Lattice paths are ubiquitous in combinatorics and algorithmics, where they are either studied per se, or as a convenient encoding of other structures. Logically, they play an important role in Philippe’s papers. For instance, they are central in his combinatorial theory of continued fractions, to which Chapter 3 of this volume is devoted. In this chapter, we present a collection of seven papers written by Philippe on lattice paths: they deal respectively with one-dimensional paths [4, 6], with models of self-avoiding polygons [34, 10, 9], and with two algorithmic questions where paths describe the evolution of an algorithm [32, 22].

1. Walks on a line: enumeration and asymptotic properties

The enumeration of lattice paths is a venerable topic in combinatorics, closely related to the study of random walks in probability theory. One of Philippe’s papers on the topic, co-authored with Cyril Banderier [6], is very complete and pleasant to read, and will certainly remain influential. It provides a uniform solution of the three following counting problems: given a finite set $S \subset \mathbb{Z}$, what is the number of bridges, meanders and excursions of length $n$ (i.e., having $n$ steps) that take their steps in $S$? By these terms, borrowed from probability theory, we mean walks on the line $\mathbb{Z}$ starting from 0, and satisfying the following properties (Fig. 1):
- bridges end at 0,
- meanders only visit non-negative points,
- excursions are those meanders that end at 0.

In particular, when $S = \{-1, +1\}$, excursions are nothing but the famous Dyck paths.

The paper first gives closed form expressions for the generating functions of these three classes of walks, as rational functions of some roots of a bivariate polynomial.
that describes the steps of $S$. In particular, these three generating functions are systematically algebraic.

The second part starts from these expressions and derives asymptotic expansions for the number of bridges, meanders, and excursions of length $n$. The exponential growth of these numbers is governed by two growth constants: one is simply the cardinality of $S$, that is, the growth constant for unrestricted walks on $\mathbb{Z}$; the other one is a smaller algebraic number $\mu$, which is the growth constant of bridges and excursions. For meanders, the growth constant is $|S|$ if the drift (the average value of a step) is non-negative, and $\mu$ otherwise.

Finally, the paper studies the limit distribution of certain parameters, like the final position of meanders, which, depending on the drift, may follow a discrete limit law, a Rayleigh or a Gaussian law (after normalization).

The exact enumerative results presented in [6] belong to a large corpus of work on path generating functions and algebraic series. The generating function of bridges is obtained by extracting a diagonal from a bivariate rational function. As in Furstenberg’s general result on diagonals of rational functions [46], this extraction is performed using a Cauchy integral and the residue formula. Alternatively, one can perform a partial fraction expansion, as in Gessel’s proof of Furstenberg’s theorem (see [49] or [69, Sec. 6.3]). The expressions for the generating functions of meanders and excursions are based on a step-by-step construction of walks and the so-called kernel method (as in [19, Example 3]). Alternatively, they can be obtained by combining two results of Gessel (Thm. 4.4 and 5.1 in [49]) based on a factorization of walks (analogous to the Wiener-Hopf decomposition in probability theory [68, Chap. IV]). Let us mention in addition that several other authors, working from a different perspective, have come up with context-free grammars that recursively describe these families of walks, and give combinatorially based algebraic equations satisfied by their generating functions (see for instance [24] and references therein).

The first order of the asymptotic expansions proved in [6] is equivalent to probabilistic results on random walks. However, the complete expansion requires the full strength of singularity analysis used by Cyril Banderier and Philippe.

This attractive paper already has a few descendants. The first of them, also co-authored by Philippe [4], is actually more a sibling than a descendant. The prototype problem of [6] was the enumeration of Dyck paths. The prototype problem of [4] is the enumeration of Łukasiewicz paths: that is, the set of steps $S$ is now infinite, but contains only finitely many positive steps, and one only studies meanders and excursions. The paper focuses on exact enumeration, and the simplest step sets again yield algebraic generating functions (Section 3 of [4]). The authentic descendants
of [6] deal for instance with walks of bounded height [14, 20, 8], or with the area under excursions [7, 66].

