = -(1 + \rho_{12} - \rho_{13} - \rho_{23})/a_{13}a_{23} = \theta_{23} \quad \text{(say)}
\]

\[
\xi_{13} = \xi_{23} = \xi_{34} = 0\quad 0 \quad 0,
\]

and it can be shown that each \(\theta\) is no larger than the product of the other two and that no more than one \(\theta\) can be positive.

Also,

\[
\Delta_{134} = 1 - \theta_{13}^{2}, \quad \Delta_{124} = 0, \quad \Delta_{234} = 1 - \theta_{23}^{2}
\]

\[
a_{1} = \frac{\theta_{13}}{\sqrt{(1 - \theta_{13}^{2})}}, \quad a_{2} = \frac{\theta_{23}}{\sqrt{(1 - \theta_{23}^{2})}}, \quad a_{3} = 0
\]

\[
b_{11} = 0, \quad b_{21} = -\infty \quad \text{sgn} \theta_{23}, \quad b_{31} = \pm \infty
\]

\[
b_{12} = -\infty \quad \text{sgn} \theta_{13}, \quad b_{22} = 0, \quad b_{32} = \pm \infty.
\]
Further,

\[ W(h,a,0) = 0 \]
\[ W(h,0,b) = 0 \]
\[ W(h,a,\infty) = -W(h,a,-\infty) \]
\[ W(h,a,\infty) = \frac{1}{2} \frac{1}{V(h|a|h)} \]
\[ V(h,a) = V(h,|a|h) \text{ sgn } a \]
\[ W(h,a,-\infty) \text{ sgn } a = -\text{ sgn } a W(h,a,\infty). \]

where \( V(a,b) \) is the \( V \)-function described by Nicholson (1943).

Consequently,

\[
P_1 = -\frac{1}{2} \left[ V \left( \frac{w}{a_{13}}, \frac{w \theta_{13}}{a_{13} \sqrt{1 - \theta_{13}^2}} \right) + V \left( \frac{w}{a_{13}}, \frac{w \theta_{23}}{a_{13} \sqrt{1 - \theta_{23}^2}} \right) \right]
\]

Any permutation of the integers 1, 2, 3, 4 which does not have 3 and 4 adjacent will have the corresponding \( P_1 = 0 \), since \( P(X_3 \leq X_1 \leq X_4) = 0 \) if \( X_3 \neq X_4 \). There are twelve permutations with 3 and 4 adjacent and these yield three sets of four equal \( P_1 \).

Thus, in the trivariate case,

\[
P(W \leq w) = -2 \left[ V \left( \frac{w}{a_{12}}, \frac{w \theta_{12}}{a_{12} \sqrt{1 - \theta_{12}^2}} \right) + V \left( \frac{w}{a_{12}}, \frac{w \theta_{13}}{a_{12} \sqrt{1 - \theta_{13}^2}} \right) \right.
\]
\[
+ V \left( \frac{w}{a_{13}}, \frac{w \theta_{13}}{a_{13} \sqrt{1 - \theta_{13}^2}} \right) + V \left( \frac{w}{a_{13}}, \frac{w \theta_{23}}{a_{13} \sqrt{1 - \theta_{23}^2}} \right) \]
\[
+ V \left( \frac{w}{a_{23}}, \frac{w \theta_{12}}{a_{23} \sqrt{1 - \theta_{12}^2}} \right) + V \left( \frac{w}{a_{23}}, \frac{w \theta_{23}}{a_{23} \sqrt{1 - \theta_{23}^2}} \right) \right].
\]
Finally, in the bivariate case one has

\[ P(W \leq W) = 2\Phi(\sqrt{\frac{2(1-\rho)}{\rho}}) - 1, \quad (W \leq \infty), \]

where \( \Phi(x) \) is the distribution function of a \( N(0,1) \) random variable. This follows since for \( n=2 \) the general case and the equally correlated case coincide and (2.6) applies.
References


References (cont)


References (cont)


On linear functions of ordered correlated normal random variables

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1. Introduction and Summary

Let $X_1, X_2, ..., X_n$ be jointly normally distributed random variables with $EX_i = 0$, $EX_i^2 = 1$ and $EX_iX_j = \rho_{ij}$ $(i \neq j = 1, 2, ..., n)$. Let $X_{(k)}$ be the $k$th order statistic when $X_i$'s are arranged in an increasing order as follows:

$$X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)} \leq ... \leq X_{(n)}.$$  \hspace{1cm} (1.1)

Many problems of statistical inference, notably the ones in multiple decisions and life testing involve the use of ordered $X_i$'s. In an earlier paper Gupta, Pillai & Steck (1964) considered the distribution of linear functions of $X_{(k)}$'s and gave closed form results for the case when the random variables are equally correlated. Also in the same paper closed form expressions for the distribution of the range $W = X_{(n)} - X_{(1)}$ were obtained for $n = 3$ and $4$ for the general case. In this paper we study the characteristic functions of individual order statistics and also the linear functions for the bivariate and trivariate cases for the general correlation matrix. Formulae for the expected values of $X_{(k)}$, $X_{(k)}^2$, and $X_{(k)}X_{(l)}$ and the first and second moments of a linear function of the $X_{(k)}$'s are obtained. The joint distribution of the range and the mid-range is given in a closed form in the trivariate case, from which the distributions of the mid-range and the mid-range/range ratio are derived, again in closed forms. Best linear unbiased estimators of the common mean of three correlated normal variables have been obtained and tabulation of the coefficients made for different sets of values of $\rho_{ij}$'s. Applications in the fields of life testing and time series analysis are discussed.

