An Improved Version of G$\acute{a}$bor Lugosi's Inequality on the Superiority of the Moment Bound

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G$\acute{a}$bor Lugosi (Concentration of Measure Inequalities, 2004) gives the following inequality which asserts that the bound on tail probabilities of a positive random variable obtained by moment methods is sharper than the classic Chernoff-Bernstein bound. Precisely, let $X \geq 0$ have a finite mgf $E(e^{sX})$ in some neighborhood of zero. Then, for $t > \mu = E(X), P(X \geq t) \leq \min_{k \geq 1} E(\frac{X}{t})^k \leq inf_{s>0}[e^{-st}E(e^{sX})]$.

We show that the moment bound is even more superior than what Lugosi's inequality says. The improved inequality is given below.

Theorem: Let $X \geq 0$ be nondegenerate with a finite mgf $E(e^{sX})$ in some neighborhood of zero. Let $t > \mu = E(X)$, and $m + 1 = argmin_k E(\frac{X}{t})^k$. Let also $s^* = arginf_{s>0}[e^{-st}E(e^{sX})]$. Then,

$$inf_{s>0}[e^{-st}E(e^{sX})] \geq [\min_{k \geq 1} E(\frac{X}{t})^k]\,[1 + (\frac{t\mu_m}{\mu_{m+1}} - 1)P(Pois(s^*t) \leq m)],$$

where $\mu_j = E(X^j)$, $Pois(\lambda)$ denotes a Poisson random variable with mean λ, and the quantity $\frac{t\mu_m}{\mu_{m+1}} - 1$ is necessarily nonnegative.