In conclusion, we note that analogous questions in two dimensions (walks in a quarter plane) or more are far from being completely understood, and are currently the topic of very active research (see for instance [12, 18, 61, 62]).

2. Solvable classes of self-avoiding polygons

The enumeration of self-avoiding walks or self-avoiding polygons on a lattice is a widely studied problem, not only in combinatorics, probability theory and computer science, but also in theoretical physics where they constitute a classical model of polymers. The topic swarms with remarkable predictions, mostly coming from physics, dealing with the number or properties of large walks and polygons. For instance, in two dimensions, it is believed that the number of polygons of perimeter $2n$ (resp. area $n$) satisfies, up to a multiplicative constant:

$$p_{2n} \sim \kappa n^{-5/2} \quad \text{(resp. } a_n \sim \mu n^{-1} \text{)},$$

where $\kappa$ and $\mu$ are positive numbers depending on the lattice, but the exponents are independent of the lattice [57, 28].

After decades of efforts, a first mathematical breakthrough was achieved in the nineties with the proof of the predicted exponents in dimension 5 or more [50, 56]. It now seems that the veil could soon be lifted on the 2-dimensional case, following two recent results: a prediction which relates large SAWs to a Schramm-Loewner evolution (SLE) process (Lawler, Schramm and Werner [54]), and the rigorous derivation of the growth constant of self-avoiding walks (and self-avoiding polygons) on the hexagonal lattice (Duminil-Copin and Smirnov [27]).

Motivated by the tremendous difficulty of this “simple” problem, many mathematicians and physicists have tried to construct subclasses of self-avoiding walks and self-avoiding polygons that are exactly solvable — meaning that the length and/or area generating functions can be determined exactly — with the subclass being as large as possible, to try to capture some properties of general walks or polygons. Two of Philippe’s papers deal with such classes of 2-dimensional polygons.

2.1. Staircase polygons, or Pólya festoons.
The boundary of these polygons consists of two North-East paths of the same length, that only meet at their endpoints. This natural class of polygons was already studied in the late fifties by Levine [55], who proved that the number of staircase polygons of perimeter $2n$ is a Catalan number. Pólya [59, Eq. (3)] was apparently the first to obtain an expression for the area (and perimeter) generating function. His expression is unusual, in that it contains arbitrarily large positive and negative powers of the area variable $q$:

$$2t + \sum_{P \text{ staircase}} t^{v(P)} \left( q^{a(P)} + q^{-a(P)} \right) = 1 - \frac{1}{\sum_{j\geq 0} t^j P_j(q)}$$

where

$$P_j(q) = \sum_{i=0}^{j} \binom{j}{i}^2 q^{-i(j-i)}$$
with

\[ \frac{j}{i_j} = \frac{(1 - q^{j-i+1}) \cdots (1 - q^i)}{(1-q) \cdots (1-q^i)}. \]

Pólya wrote that the proof of this formula would “be given in a continuation of (t)his paper”, which never appeared. In a subsequent note [34], Philippe gave what was “almost certainly” Pólya’s proof, but did not publish it either, probably because he realized that Gessel had already written this proof [48, Section 11].

A few years after Pólya’s paper, another formula was found [51] for the area generating function of staircase polygons, involving only non-negative powers of \( q \). Similar formulas now exist for many classes of polygons having a convexity property [13]. An analogue of Pólya’s formula for the more general class of directed column-convex polygons was found by Feretić [31]. Pólya’s approach also inspired the perimeter enumeration of \( d \)-dimensional convex polygons [17].

The critical exponents describing the perimeter or area enumeration of staircase polygons are not those of general self-avoiding polygons: the counterpart of (1) is

\[ p_{2n} \sim 4^n n^{-3/2} \quad (\text{resp.} \quad a_n \sim (2.309 \cdots)^n n^0) \]

(see for instance [45, Example IX.14] for the area related result). However, random staircase polygons and general self-avoiding polygons of fixed (and large) perimeter seem to share a common feature: the limit law of their area seems to be, after normalization by \( n^{3/2} \), an Airy area distribution (see Chapter 2 of Volume II). This is proved for staircase polygons [63], but only predicted for general self-avoiding polygons [65, 64].