2. The Bivariate Case

In the bivariate case, the characteristic function is given by

$$\phi_{X_{(1)}, X_{(2)}}(t_1, t_2) = E(\exp(\{it_1X_{(1)} + it_2X_{(2)}\}) = 2E[\exp(\{it_1X_1 + it_2X_2\})/X_1 \leq X_2].$$  \hspace{1cm} (2.1)

Hence

$$\phi_{X_{(1)}, X_{(2)}}(t_1, t_2) = \frac{1}{\sqrt{(1-\rho^2)}} \int_{\infty}^{-\infty} \int_{\infty}^{-\infty} \exp(\{it_1X_1 + it_2X_2\})$$

$$\times \exp\left[-\frac{X_1^2 + X_2^2 - 2\rho X_1X_2}{2(1-\rho^2)}\right] dX_1 dX_2.$$  \hspace{1cm} (2.2)

Now we state a lemma which simplifies (2.2).

**Lemma 1.** For any real numbers $\alpha$ and $\beta$,

$$\frac{1}{\sqrt{(2\pi)}} \int_{\infty}^{-\infty} \Phi(\alpha X + \beta) e^{-\frac{1}{2}X^2} dX = \Phi\left(\frac{\beta}{\sqrt{1 + \alpha^2}}\right)$$  \hspace{1cm} (2.3)

where $\Phi(\cdot)$ denotes the c.d.f. of the random variable $N(0, 1)$.

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After completing squares, performing integrations on $X_1$ and using the lemma (2.3), one obtains from the moment generating function corresponding to (2.2), the following explicit closed form for the characteristic function.

$$
\phi_{X(t_1, \tau)}(t_1, t_2) = 2 \exp \left\{-\frac{1}{2}(t_1^2 + 2t_1 t_2 + t_2^2)\right\} \Phi \left(\frac{1}{2} \sqrt{(1 - \rho)} (t_2 - t_1)\right). \tag{2.3a}
$$

(The function $\Phi(z)$ is well defined and analytic in the complex plane.) Note that the formula (2.3a) gives the answer for the characteristic function in an explicit form and has been obtained directly. The formula (5) in the paper by Owen & Stock (1962) after evaluation of the characteristic function of the two ordered random variables (arising from independent case) reduces to the right hand side of (2.3a).

It follows from (2.3a) that

$$
\phi_{X(\tau)}(t_2) = 2 \exp \left(-\frac{1}{2}t_2^2\right) \Phi \left(\frac{1}{2} \sqrt{(1 - \rho)} t_2\right) \tag{2.4}
$$

and by substituting $t_2 = t_1$, we get the characteristic function for $X(\tau)$. Note that the characteristic function of a linear function of $X(\tau)$ and $X(\omega)$ can be obtained from (2.3a) by substituting $t_i = a_i t$ ($i = 1, 2$) and is given by

$$
\phi_{X(\tau) + X(\omega)}(t) = 2 \exp \left(-\frac{1}{2}t^2(a_1^2 + 2\rho a_1 a_2 + a_2^2)\right) \Phi \left(\frac{1}{2} \sqrt{(1 - \rho)} t(a_2 - a_1)\right). \tag{2.5}
$$

From (2.5) one can obtain all the moments of $X(\tau) + X(\omega)$ and verify the usual formula for $a_1 = a_2 = 1$, i.e. the case of the mid-range. The characteristic function for $a_1 = a_2 = 1$ is

$$
\phi_{X(\tau)} + X(\omega)(t) = \exp \left(-\frac{1}{2}t^2(1 + \rho)\right), \tag{2.6}
$$

and hence mid-range follows the normal distribution $N(0, 2(1 + \rho))$.

From (2.5) we obtain the density function $f(y)$ of $Y = X(\tau) + X(\omega)$ as

$$
f(y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_y(t) e^{-ty} dt \tag{2.7}
$$

which after some simplification gives

$$
f(y) = \sqrt{\frac{2\pi}{\xi}} \exp \left(-\frac{y^2}{2\xi}\right) \Phi \left(\eta y\right) \tag{2.8}
$$

where $\xi = a_1^2 + a_2^2 + 2\rho a_1 a_2$ and $\eta = \sqrt{2(1 - \rho)} (a_2 - a_1) / (1 + \rho)(a_2 + a_1)^2$.

By substituting $a_1 = 0$ and $a_2 = 1$ in (2.8), we obtain the density function for $X(\omega)$, which is,

$$
f(X(\omega)) = \frac{1}{\sqrt{\pi}} \exp \left(-\frac{1}{2} X(\omega)^2\right) \Phi \left(\sqrt{2(1 - \rho)} X(\omega) / (1 + \rho)\right) \tag{2.9}
$$

which is a special case of the distribution of the maximum of several equally correlated normal random variables and is discussed and tabulated by Gupta (1963). Now we discuss the evaluation of the density function of the statistic $a_1 X(\tau) + a_2 X(\omega) = X_0$ where $X_0$ is $N(0, 1)$ distributed independently of $X_1$ and $X_2$. This statistic arises in connexion with some multiple decision problems (Gupta & Sobel, 1957).

