Last time (Session 2)
Conditional probability \& independent.

This time (Session 3)
1. Counting permutations
2. Electrical circuits (independence)
3. Geometric distribution
4. Counting combinations
5. Binomial distribution

\( C_i = \text{event component } i \text{ conducts elec.} \)
\( F = \text{event current flows thru circuit} \)
Assume \( \{ C_i \}'s \) independent \( \text{with indicated prob's} \)

(a)
\( F = \)

\[ \begin{array}{c}
\text{#1} \\
\text{#2} \\
.7 \\
.6 \\
\end{array} \]

\( P(F) = \)

(b)
\( F = \)

\[ \begin{array}{c}
\text{#1} \\
\text{#2} \\
.7 \\
.6 \\
\end{array} \]

\( P(F) = \)

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**HW Comments**
1. Put problems in order
2. "U" for sets
   "\(" for numbers
\{men\} \cup \{women\} = \{people\}
3 \text{ + 4 = 7}
(b) $P(C_1|F) =$

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(b) $P(C_1|F) =$

---

(c) $P(C_1|F^c) =$

---

(c) $P(C_1|F^c) =$

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More counting: Permutations & Combinations

Example: Club has 10 members

(a) How many ways to choose officer slate (Pres., Sec., Treas.)?

= permutation, size 3.
(b) How many ways to choose a committee of size 3?

\[\text{number of committees} = \binom{10}{3}\]

For \( n \) elements,

- **Permutations, size \( k \)**
  \[\text{permutations, size } k = \frac{n!}{(n-k)!} \]

- **Combinations, size \( k \)**
  \[\binom{n}{k} = \frac{n!}{k! (n-k)!} \]

How many unordered 5-card hands?

How many are club flushes?

Grid-representation of the 52 cards in a standard deck.
\[ P\{ \text{Ace-high straight} \} \\
= P\{ \text{Hand has 10, J, Q, K, A} \} \\
= \_ \times \text{above} \]

How many arrangements of eeeeennnn?

5.9

5.10

Binomial Distribution
Roll a die 4 times.
\[ \Omega = \{ \text{aaaa, aaan, \ldots, nnnn} \} \]
(or \[ \Omega' = \{ 1111, 1112, \ldots, 6666 \} \])
\[ X = \# \text{aces in 4 rolls} \]
\[ P\{ X = 0 \} = P\{ 0 \text{ aces} \} = \_ \]

How many routes from A to B? (always going east or north)

5.11

5.12
\[ P\{X = 1\} = P\{1 \text{ ace}\} \]
\[ = P\{aann, nann, nnan, nnna\} \]
\[ = \]

\[ P\{X = 3\} = P\{3 \text{ aces}\} \]
\[ = P\{aaan, aana, anaa, naaa\} \]
\[ = \]

\[ P\{X = 4\} = P\{4 \text{ aces}\} \]
\[ = P\{aaaa\} = \]

5.12

5.14

5.15

Probability histogram for the number of aces in 4 die rolls
Bernoulli ($p$) trials: $\Box$ 79

(independent)

success-failure trials,

$p =$ common success probability

Examples: Success

toss coin Heads

roll die Ace

have child Boy

Randomly sample 4 balls, one-by-one
(a) with repl.
(b) without repl.

$X =$ # White draws

is Binomial ($n =$, $p =$)
in scenario

(a) both

(b) $\Box$ 3.22

$X = \begin{cases} \# \text{successes} & \Box 81 \\
in \frac{n}{\text{independent}} \\
"\text{Bernoulli}(p)\text{ trials (i.e., 5-F)}" \\
\text{is Binomial}(n, p) \text{ r.v.} \\
\text{and} \\
P\{X = k\} = \binom{n}{k} p^k q^{n-k} \\
k = 0, 1, \ldots, n \\
q = 1 - p
\end{cases}

\sum \{b b b b, b b b w, \ldots, w w w w\}

# \sum = \Box \text{Equally likely?}

P\{X = 0\} = P\{b b b b\}$ 0 0 0 0 0

\begin{cases} \{\text{Scenario (a)}\} \\
\{\text{with repl.}\} \\
\{\text{Scenario (b)}\} \\
\{\text{w/o repl.}\}
\end{cases}$ 0 0 0 0 0
\[ P(X=1) = P(\text{bbbw}, \text{bbwb}, \text{bwbb}, \text{wbbb}) \]
\[ = \binom{4}{1} P(\text{bbbw}) \text{ in (a) or (b) !!} \]
\[ = \binom{4}{1} \left( \frac{4}{7} \right)^3 \cdot 0.756 \text{ in (a)} \]
\[ = \binom{4}{1} \frac{3}{10} \cdot \frac{2}{9} \cdot \frac{1}{8} \cdot \frac{1}{7} \]
\[ = 0.0333 \text{ in (b)} \]

\[ \Omega = \{ \text{bbbbb, bbbw, bbwb, bwbb, wbbb, bbww, bwwb, bwbb, wbbw, wwww, wbww, wwww} \} \]

Likewise,
\[ P(X=2) = \binom{4}{2} P(\text{bbww}) \]
\[ P(X=3) = \binom{4}{3} P(\text{bwww}) \]
\[ P(X=4) = \binom{4}{4} P(\text{wwww}) \]

Probabilities for \( X = \#\text{ white} \)

<table>
<thead>
<tr>
<th>X</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>w.c.</td>
<td>.008</td>
<td>.076</td>
<td>.265</td>
<td>.412</td>
<td>.240</td>
</tr>
<tr>
<td>(b) w/o r.</td>
<td>0</td>
<td>.033</td>
<td>.300</td>
<td>.500</td>
<td>.166</td>
</tr>
</tbody>
</table>

In (b),
\[ P(X=k) = \frac{\binom{7}{k} \binom{3}{4-k}}{\binom{10}{4}} \]
Draw balls until 3rd black draw.

T = # draws needed.

\[ P\{T = 5\} = \]

<table>
<thead>
<tr>
<th>Number</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
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<tbody>
<tr>
<td>Page</td>
<td>87</td>
<td>8</td>
<td>7</td>
<td>6</td>
</tr>
</tbody>
</table>

\[ p = \frac{1}{8} \quad \frac{1}{4} \quad \frac{1}{2} \quad \frac{3}{4} \quad \frac{7}{8} \]

If \( X \sim \text{Binomial}(n=13, p = \frac{1}{6}) \),

(Story?)

then \( Y = 13 - X \)

is \( \text{Binomial}(n=__, p=__) \)

and

\[ P\{X=4\} = P\{Y = __\} \]

**Ratios of consecutive probabilities**

For \( X \sim \text{Bin}(n, p); \ 0 < k < n , \)

\[ \frac{P\{X=k\}}{P\{X=k-1\}} = \frac{n!}{k!(n-k)!} \cdot \frac{p^k q^{n-k}}{p^{k-1} q^{n-k+1}} \]

\[ = \left[ \frac{n-k+1}{k} \right] \frac{p}{q} \]

\[ \downarrow \text{as } k \uparrow \]
For $X \sim \text{Binomial}(n, p)$, $P(X = k)$ is the probability of obtaining exactly $k$ successes in $n$ independent Bernoulli trials, each with success probability $p$. The probability mass function is given by:

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$$

One of the integers $k$ for which $P(X = k)$ is maximum is given by:

$$k = \left\lfloor np \right\rfloor$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to $x$. This is one of the sides of the Pascal triangle. For $n = 100$, $p = 0.5$, the distribution of $X$ is shown in the figure, with the maximum probability occurring at $k = 50$. The number of trials $n$ can be found by applying the binomial distribution formula to the observed data.