2.2. Prudent self-avoiding polygons. A self-avoiding walk is prudent if it never takes a step towards a vertex it has already visited. This class of walks has been considered, often independently, by several authors, starting with Turban and Debierre in 1986 [70] (see also [60]). Their enumeration is not completely achieved, but some subclasses with (conjecturally) the same growth constant have been solved exactly [16, 23].

A prudent self-avoiding polygon is a prudent walk ending at distance 1 from its starting point [47, 67]. Once polygons are defined, it becomes natural to count them by their area, rather than perimeter. In collaboration with Nick Beaton and Tony Guttmann [10], Philippe proved that the area generating function of three-sided prudent polygons is given by the following rather formidable expression:

\[
A(q) := \sum_{P \text{ prudent}} q^{a(P)} = \frac{2q(3 - 10q + 9q^2 - q^3)}{(1-2q)^2(1-q)} + 2q^3(1-q)^2 \sum_{m \geq 1} \frac{(-1)^{m+1} q^{2m}}{(1-2q)^m(1-q-q^{m+1})} \prod_{k=1}^{m-1} \frac{1-q-q^k+q^{k+1}-q^{k+2}}{1-q-q^{k+1}}.
\]

A delicate asymptotic study then gives, for the number of polygons of area \( n \), the following asymptotic behaviour:

\[
a_n = \kappa (\log_2 n) 2^n n^\gamma + O(n^{2} n^{\gamma-1}),
\]
where the exponent $\gamma = \log_2 3$ is irrational, and $\kappa(x)$ is a periodic function of $x$. The irrationality of the exponent $\gamma$, and the periodic fluctuations in $\log n$ are very unusual phenomena for 2-dimensional lattice objects. However, periodic fluctuations occur in several problems related to words or trees, having a (slight) number theoretic flavour (see for instance [36, 26, 53, 58], or [45, p. 308], and Chapters 4 and 5 in Volume III.

Several central questions on prudent walks and polygons remain open, for instance the enumeration of all prudent walks (or polygons) by length (or perimeter and/or area), and their asymptotic properties. In particular, uniform prudent walks with many steps are conjectured to stick to a diagonal. A very different behaviour has been proved for a non-uniform (but still very natural) distribution [11].

3. Walks and algorithms: two examples

In each of the last two papers of this chapter, walks play an instrumental role in the analysis of an algorithm.

In the first one [32], entries are inserted and deleted randomly in/from a two-end stack of finite size $m$, until the entries inserted from one end meet the entries inserted from the other end, somewhere in the middle of the stack. This simple allocation scheme — which is in fact a finite state Markov chain — was first described by Knuth [52, Ex. 2.2.2.13], who asked about the stopping time and the law of the final state. The evolution of the stack can be described by a square lattice walk confined to the triangle $x \geq 0, y \geq 0, x + y \leq m$, and the analysis of the chain can be reduced to the enumeration of such walks. This is achieved via repeated applications of the reflection principle, the basic ingredient being the enumeration of walks confined to an interval (in terms of trigonometric functions). A similar walk problem is known to describe the number of inversions in the adjacent transposition random walk on permutations [30, 29, 15]. It can be solved using a bivariate extension of the kernel method mentioned in Section 1, which is likely to apply to [32] as well.

Finally, in [22] a rectangle packing algorithm is analyzed via the study of the expected maximal height of a simple 1-dimensional random walk $S_n$, taking steps 1 and $-1$ with probability $1/2$. The paper improves on the result given by the central limit theorem, namely

$$E\left(\max_{k \leq n} S_k\right) \sim \sqrt{\frac{2n}{3\pi}},$$

by establishing the following refined estimate:

$$E\left(\max_{k \leq n} S_k\right) = \frac{2n}{3\pi} - c + \frac{1}{5} \sqrt{\frac{2}{3\pi n}} + O(n^{-3/2}),$$

where the constant $c$ can be computed to great accuracy. Lower order terms could in principle be calculated as well.

Et maintenant, on rend la plume à Philippe.
Bibliography