A convolution of the earlier derived distribution of $a_1 X(\tau) + a_2 X(\omega)$ with $X_0$ gives the following result for the density, $g(Z)$, of $Z = a_1 X(\tau) + a_2 X(\omega) - X_0$.

$$
g(Z) = \frac{2}{\sqrt{2\pi(1 + \xi)}} \exp \left[-\frac{Z^2}{2(1 + \xi)}\right] \Phi \left((1 + \xi) \frac{Z}{(1 + \xi)(1 + \eta^2)}\right), \tag{2.10}
$$
3. Trivariate Case

$$E[\exp \left\{ i(t_1 X_1 + t_2 X_2 + t_3 X_3) \right\}]$$

$$= \sum_{i,j,k} \iiint_{x_1 < x_2 < x_3} \exp \left\{ i(t_1 X_1 + t_2 X_2 + t_3 X_3) \right\} \frac{\exp -\frac{1}{2}(X^TA^{-1}X)}{|A|^{\frac{1}{2}}(2\pi)^{\frac{3}{2}}} dX_1 dX_2 dX_3$$

(3.1)

where \((i,j,k)\) represents a permutation of the positive integers \((1, 2, 3)\) and where the summation extends over all the six permutations of \((i,j,k)\). We shall evaluate the one term of (3.1) corresponding \(X_1 < X_2 < X_3\). We transform from \(X_1, X_2, X_3 (X_1 < X_2 < X_3)\) to

\[
\begin{align*}
W &= X_3 - X_1, \\
M &= X_2 - X_1, \\
U &= X_3 + (A^{12} X_3 + A^{13} X_3)/A^{23},
\end{align*}
\]

(3.1a)

where \(A^{12}, A^{23}\), etc., are elements of \(A^{-1}\) (a \(3 \times 3\) matrix) whose positions are denoted by respective upper suffixes. Then

\[
I = \iiint_{X_1 < X_2 < X_3} \exp \left\{ i(t_1 X_1 + t_2 X_2 + t_3 X_3) \right\} \frac{\exp -\frac{1}{2}(X^TA^{-1}X)}{(2\pi)^{\frac{3}{2}}|A|^\frac{1}{2}} dX_1 dX_2 dX_3
\]

(3.2)

\[
= \frac{1}{2|A|^{\frac{1}{2}}(2\pi)^{\frac{3}{2}}} \int_{-\infty}^{\infty} dW \int_{-\infty}^{\infty} dM \int_{C_0 M + d} dU \exp \left\{ i t_2 (M - W) + i t_2 \frac{1}{2} (M - W) \right\}
\]

(3.3)

where

\[
\begin{align*}
a_0 &= -\frac{A^{12} + A^{23}}{2A^{23}} + \frac{\rho_{12} - \rho_{23}}{2(1 + \rho_{12})}, \\
b &= \frac{A^{12} - A^{23}}{2A^{23}} = -\frac{\rho_{12} + \rho_{23}}{2(1 - \rho_{12})},
\end{align*}
\]

\[
C_0 = \frac{1}{2} - a_0, \\
d_1 = -\frac{1}{2} - b, \\
d_2 = \frac{1}{2} - b,
\]

(3.4)

\[
A = 1 - \rho_{12}^2 - \rho_{23}^2 - \rho_{31}^2 + 2\rho_{12}\rho_{23}\rho_{31}.
\]

After some simplification (completing the squares) and integrating out \(U\), we get

\[
I = \frac{\exp \left\{ -\frac{t_2^2}{4(2A^{23})} \right\}}{4\pi |A|^{\frac{1}{2}} \sqrt{4A^{23}}} \int_{0}^{\infty} \Psi_1(W) dW \int_{-\infty}^{\infty} \left[ \Phi \left( \sqrt{A^{23}}(C_0 M + d) W - \frac{i t_2}{\sqrt{A^{23}}} \right) - \Phi \left( \sqrt{A^{23}}(C_0 M + d) W - \frac{i t_2}{\sqrt{A^{23}}} \right) \right] \Psi_1(M) dM,
\]

(3.5)

where

\[
\Psi_1(M) = \exp \left\{ -\frac{M^2}{4(1 + \rho_{12})} + \frac{1}{2} i M (t_1 + t_3 + 2a_0 t_2) \right\}
\]

(3.6)

\[
\Psi_1(W) = \exp \left\{ -\frac{W^2}{4(1 - \rho_{12})} + \frac{1}{2} i W (-t_1 + t_3 + 2b t_2) \right\}
\]

(3.7)

Now integrating out \(M\) by applying the Lemma (2.3), we obtain

\[
I = \frac{1}{2} \sqrt{\left( \frac{1}{(1 - \rho_{12})} \right)} \exp \left\{ -\frac{t_2^2}{2A^{23}} - \frac{(1 + \rho_{12})}{4} (t_1 + t_3 + 2a_0 t_2) \right\} I_1,
\]

(3.8)

where

\[
I_1 = \int_{0}^{\infty} \exp \left\{ -\frac{W^2}{4(1 - \rho_{12})} + \frac{1}{2} i W (-t_1 + t_3 + 2b t_2) \right\} [\Phi(d_2) - \Phi(d_1)] dW
\]

(3.9)

and

\[
\Phi(d) = \Phi \left( C_0 \sqrt{A^{23}(1 + \rho_{12})} (t_1 + t_3 + 2a_0 t_2) - \left( i t_2 \sqrt{A^{23}} + \sqrt{A^{23}} d_1 W \right) \right).
\]

(3.10)
Thus, formula (3-6) provides the characteristic function $\phi(t_1, t_2, t_3)$ for $X_1 \leq X_2 \leq X_3$. By permuting the indices 1, 2, 3 in (3-6), we can obtain the other cases. Note (3-6) involves an integral $I_t$.

It should be pointed out that the characteristic function for the linear function

$$a_1 X_{(1)} + a_2 X_{(2)} + a_3 X_{(3)}$$

is given by

$$\phi_{\alpha_1 X_{(1)} + \alpha_2 X_{(2)} + \alpha_3 X_{(3)}} = \phi_{X_r, X_r, X_r}(a_1 t, a_2 t, a_3 t). \quad (3-8a)$$

**Special case**

If we are interested in $\phi_{X_{(1)}}(t_1)$ we can find this by substituting $t_2 = t_3 = 0$ in the six terms of $\phi_{X_{(1)}, X_{(2)}, X_{(3)}}(t_1, t_2, t_3)$. Thus

$$\phi_{X_{(1)}}(t_1) = \sum_{i,j,k} I_{i,j,k}(t_1, 0, 0), \quad (3-8b)$$

where

$$I_{i,j,k}(t_1, t_2, t_3) = \text{integral similar to } I \text{ of (3-6)} \quad (3-9)$$

which corresponds to the permutation $(i, j, k)$ instead of $(1, 2, 3)$ appearing there, the subscript $1$ in $I_t$ remaining unchanged. Now we write one term in (3-8b), viz.

$$I_{1,2,3}(t_1, 0, 0) = \frac{1}{2} \frac{1}{\pi (1 - \rho_{13})} \exp \left[ - \frac{W^2}{4(1 - \rho_{13})} \right] I_{1,2,3}(t_1, 0, 0), \quad (3-10)$$

where

$$I_{1,2,3}(t_1, 0, 0) = \int_0^\infty \exp \left[ - \frac{W^2}{4(1 - \rho_{13})} - \frac{it_1 \omega}{2} \right] \Phi \left( iC_0 \sqrt{A_{22}^2(1 + \rho_{13}) t_1 + \sqrt{A_{22}^2 d_4}} \right) \frac{\sqrt{1 + 2A_{22}^2(1 + \rho_{13}) C_0}}{\sqrt{1 + 2A_{22}^2(1 + \rho_{13}) C_0^2}} dW. \quad (3-11)$$

From (3-11), we obtain

$$E(X_1 | X_1 \leq X_2 \leq X_3) = \frac{\partial I_{1,2,3}(t_1, 0, 0)}{\partial t_1} \bigg|_{t_1 = 0}, \quad (3-12)$$

$$E(X_1 | X_1 \leq X_2 \leq X_3) = -\frac{1}{2} \frac{1}{\pi (1 - \rho_{13})} I_{1,2,3}(t_1, 0, 0), \quad (3-13)$$

$$I_{1,2,3} = \frac{1}{2} \frac{1}{\pi (1 - \rho_{13})} I_{1,2,3}^2 + \frac{1}{2} \sqrt{\pi (1 - \rho_{13})} I_{1,2,3}^2, \quad (3-14)$$

$$\theta(d_1) = d_1 \sqrt{\frac{A_{22}^2}{1 + 2A_{22}^2(1 + \rho_{13}) C_0^2}} (i = 1, 2), \quad (3-15)$$

$$A_{11} = A_{22}^2 = A_{33}^2 = (1 + \rho)/((1 - \rho)(1 + 2\rho)); \quad A_{12} = A_{23} = A_{13} = -\rho/[(1 - \rho)(1 + 2\rho)]. \quad (3-16)$$

Again from (3-13), by permuting the indices 1, 2, 3 of $p_{ij}$, one obtains the other five terms.

**Special case**

If $\rho_{ij} = \rho, i \neq j = 1, 2, 3$, we have

$$d_1 = -d_2 = -\frac{1}{2}, \quad C_0 = \frac{1 - \rho}{2(1 + \rho)}; \quad \frac{1}{2} \frac{1}{\pi (1 - \rho_{13})} I_{1,2,3}^2 + \frac{1}{2} \sqrt{\pi (1 - \rho_{13})} I_{1,2,3}^2$$

$$A_{11} = A_{22}^2 = A_{33}^2 = (1 + \rho)/((1 - \rho)(1 + 2\rho)); \quad A_{12} = A_{23} = A_{13} = -\rho/[(1 - \rho)(1 + 2\rho)]. \quad (3-16)$$
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so that

\[ I_{i,j,k} = 0 \quad \text{and} \quad I_{i,j,k} = \frac{1}{2} \quad \text{for all } i, j, k, \]

\[ E X_{ij} = 6 E (X_1 | X_1 \leq X_2 \leq X_3) = \frac{3}{2} \sqrt{\left( \frac{1 - \rho}{\pi} \right)}, \]

which checks with the result obtained directly for this case and further for \( \rho = 0 \) checks with the well-known values.

From (3-8a), following similar methods as for obtaining the \( E (X_1 | X_1 \leq X_2 \leq X_3) \), we can obtain

\[ E(a_1 X_1 + a_2 X_2 + a_3 X_3 | X_1 \leq X_2 \leq X_3) = \frac{1}{2} \sqrt{\left( \frac{1 - \rho_{13}}{\pi} \right)} I_{i,j,k}^1 + \frac{1}{2} \sqrt{\left( \frac{1}{\pi (1 - \rho_{13})} \right)} I_{i,k}^2, \]

where

\[ I_{i,j,k}^1 = (-a_1 + a_2 + 2a_3 \rho_{13} I_{3}^2,3, \]

and

\[ I_{i,k}^2 = \left( a_1 + a_2 + 2a_3 \rho_{13} - \frac{a_2}{C_0 (1 + \rho_{13})} \right) I_{4}^2,3. \]

From (3-19) we obtain

\[ E(X_3 | X_1 \leq X_2 \leq X_3) = \sqrt{\left( \frac{1 - \rho_{13}}{\pi} \right)} I_{3}^2,3, \]

and

\[ E(X_3 | X_1 \leq X_2 \leq X_3) = \frac{1}{2} \sqrt{\left( \frac{1 - \rho_{13}}{\pi} \right)} I_{i,j,k}^1 + \frac{1}{2} \sqrt{\left( \frac{1}{\pi (1 - \rho_{13})} \right)} I_{i,k}^2. \]

Again, putting \( a_1 = a_2 = 0 \) and \( a_3 = 1 \), we get

\[ E(X_3 - X_1 | X_1 \leq X_2 \leq X_3) = \sqrt{\left( \frac{1 - \rho_{13}}{\pi} \right)} I_{3}^2,3. \]

Similarly,

\[ E(X_3 + X_1 | X_1 \leq X_2 \leq X_3) = \sqrt{\left( \frac{1}{\pi (1 - \rho_{13})} \right)} I_{i,j,k}^1. \]

Covariance between \( X_{i,j} \) and \( X_{i,0}, j = 1, 3 \)

From (3-8) proceeding as usual we obtain

\[ E(X_j X_2 | X_1 \leq X_2 \leq X_3, j = 1, 3) = \psi_j, \]

where

\[ \psi_j = \frac{1}{2 \pi} \left[ a_0 (1 + \rho_{13}) + b \gamma_j (1 - \rho_{13}) \right] \arctan \left[ \frac{\theta(d_4) - \theta(d_1)}{\sqrt{2(1 - \rho_{13})} \left( \frac{1}{2(1 - \rho_{13})} + \frac{\theta(d_2) \theta(d_1)}{d_2} \right)} \right] + \frac{1}{2 \pi \sqrt{2(1 - \rho_{13})} \left( \frac{1}{2(1 - \rho_{13})} + \theta(d_1) \right)} \left[ \frac{b \gamma_j (1 - \rho_{13}) + b C_0 (1 + \rho_{13}) \theta(d_2)}{d_2} + \frac{\gamma_j \lambda_9}{2} - \frac{C_0 \theta(d_2) \theta(d_1)}{d_2} \right] \]

\[ - \frac{1}{2 \pi \sqrt{2(1 - \rho_{13})} \left( \frac{1}{2(1 - \rho_{13})} + \theta(d_1) \right)} \left[ \frac{b \gamma_j (1 - \rho_{13}) + b C_0 (1 + \rho_{13}) \theta(d_2)}{d_2} + \frac{\gamma_j \lambda_9}{2} - \frac{C_0 \theta(d_2) \theta(d_1)}{d_2} \right], \]
where $\lambda_0$ is obtained by putting $\alpha_1 = \alpha_2 = 0$ and $\alpha_3 = 1$ in $\lambda$ defined in (3.28) below. It should be noted that $\gamma_j = -1$ or $+1$ according as $j = 1$ or 3. Further, in the case of equal correlation, we obtain from (3.25)

$$6E(X_1X_2|X_1 < X_2 < X_3) = E(X_2X_3|X_1 < X_2 < X_3) = \rho + (\sqrt{3}/2\pi)(1-\rho). \quad (3.26)$$

Now we obtain $E[(a_1X_1 + a_2X_2 + a_3X_3)^t|X_1 < X_2 < X_3]$. Starting from (3.8a) and differentiating the function $E[\exp\{i\beta(a_1X_1 + a_2X_2 + a_3X_3)\}]$ twice partially with respect to $t$ and putting $t = 0$, we obtain after simplification (details omitted)

$$E[(a_1X_1 + a_2X_2 + a_3X_3)^t|X_1 < X_2 < X_3] = \frac{1}{\pi} \left[ \frac{\beta + \delta^2(1-\rho_{12})}{[2(1-\rho_{13})]^2 - \delta^2(1-\rho_{12}) + 1} \right] \left[ \frac{[2(1-\rho_{12})]^{\delta^2/2} \theta'(d_2) + \delta \theta'(d_2)}{[2(1-\rho_{12})]^{\delta^2/2} \theta'(d_1) + \delta \theta'(d_1)} \right] \quad (3.27)$$

where

$$\beta = \frac{2a_1}{1 + \rho_{12}} \frac{A^{22}(a_1 + a_2 + 2a_0 a_2)}{4A^{22}},$$
$$\delta = \frac{1}{2} \left( -a_1 + a_2 + 2a_0 \right),$$
$$\lambda = C_0 \sqrt{A^{22}(1 + \rho_{12})(a_1 + a_2 + 2a_0 a_2) - (a_0 A^{22})},$$

and

$$\rho_{12} = \frac{C_0}{\sqrt{1 + 2A^{22}(1 + \rho_{12}) C_0^2}},$$

where $a_0, C_0, b, d_1, d_2$ are defined in (3.3), $\theta(d_1)$ and $\theta(d_2)$ are defined by (3.14) and where $A^{22} = (1-\rho_{12})/|A|$. As a particular case take $a_4 = a_2 = 0, a_1 = 1$ and $\rho_{12} = \rho$ (all $i, j, i + j$), then we obtain

$$6E(X_1^2 | X_1 < X_2 < X_3) = 1 + (1-\rho)[1 + \sqrt{(3)/(2\pi)}] \quad (3.9)$$

which agrees with the well-known result for this case (see, for example, Owen & Steck, 1962).

4. Distributions of the range, mid-range and the ratio of mid-range to the range

The distribution of the range for $n = 3$ and 4 normal random variables for the general case has been obtained by Gupta, et al. (1964). Now we shall obtain closed-form expressions for the distribution of the ratio of mid-range to the range.

For the case of $n = 3$ correlated normal random variables, it is easy to show that the joint density function of $W, M, U$, defined in (3.1a), is given by

$$f(W, M, U) = \frac{\exp\left[ -\frac{W^2}{4(1-\rho_{12})} - \frac{M^2}{4(1+\rho_{12})} - \frac{A^{22} U^2}{2} \right]}{2(2\pi)^{1/2} |A|^{1/2}}. \quad (4.1)$$

Integrating out $U$ in (4.1) and writing

$$W_1 = \frac{1}{2} \sqrt{A^{22}} W \quad \text{and} \quad z_1 = \frac{1}{2} \sqrt{A^{22}} M (1 + \rho_{12} - \rho_{23})/(1 + \rho_{12})$$

we obtain

$$f(W_1, z_1) = C \exp\left[ -\frac{W_1^2}{4(1-\rho_{12}) A^{22}} - \frac{Dz_1^2}{2} \right] \left[ \Phi(z_1 + W_1(a + 1)) - \Phi(z_1 + W_1(a - 1)) \right] \quad (4.2)$$
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where
\[ a = (\rho_{12} - \rho_{13})/(1 - \rho_{13}) \quad D = 2 \left[ A^{22}(1 + \rho_{13}) \left( 1 - \frac{\rho_{12} + \rho_{13}}{1 + \rho_{13}} \right)^2 \right]^{-1} \]
\[ C = 1/\sqrt{D/(2A^{22}(1 - \rho_{13}))} \].

(4.3)

From (4.2) we obtain (by putting \( z_i/W_1 = U_1 \) and \( W_1 = W_1 \) and integrating out \( W_i \)), the density function of \( U_1 \) as

\[
f(U_1) = C \left[ \frac{\Phi \left( \frac{U_1}{\sqrt{1 + \lambda_i/(a + 1)^2}} \right)}{\sqrt{1 + \lambda_i/(a + 1)^2}} \exp \left[ -\frac{U_1^2}{2} \left( 1 - \frac{1}{1 + \lambda_i/(a + 1)^2} \right) \right] \right. 
\[ - \frac{\Phi \left( \frac{-U_1}{\sqrt{1 + \lambda_i/(a + 1)^2}} \right)}{\sqrt{1 + \lambda_i/(a + 1)^2}} \exp \left[ -\frac{U_1^2}{2} \left( 1 - \frac{1}{1 + \lambda_i/(a + 1)^2} \right) \right] \],
\]

(4.4)

where
\[ \lambda_i = DU_i^2 + 2/(A^{22}(1 - \rho_{13})) \].

Equation (4.4) provides the density function of

\[ U_1 = \frac{M (1 + \rho_{13} - \rho_{12} - \rho_{23})}{(1 + \rho_{13})}. \]

(4.5)

Again from (4.2), we obtain the distribution of \( z_1 \) by making use of the following lemma.

**Lemma 2.** For any real numbers \( \alpha_i \) and \( \beta_i \)

\[
\frac{1}{\sqrt{2\pi}} \int_0^\infty \Phi(\alpha_i + \beta_i x) e^{-x^2} dx = \frac{1}{2} \Phi(\alpha_i/\sqrt{(1 + \beta_i^2)}) + (\text{arc tan} \beta_i)/(2\pi) - V(\alpha_i/\sqrt{(1 + \beta_i^2)}, \alpha_i \beta_i/\sqrt{(1 + \beta_i^2)}),
\]

(4.6)

where \( V(h, q) \) is the \( V \)-function of Nicholson (1943). Applying Lemma 2 to (4.2) for integration with respect to \( W_i \), we get

\[
f_2(z_1) = \{ \sqrt{D} e^{-z_1^2}/\sqrt{(8\pi)} \} \left[ \Phi(z_1/\sqrt{(1 + B_1^2)}) - \Phi(z_1/\sqrt{(1 + B_2^2)}) + [(\text{arc tan} B_1) - (\text{arc tan} B_2)]/\pi 
\[ - 2 V(z_1/\sqrt{(1 + B_1^2)}, z_1 B_1/\sqrt{(1 + B_1^2)}) + 2 V(z_1/\sqrt{(1 + B_2^2)}, z_1 B_2/\sqrt{(1 + B_2^2)}) \right], \]

(4.7)

where
\[ B_1 = (a + 1) \{ \frac{1}{2} (1 - \rho_{13}) A^{22} \}^{1/2} \quad \text{and} \quad B_2 = (a - 1) \{ \frac{1}{2} (1 - \rho_{13}) A^{22} \}^{1/2}. \]

**5. Best Linear Unbiased Estimators of the Common Mean of the Three Correlated Normal Random Variables**

Now we consider the use of the results in the previous sections for the construction of the best linear unbiased estimators of the common mean \( \mu \) of three correlated normal random variables. Let \( X_1', X_2', X_3' \) be normal random variables with the common mean \( \mu \), common variance unity and \( E(X_1' X_2') = \rho_{12} \mu^2 \). Then we are interested in finding the linear function based on \( X_i' \) which is an unbiased estimator of \( \mu \) and which has the smallest variance in the class of all linear unbiased estimators. Let \( \Sigma a_i X_i' \) be the estimator that we are looking for. Then the condition for unbiasedness implies

\[
\Sigma a_i = 1, \quad \Sigma a_i E(X_i') = 0.
\]

(5.1)

It should be pointed out that (5.1) gives two necessary conditions for the unbiasedness of an estimator based on any number of random variables. Now we have to minimize the variance of \( \Sigma a_i X_i' \) which is the same as the variance of \( E(X_i') \) and can be computed by
using the formula (3.27) for all the six permutations of the subscripts 1, 2, 3. Thus one minimizes the variance of \( \sum a_i X_{ij} \) subject to the condition \( \sum a_i = 1 \), and the condition that the sum of the terms obtained by interchanging the subscripts of \( \rho_{ij} \) in (3.19) equals zero. The equations that determine the coefficients corresponding to the minimum value are given in the Appendix. Table 1 gives the values of \( a_i \)'s for selected set of values of \( \rho_{12}, \rho_{13}, \rho_{23} \). The computations were carried out on IBM 7094. It should be pointed out that the selected values of \( \rho_{ij} \) have to satisfy certain linear restrictions.

6. Applications

(i) Test of equality of the means of a multivariate population having a common \( \rho \) and a common standard deviation

Let \( X_{ij}^{(1)} \leq X_{ij}^{(2)} \leq \ldots \leq X_{ij}^{(k)} \) \((j = 1, 2, \ldots, k)\), be the \( n \) ordered random variables which have come from a multivariate normal population with mean vector \((\mu_1, \mu_2, \ldots, \mu_n)\) and a common correlation coefficient \( \rho \) and a common standard deviation, \( \sigma \). Suppose we are interested in testing the hypothesis \( H \) against the alternative \( A \), where

\[
H: \mu_1 = \mu_2 = \ldots = \mu_n, \quad A: \text{not } H, \tag{6.1}
\]

then the following simple test can be applied. Let \( W_j = X_{(n)}^{(j)} - X_{(1)}^{(j)} \) and \( W_{\text{max}} = \max_j W_j \). Test: reject \( H \) if

\[
W_{\text{max}} \sqrt{\frac{1}{n}} \sum_{j=1}^{k} W_j \geq C(\alpha, n, k),
\]

where \( C(\alpha, n, k) \) is a constant which depends on the size of the test and on \( n \) and \( k \). In fact, \( C(\alpha, n, k) \) is such that

\[
P\left( W_{\text{max}} \sqrt{\frac{1}{n}} \sum_{j=1}^{k} W_j \geq C(\alpha, n, k) \mid \mu_1 = \ldots = \mu_k \right) = \alpha. \tag{6.2}
\]

It should be pointed out that the fact that the distribution of \( W_{\text{max}} \sqrt{\frac{1}{n}} \sum_{j=1}^{k} W_j \) is independent of \( \rho \) has already been shown in the paper by Gupta et al. (1964). The upper 5% points for this test are tabulated by Bliss, Cochran & Tukey (1956).

It should be noted that the above has an important application to the situations in life testing where the observations are ordered and we can perform the test without knowing the unordered random samples from the multivariate normal population for which the common correlation coefficient and the common standard deviation are unknown.

(ii) Confidence interval for the common \( \rho \) in a multivariate normal population with common mean and common known standard deviation

It is clear that if \( X_{(n)} \) and \( X_{(1)} \) are the largest and smallest of \( n \) equally correlated random variables with a common mean and common standard deviation, say unity, then one- or two-sided 100\( \alpha \)% confidence bounds for \( \rho \) are obtained from

\[
\frac{X_{(n)} - X_{(1)}}{C_2} \leq (1 - \rho) \leq \frac{X_{(n)} - X_{(1)}}{C_1}, \tag{6.3}
\]

where \( C_1 \) and \( C_2 \) are the percentiles of the distribution of the range of \( n \) independent and identically distributed normal random variables. It may be pointed out that in life testing where the observations are naturally ordered, use of (6.3) gives a confidence interval statement for \( \rho \) even when the unordered sample is not known.
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(iii) Test of equality of two oscillations in non-overlapping intervals of equal length in time series

Tests based on the ratio of differences between peaks and troughs from two non-overlapping intervals of equal length might be indicative of the nature of the oscillations involved in the time series. If we assume the variables to follow a multivariate normal distribution within intervals of equal length, this test would amount to the one based on the ratio of two independent ranges in the correlated case. (The distribution of this ratio for common mean and common standard deviation can be obtained as a series of beta functions using the distribution of range developed by the authors as a series of gamma functions which is available in Mimeograph series no. 13 Department of Statistics, Purdue University. For the case of means not necessarily equal the distribution has not been worked out.) To establish the equality of two oscillations, in addition, tests based on the differences of mid-ranges may have to be carried out. However, since tests based on the differences of mid-ranges cannot be easily performed, the ratio of mid-range to the range can be used to test any assigned value of the average of two population means (corresponding to the peak and trough) and this test can be repeated for each interval. In all the discussions above it is assumed that the variables have a common standard deviation, otherwise adjustments have to be made at various stages. Also the assumption of a multivariate normal distribution means that the set of correlation coefficients are known or otherwise have to be estimated, possibly through the use of serial correlation coefficients. However, if the correlation coefficients are neither known nor evaluated, some simulation process could be adopted for the performance of the tests.

The authors are grateful to the referee for bringing to their attention the result in Lemma 2. They also wish to thank Mrs Louise Lui, Statistical Laboratory, Purdue University, for the excellent programming of the material in the Appendix of this paper required for computing Table 1 on IBM 7094 computer, Purdue University's Computer Sciences Center.

APPENDIX

The condition $\Sigma_{ii} EX_{i0} = 0$ given in (3-1) reduces to the following equation by using (3-19) and $\Sigma_{ii} = 1$.

$$k_1 a_3 = k_2 a_4 + k_3,$$

(A 1)

where

$$k_1 = \sqrt{\left(\frac{1-\rho_{12}}{\pi}\right) I_{11}^u + \sqrt{\left(\frac{1-\rho_{12}}{\pi}\right) I_{22}^u + \sqrt{\left(\frac{1-\rho_{12}}{\pi}\right) I_{33}^u}}},$$

$$k_2 = -\sqrt{\left(\frac{1-\rho_{12}}{\pi}\right) d_{11} I_{11}^u + \sqrt{\left(\frac{1-\rho_{12}}{\pi}\right) d_{22} I_{22}^u + \sqrt{\left(\frac{1-\rho_{12}}{\pi}\right) d_{33} I_{33}^u}}},$$

$$[\pi(1-\rho_{12})^{-1} + [C_{11} + (2A^{12}C_{11}(1+\rho_{12}))^{-1}] I_{11}^u + [\pi(1-\rho_{12})^{-1} [C_{11} + (2A^{12}C_{11}(1+\rho_{12}))^{-1}] I_{22}^u + [\pi(1-\rho_{12})^{-1} [C_{11} + (2A^{12}C_{11}(1+\rho_{12}))^{-1}] I_{33}^u},$$

$$k_3 = \frac{1}{2} \left[\sqrt{\left(\frac{1-\rho_{12}}{\pi}\right) I_{11}^u + \sqrt{\left(\frac{1-\rho_{12}}{\pi}\right) I_{22}^u + \sqrt{\left(\frac{1-\rho_{12}}{\pi}\right) I_{33}^u}}},$$

$$[\pi(1-\rho_{12})^{-1} I_{11}^u + [\pi(1-\rho_{12})^{-1} I_{22}^u + [\pi(1-\rho_{12})^{-1} I_{33}^u}].$$

where $I_{i}^{u}$ is a simpler notation for $I_{i}^{1-3}$ in (3-14), $d_{i1}, d_{i2}, d_{i3}, C_{i1}, \theta(<d_{i1})$ and $\theta(>d_{i2})$ respectively in (3-14). $I_{i}^{u}$ is obtained from $I_{i}^{u}$ by interchanging in the latter $\rho_{12}$ and $\rho_{12}$ and so also $d_{i1}$ from $d_{i1}$, $d_{i2}$ from $d_{i2}$, $C_{i1}$ from $C_{i1}$, etc. Similarly, $I_{i}^{u}$ is a briefer notation for
$I_{11}^{3,2}$ in (3-15). $I_{11}^{3}$ is obtained from $I_{11}^{1}$ by interchange of $\rho_{13}$ and $\rho_{14}$. Further, $I_{11}^{3}$ is obtained from $I_{11}^{1}$ by interchanging in the latter $\rho_{12}$ and $\rho_{13}$ and similarly $I_{11}^{3}$ from $I_{11}^{1}$. $d_{13}, d_{23}, C_{12},$ etc., are obtained in a similar manner.

The value of $a_4$ corresponding to the minimum variance is given by the equation

$$l_1 a_4 = l_2,$$

where

$$l_1 = 2 \left[ A_1 \left( \frac{(A_{33})^{-1}}{2} + C_{55} (1 + \rho_{13}) \right) + A_1 \left( A_{33}^{-1} - \frac{1}{2} + C_{55} (1 + \rho_{13}) \right) + A_1 \left( \frac{(A_{11})^{-1}}{2} + C_{55} (1 + \rho_{13}) \right) \right. $n_k \left( d_{11} + \frac{k_3}{k_1} \right)^2 + B_1 \left( d_{13} + \frac{k_3}{k_1} \right)^2 + B_2 \left( d_{23} + \frac{k_3}{k_1} \right)^2 + B_3 \left( d_{33} + \frac{k_3}{k_1} \right)^2$

$$+ C_1 \theta(d_{13}) \left( 2C_{66}^2 (1 + \rho_{13}) + \frac{1}{A_{33}} \right) \left( d_{11} + \frac{k_3}{k_1} \right)^2 + C_2 \theta(d_{13}) \left( 2C_{66}^2 (1 + \rho_{13}) + \frac{1}{A_{33}} \right) \left( d_{23} + \frac{k_3}{k_1} \right)^2$\n
$$+ D_1 \theta(d_{13}) \left( 2C_{66}^2 (1 + \rho_{13}) + \frac{1}{A_{33}} \right) \left( d_{33} + \frac{k_3}{k_1} \right)^2 + D_2 \theta(d_{13}) \left( 2C_{66}^2 (1 + \rho_{13}) + \frac{1}{A_{33}} \right) \left( d_{33} + \frac{k_3}{k_1} \right)^2$\n
$$+ D_3 \theta(d_{13}) \left( 2C_{66}^2 (1 + \rho_{13}) + \frac{1}{A_{33}} \right) \left( d_{33} + \frac{k_3}{k_1} \right)^2 \left[ A_1 \left( 1 + \rho_{13} \right) C_{01} + A_1 \left( 1 + \rho_{13} \right) C_{02} + A_2 \left( 1 + \rho_{23} \right) C_{03} \right]$

$$\left. + \left( 1 - \frac{2k_3}{k_1} \right) \left( B_1 \left( d_{11} + \frac{k_3}{k_1} \right)^2 + B_2 \left( d_{22} + \frac{k_3}{k_1} \right)^2 + B_3 \left( d_{33} + \frac{k_3}{k_1} \right)^2 \right) \right] \right.$$

and where

$$A_1 = \frac{1}{\pi} \arctan \left[ \frac{\theta(d_{13}) - \theta(d_{11})}{2(1 - \rho_{13}) + \theta(d_{12}) \theta(d_{13})} \right],$$

$$B_1 = \left( 1 - \rho_{13} \right) A_1 + \frac{\sqrt{2}}{\pi} (1 - \rho_{13}) \frac{\theta(d_{13})}{\sqrt{2} \theta(d_{11})},$$

$$C_1 = \frac{1}{\sqrt{2} (1 - \rho_{13})} \left( \frac{\theta(d_{13})}{\sqrt{2} \theta(d_{11})} - \frac{1}{\theta(d_{11})} \right),$$

$$D_1 = \frac{1}{\pi} \left( \frac{1}{2} - \rho_{13} \right) \frac{\theta(d_{13})}{\sqrt{2} \theta(d_{11})} + \frac{1}{2} \theta(d_{11}) + \frac{1}{\theta(d_{11})}.$$
Linear functions of ordered correlated normal random variables

REFERENCES


### Table 1. Values* of $a_i$'s $(i = 1, 2, 3)$ corresponding to the minimum variance for given values of $\rho_{ij}$'s

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